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On the distribution of the nontrivial zeros of quadratic L-functions of imaginary quadratic number fields close to the real axis

by

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1. Introduction. The distribution of zeros near $s = 1/2$ for various families of L-functions has received widespread attention since the appearance of the seminal joint work of Katz and Sarnak, [8] and [9]. One aspect of the basic conjectures which came to light through their research relates the one-level normalized spacings of "low-lying" zeros of certain families of L-functions, when ordered by their conductors, to classical symmetry groups associated with each family.

In the case of the family of quadratic Dirichlet L-functions (over $\mathbb{Q}$), partial results (cf. [8], [13], [14]) suggest that the symmetry group associated with this family should be symplectic; see [8] for more details. In this case the functional equations for the completed L-functions are "self-dual", i.e. remain invariant under the substitution $s \mapsto 1 - s$. In the function field analogue, where the distribution conjectures become theorems, Katz and Sarnak have shown that for certain families of zeta functions whose functional equations are self-dual, the distribution of the zeros is governed by a symplectic symmetry group (cf. [8], [9]).

In the present note, we consider the family of quadratic L-functions over an arbitrary imaginary quadratic base field. We study the distribution of their nontrivial zeros close to the real axis. Here, as above, we find that the completed L-functions have self-dual functional equations; and once again we see, as in the case over $\mathbb{Q}$, that the distribution of the zeros indicates a symplectic symmetry group, which is as expected. It should be pointed out, however, that self-duality of the functional equations associated with a family of L-functions need not indicate a symplectic distribution of low-lying zeros. Peter Sarnak has pointed out to us that there are families of L-functions with associated self-dual functional equations, but for which the

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distribution of the low-lying zeros seems to suggest an orthogonal symmetry group.

The method of proof is similar to that in our previous article, [14], but there were obstacles to overcome. The main problem was figuring out the "main term" in the "form factor" in the case where $D = o(x)$. This impasse was overcome by using quadratic reciprocity for the 2-power residue symbol (generalized Legendre symbol) over an arbitrary number field as described in Hecke's text [6].

2. Statements of main results. At this point we state our main theorem and its two corollaries. Before doing so we need to introduce some notation and concepts. Let $K(s)$ be a function analytic in the strip $-1 < \mathrm{Re}(s) < 2$ such that $|K(\sigma + it)| \ll t^{-2}$ as $t \to \infty$ and such that the function

$$a(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s) x^{-s}\, ds$$

is absolutely convergent for $-1 < c < 2$ and all $x > 0$, continuous, differentiable for all but finitely many points, of bounded variation, real-valued, nonnegative, of compact support on the interval $(0, \infty)$, and such that $a(1) \neq 0$. Furthermore, assume $K(1/2 + it) = K(1/2 - it)$ for all real t. A particular choice of $K(s)$ is given by

$$K(s) = \left(\frac{e^{s-1/2} - e^{-s+1/2}}{2s - 1} \right)^2,$$

in which case

$$a(t) = \begin{cases} \frac{1}{2} t^{-1/2} \big(1 - \frac{1}{2} |\log t|\big) & \text{if } e^{-2} < t < e^2, \\ 0 & \text{otherwise} \end{cases}$$

(see [12]).

Next, let k be a complex quadratic number field with discriminant d_k and ring of integers $\mathcal{O} = \mathcal{O}_k$. If α is any nonzero integer in k, then denote by χ_α the 2-power residue symbol with respect to α. Let $L(s, \chi_\alpha)$ be the (quadratic) L-series attached to χ_α. Also let $N\alpha$ be the norm of α from k to $\mathbb{Q}$.

The point of our main theorem is to investigate the behavior of the sum

$$\sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha / D} \sum_{\varrho(\alpha)} K(\varrho) x^{\varrho}$$

as $D \to \infty$, where the outer sum ranges over the nonzero elements of $\mathcal{O}$ and the inner sum is over the nontrivial zeros of $L(s, \chi_\alpha)$.

Main Theorem. *Assume all the notation above and suppose that the Generalized Riemann Hypothesis* (*GRH*) *holds for all abelian L-functions* *over k. Then as $D \to \infty$ and either $x \to \infty$ or $x = 1$,*

$$\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^{\varrho} =$$

$$\begin{cases} -\frac{1}{2}K\left(\frac{1}{2}\right)Dx^{1/2} + \frac{1}{2}K(1)\pi(4|d_k|)^{-1/4}xD^{1/2} + a\left(\frac{1}{x}\right)D\log D \\ \qquad + O\big(D\log D\log x + Dx^{1/3}\log x + a\left(\frac{1}{x}\right)D\big) & \text{if } x = o(D), \\ 0 + O(x\log^2 x + Dx^{1/3}\log x) & \text{if } D = o(x). \end{cases}$$

All the implied constants depend only on the base field k and the kernel K.

Now define, for $y \in \mathbb{R}$,

$$F_K(y, D) = \left(\frac{1}{2}K\left(\frac{1}{2}\right)D\right)^{-1} \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)D^{iy\gamma},$$

where $\varrho = 1/2 + i\gamma$. Then we have the following corollary, which is just a special case of the Main Theorem.

First Corollary. *Assuming GRH for all abelian L-functions over k, as $D \to \infty$,*

$$F_K(y, D) = \begin{cases} -1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2}a(D^{-y})\log D + o(1) & \text{if } |y| < 1, \\ 0 + o(1) & \text{if } 1 < |y| < 2, \end{cases}$$

uniformly on compact subsets of $(-2,-1) \cup (-1,1) \cup (1,2)$.

The second corollary concerns the distribution of the nontrivial zeros of our family of L-functions near the real axis.

Second Corollary. *Suppose $r(y)$ is an even continuous function with $r(y)$ and $yr(y)$ in $L^1(\mathbb{R})$, such that its Fourier transform,*

$$\widehat{r}(y) = \int_{-\infty}^{\infty} r(u)e^{-2\pi iyu}\,du,$$

is also continuous and in $L^1(\mathbb{R})$, and has compact support in $(-2,2) \setminus \{\pm 1\}$. Then under GRH for all abelian L-functions over k, an imaginary quadratic number field, as $D \to \infty$,

$$D^{-1} \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} \sum_{\varrho(\alpha)} K(\varrho)r\left(\frac{\gamma\log D}{2\pi}\right)$$
$$= 2\int_{-\infty}^{\infty}\left(1 - \frac{\sin 2\pi y}{2\pi y}\right)r(y)\,dy + o(1),$$

where the implied constant depends only on the field k and the kernel K.

3. Preliminaries. Let k be an imaginary quadratic number field. Denote by $\mathcal{O} = \mathcal{O}_k$ the ring of integers of k and by d_k its discriminant. If α is a nonzero algebraic integer of k, then let χ_α be the 2-power residue symbol; i.e. if $\mathfrak{p}$ is any (nonzero) prime ideal of $\mathcal{O}$ not dividing (2α), then let

$$\chi_\alpha(\mathfrak{p}) = \left(\frac{\alpha}{\mathfrak{p}}\right) = \begin{cases} 1 & \text{if } x^2 \equiv \alpha \bmod \mathfrak{p} \text{ has a solution,} \\ -1 & \text{if } x^2 \equiv \alpha \bmod \mathfrak{p} \text{ has no solutions.} \end{cases}$$

Define $\chi_\alpha(\mathfrak{p}) = 0$ if $\mathfrak{p}$ divides (2α), and extend this definition to all nonzero ideals $\mathfrak{a}$ of $\mathcal{O}$ by multiplicativity. As is well known, χ_α is induced by a primitive character χ_α^* which may be identified with the Artin symbol $(\,\cdot\,, K/k)$ where $K = k(\sqrt{\alpha})$. Indeed, $(\mathfrak{p}, K/k)(\sqrt{\alpha}) = (\alpha/\mathfrak{p})\sqrt{\alpha}$ for all $\mathfrak{p}$ relatively prime to 2α.

Let

$$L(s, \chi_\alpha) = \sum_{\mathfrak{a}} \frac{\chi_\alpha(\mathfrak{a})}{N\mathfrak{a}^s} \qquad (\mathrm{Re}(s) > 1),$$

where the sum is over all nonzero integral ideals $\mathfrak{a}$ of $\mathcal{O}_k$ and $N\mathfrak{a} = \#(\mathcal{O}/\mathfrak{a})$. The L-function associated with χ_α^* is defined analogously. As is well known, these L-series have Euler product expansions by the unique factorization property of ideals in $\mathcal{O}$. Moreover, the Dedekind zeta function of K satisfies

$$\zeta_K(s) = \zeta_k(s) L(s, \chi_\alpha^*) \qquad (\mathrm{Re}(s) > 1).$$

All of the functions above can be analytically continued to the whole complex plane (with simple pole at $s = 1$ for zeta functions), thanks to Hecke. Thus, in particular, $L(s, \chi_\alpha^*)$ and $L(s, \chi_\alpha)$ differ by a finite Euler factor and so have the same set of nontrivial zeros.

Next, we apply Weil's explicit formula to $L(s, \chi)$ where $\chi = \chi_\alpha^*$. To this end, we refer to the article [1] of Barner. Suppose $F : \mathbb{R} \to \mathbb{C}$ is a function of bounded variation; let

$$\Phi(s) = \int_{-\infty}^{\infty} F(z) e^{(s-1/2)z}\, dz.$$

By a change of variable, $z = \log t$, we see that

$$\Phi(s) = \int_0^{\infty} F(\log t) t^{s-1/2} \,\frac{dt}{t}.$$

If $K(s)$ is a function on $\mathbb{C}$ and x is any real number greater than 2, formally set $K(s)x^s = \Phi(s) =: \Phi_x(s)$. Then

$$K(s) = \int_0^{\infty} F(\log t) x^{-s} t^{s-1/2} \,\frac{dt}{t} = \int_0^{\infty} x^{-1/2} F(\log t)(x^{-1}t)^{s-1/2} \,\frac{dt}{t}.$$

Replacing $x^{-1}t$ by t we obtain

$$K(s)=\int_0^\infty (xt)^{-1/2}F(\log(xt))t^s\,\frac{dt}{t}=\int_0^\infty a(t)t^s\,\frac{dt}{t},$$

where $a(t)=(xt)^{-1/2}F(\log(xt))$. Then notice that $F(\log z)=z^{1/2}a(z/x)$.

Now assume that $K(s)$ is rapidly decreasing in t where $s=\sigma+it$ (i.e. $|K(\sigma+it)|\ll t^{-2}$ as $t\to\infty$) and $K(1/2+it)=K(1/2-it)$ such that $a(t)$ is nonnegative and has compact support in $(0,\infty)$. In particular, assume that its support lies in $[A,B]$. Then Weil's explicit formula takes the form

$$\begin{aligned}\sum_{\varrho}K(\varrho)x^{\varrho}&=\varepsilon_0(K(0)+K(1)x)+a\Big(\frac{1}{x}\Big)\log\frac{(Nd_{K/k})|d_k|}{4\pi^2}\\&\quad-\sum_{\mathfrak p}\sum_{n=1}^{\infty}(\log N\mathfrak p)a\Big(\frac{N\mathfrak p^n}{x}\Big)\chi(\mathfrak p^n)\\&\quad-\sum_{\mathfrak p}\sum_{n=1}^{\infty}\frac{\log N\mathfrak p}{N\mathfrak p^n}\,a\Big(\frac{1}{xN\mathfrak p^n}\Big)\chi(\mathfrak p^n)+W_\infty(a,\chi),\end{aligned}$$

where the $\sum_\varrho$ is over the nontrivial zeros of $L(s,\chi)$, $\sum_{\mathfrak p}$ is over the nonzero prime ideals of $\mathcal O$, $\varepsilon_0=\varepsilon_0(\chi)=1$ if χ is the principal character, and $\varepsilon_0=0$ if not. Finally,

$$\begin{aligned}W_\infty(a,\chi)&=2\,\frac{\Gamma'}{\Gamma}\Big(\frac12\Big)a\Big(\frac1x\Big)\\&\quad-\int_1^\infty\Big(t^{1/2}a\Big(\frac tx\Big)+t^{-1/2}a\Big(\frac{t^{-1}}{x}\Big)-2a\Big(\frac1x\Big)\Big)\frac{t^{-1/2}}{1-t^{-1}}\,\frac{dt}{t}.\end{aligned}$$

This implies that

$$\begin{aligned}\sum_{\varrho}K(\varrho)x^{\varrho}&=\varepsilon_0K(1)x+a\Big(\frac1x\Big)\log\frac{(Nd_{K/k})|d_k|}{4\pi^2}\\&\quad-\sum_{\mathfrak p}\sum_{n=1}^{\infty}(\log N\mathfrak p)a\Big(\frac{N\mathfrak p^n}{x}\Big)\chi(\mathfrak p^n)+O(1).\end{aligned}$$

We need to extend this result to the (not necessarily primitive) characters χ_α.

PROPOSITION 1. *If $L(s,\chi_\alpha)$ is the L-function associated with χ_α, then the following explicit formula holds*:

$$\begin{aligned}\sum_{\varrho}K(\varrho)x^{\varrho}&=\varepsilon_0K(1)x+a\Big(\frac1x\Big)\log\frac{(N\alpha)|d_k|}{4\pi^2}\\&\quad-\sum_{\mathfrak p}\sum_{n=1}^{\infty}(\log N\mathfrak p)a\Big(\frac{N\mathfrak p^n}{x}\Big)\Big(\frac{\alpha}{\mathfrak p^n}\Big)+O((\log N\alpha)(\log x)).\end{aligned}$$

Proof. Suppose χ is the primitive character of conductor $\mathfrak{f}$ that induces χ_α. Then as noted earlier $L(s,\chi)$ and $L(s,\chi_\alpha)$ share the same set of nontrivial zeros. Then the left-hand side of the equation in the proposition is identical with that in the explicit formula above. Now consider the difference of the right-hand sides of the two formulas:

$$\begin{aligned}\sum_{\mathfrak{p},n}(\log N\mathfrak{p})a\left(\frac{N\mathfrak{p}^n}{x}\right)(\chi(\mathfrak{p}^n)-\chi_\alpha(\mathfrak{p}^n)) &\ll \sum_{\mathfrak{p}\mid(2\alpha)}(\log N\mathfrak{p})a\left(\frac{N\mathfrak{p}}{x}\right)\\ &\ll \sum_{\substack{\mathfrak{p}\mid(2\alpha)\\ N\mathfrak{p}\le x}}\log N\mathfrak{p} \ll (\log N\alpha)(\log x),\end{aligned}$$

since if $(2\alpha)=\mathfrak{p}_1^{a_1}\cdots\mathfrak{p}_m^{a_m}$ is the prime ideal factorization of 2α, then

$$\log N\alpha = a_1\log N\mathfrak{p}_1+\cdots+a_m\log N\mathfrak{p}_m \gg m.$$

Now notice that

$$a\left(\frac{1}{x}\right)\log\frac{(N\alpha)|d_k|}{4\pi^2} - a\left(\frac{1}{x}\right)\log\frac{(N\mathfrak{f})|d_k|}{4\pi^2} = O(\log N\alpha).$$

This establishes the proposition. ∎

4. Technical lemmas. We start by stating a 2-variable version of Euler–Maclaurin summation. Let $P_m(x)$ denote the mth periodic Bernoulli function. Hence $P_m(x)=B_m(x-[x])$, where as usual $B_m(x)$ is defined by

$$\frac{e^{tx}}{e^t-1}=\sum_{m=0}^{\infty}B_m(x)\,\frac{t^m}{m!}.$$

LEMMA 1 (Euler–Maclaurin summation). *Let N be a positive integer and $f(u,v)$ be a function such that any $2N$th-order partial derivative of f is continuous. Then for integers a,b,c,d with $a\le b$ and $c\le d$ we have*

$$\begin{aligned}\sum_{n=c}^{d}\sum_{m=a}^{b}f(m,n) &= \int_c^d\int_a^b f(u,v)\,du\,dv\\ &\quad+\sum_{\mu,\nu=1}^{N}\frac{(-1)^{\mu+\nu}}{\mu!\nu!}\,P_\mu(u)P_\nu(v)\,\frac{\partial^{\mu+\nu-2}f(u,v)}{\partial u^{\mu-1}\partial v^{\nu-1}}\bigg|_{a^-}^{b^+}\bigg|_{c^-}^{d^+}\\ &\quad+\sum_{\mu=1}^{N}\frac{(-1)^{\mu}}{\mu!}\,P_\mu(u)\int_c^d\frac{\partial^{\mu-1}f(u,v)}{\partial u^{\mu-1}}\,dv\bigg|_{a^-}^{b^+}\\ &\quad+\sum_{\nu=1}^{N}\frac{(-1)^{\nu}}{\nu!}\,P_\nu(v)\int_a^b\frac{\partial^{\nu-1}f(u,v)}{\partial v^{\nu-1}}\,du\bigg|_{c^-}^{d^+}\end{aligned}$$

$$+\sum_{\mu=1}^{N}\frac{(-1)^{N+\mu-1}}{\mu!N!}\,P_\mu(u)\int_c^d P_N(v)\,\frac{\partial^{N+\mu-1}f(u,v)}{\partial u^{\mu-1}\partial v^N}\,dv\,\Big|_{a^-}^{b^+}$$
$$+\sum_{\nu=1}^{N}\frac{(-1)^{N+\nu-1}}{\nu!N!}\,P_\nu(v)\int_a^b P_N(u)\,\frac{\partial^{N+\nu-1}f(u,v)}{\partial v^{\nu-1}\partial u^N}\,du\,\Big|_{c^-}^{d^+}$$
$$+\int_c^d\int_a^b\bigg[\frac{(-1)^{N-1}}{N!}\,P_N(u)\,\frac{\partial^N f}{\partial u^N}+\frac{(-1)^{N-1}}{N!}\,P_N(v)\,\frac{\partial^N f}{\partial v^N}$$
$$+\frac{1}{(N!)^2}\,P_N(u)P_N(v)\,\frac{\partial^{2N}f}{\partial u^N\partial v^N}\bigg]\,du\,dv,$$

where, e.g., $u=a^-$ and $v=b^+$ denote the appropriate one-sided limits.

The proof of the lemma follows by applying the one-variable version twice and is left to the reader. The one-variable version may be found on page 490 of [7], for example.

Corollary 1. *Let $m\in\mathbb{Z}^{>0}$, $\alpha,\beta\in\mathbb{C}$ with $\alpha\overline{\beta}-\overline{\alpha}\beta\neq 0$, and $c>0$. Then*

$$\sum_{(m_1,m_2)\in\mathbb{Z}^2} e^{-c|m_1\alpha+m_2\beta|^{2m}y^{-2m}}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-c|u\alpha+v\beta|^{2m}y^{-2m}}\,du\,dv+O(y^{-M})$$

as $y\to\infty$ for any positive integer M.

Proof. Let
$$f(u,v)=e^{-c|u\alpha+v\beta|^{2m}y^{-2m}}.$$
Then f is a rapidly decreasing function and thus we may apply Lemma 1, noting that all the relevant sums and integrals converge, obtaining

$$\sum_{(m_1,m_2)\in\mathbb{Z}^2} f(m_1,m_2)$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\,du\,dv+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bigg[\frac{(-1)^{N-1}}{N!}\,P_N(u)\,\frac{\partial^N f}{\partial u^N}$$
$$+\frac{(-1)^{N-1}}{N!}\,P_N(v)\,\frac{\partial^N f}{\partial v^N}+\frac{1}{(N!)^2}\,P_N(u)P_N(v)\,\frac{\partial^{2N}f}{\partial u^N\partial v^N}\bigg]\,du\,dv.$$

We consider the second integral. An easy induction argument shows that
$$\frac{\partial^M f(u,v)}{\partial u^\mu\partial v^\nu}=f(u,v)\,\sum_{j=1}^{M} y^{-2mj}h_{2mj-M}^{(\mu,\nu)}(u,v),$$
where $h_n(u,v)$ is a homogeneous polynomial of degree n and where we set this form equal to 0 if the "degree" is negative. Now let $u=yu_1$ and $v=yv_1$;

then, in particular,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_N(u)\,\frac{\partial^N f}{\partial u^N}\,du\,dv$$
$$= y^{2-N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_N(yu_1)e^{-c|u_1\alpha+v_1\beta|^{2m}}\sum_{j=1}^{N} h^{(N,0)}_{2mj-N}(u_1,v_1)\,du_1\,dv_1.$$

Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P_N(u)\,\frac{\partial^N f}{\partial u^N}\,du\,dv \ll y^{2-N}.$$

The other two terms are similarly $\ll y^{2-N}$. If we let $N = M+2$, the corollary is established. ∎

LEMMA 2. *Let $\mathfrak{a}$ be a nonzero fractional ideal of k, and let $m \in \mathbb{Z}^{>0}$. Then*

$$\sum_{\mu\in\mathfrak{a}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N(\mu)^m y^{-1}} = \frac{y^{1/m}\mathcal{I}}{N\mathfrak{a}\sqrt{|d_k|}^{\,1-1/m}} + O(y^{-M})$$

as $y\to\infty$, for any $M\in\mathbb{Z}^{>0}$, and where

$$\mathcal{I} = \frac{(2\pi)^{1-1/m}}{m}\,\Gamma\Big(\frac{1}{m}\Big).$$

The sum is over all elements of $\mathfrak{a}$ and $N\mu$ represents the norm from k to $\mathbb{Q}$ of μ. In particular

$$\sum_{\mu\in\mathfrak{a}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N(\mu)^m y^{-1}} = \begin{cases} \dfrac{y}{N\mathfrak{a}} + O(y^{-M}) & \text{if } m=1,\\[2mm] \dfrac{y^{1/2}}{N\mathfrak{a}}\,\pi 2^{-1/2}|d_k|^{-1/4} + O(y^{-M}) & \text{if } m=2.\end{cases}$$

Proof. Let $\{\alpha,\beta\}$ be an integral basis of $\mathfrak{a}$. Then

$$\sum_{\mu\in\mathfrak{a}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu^m y^{-1}} = \sum_{(m_1,m_2)\in\mathbb{Z}^2} e^{-\frac{2\pi}{\sqrt{|d_k|}}|m_1\alpha+m_2\beta|^{2m}y^{-1}}.$$

By the corollary, we need only show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}N(x_1\alpha+x_2\beta)^m y^{-1}}\,dx_1\,dx_2 = \frac{y^{1/m}\mathcal{I}}{N\mathfrak{a}\sqrt{|d_k|}^{\,1-1/m}}.$$

To this end, let

$$u+vi = \frac{x_1\alpha+x_2\beta}{(y\sqrt{|d_k|})^{1/2m}}, \quad \text{so} \quad u-vi = \frac{x_1\bar{\alpha}+x_2\bar{\beta}}{(y\sqrt{|d_k|})^{1/2m}}.$$

Now

$$\begin{pmatrix} u+vi \\ u-vi \end{pmatrix} = \frac{1}{(y\sqrt{|d_k|})^{1/2m}} \begin{pmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and

$$\left| \det \begin{pmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix} \right| = N(\mathfrak{a})\sqrt{|d_k|}$$

(see, e.g., [5, p. 188]). By elementary row reduction we get $(u,v)^{\mathrm{t}} = B(x_1,x_2)^{\mathrm{t}}$ where

$$B = \frac{1}{(y\sqrt{|d_k|})^{1/2m}} \begin{pmatrix} (\alpha+\overline{\alpha})/2 & (\beta+\overline{\beta})/2 \\ (\alpha-\overline{\alpha})/2i & (\beta-\overline{\beta})/2i \end{pmatrix}.$$

But then notice that

$$|\det(B)| = \frac{1}{2(y\sqrt{|d_k|})^{1/m}} \left| \det \begin{pmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix} \right|.$$

Hence

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}|x_1\alpha+x_2\beta|^{2m}y^{-1}}\,dx_1\,dx_2 = |\det B^{-1}| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(u^2+v^2)^m}\,du\,dv.$$

But

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi(u^2+v^2)^m}\,du\,dv = \int_{0}^{2\pi}\int_{0}^{\infty} e^{-2\pi r^{2m}} r\,dr\,d\theta = 2\pi \int_{0}^{\infty} e^{-2\pi r^{2m}} r\,dr$$
$$= \frac{(2\pi)^{1-1/m}}{2m}\,\Gamma\left(\frac{1}{m}\right),$$

by changing first to polar coordinates and then changing to $t = 2\pi r^{2m}$.

This establishes the lemma. ■

LEMMA 3. *The series*

$$\sum_{\substack{\mu\in\mathcal{O} \\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y} \log N\mu = y\log y + O(y)$$

as $y \to \infty$.

Proof. Write

$$\sum_{\substack{\mu\in\mathcal{O} \\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y} \log N\mu = S_1 + S_2,$$

where

$$S_1 = \sum_{\substack{\mu \in \mathcal{O} \\ \mu \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mu/y} \log y, \qquad S_2 = \sum_{\substack{\mu \in \mathcal{O} \\ \mu \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mu/y} \log\left(\frac{N\mu}{y}\right).$$

Notice that

$$S_1 = y \log y + O(\log y)$$

by Lemma 2.

Now consider S_2. By Riemann–Stieltjes integration,

$$S_2 = \int_{1^-}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right) dJ(u),$$

where

$$J(u) = \sum_{\substack{\mu \in \mathcal{O} \\ N\mu \leq u}} 1.$$

But $J(u) = \varrho u + O(u^{1/2})$ where ϱ is a constant depending only on the field k; see, e.g., Lang [11, p. 132]. Hence,

$$S_2 = \int_{1}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right) d\varrho u - \int_{1^-}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right) d(J(u) - \varrho u).$$

The first integral is $\ll y$, by direct computation. In the second integral, we integrate by parts and then make a change of variable, obtaining

$$\begin{aligned}
&\int_{1^-}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right) d(J(u) - \varrho u) \\
&\qquad = (J(u) - \varrho u) e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right)\Bigg|_{1^-}^{\infty} \\
&\qquad\quad - \int_{1}^{\infty} (J(u) - \varrho u)\, d\left(e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right)\right) \\
&\qquad \ll \log y \int_{1}^{\infty} (J(u) - \varrho u) e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \frac{du}{u} \\
&\qquad\quad + \frac{2\pi}{y\sqrt{|d_k|}} \int_{1}^{\infty} (J(u) - \varrho u) e^{-\frac{2\pi}{\sqrt{|d_k|}} u/y} \log\left(\frac{u}{y}\right) du \\
&\qquad \ll \log y \int_{y^{-1}}^{\infty} (J(yv) - \varrho yv) e^{-\frac{2\pi v}{\sqrt{|d_k|}}} \frac{dv}{v} \\
&\qquad\quad + \frac{2\pi}{\sqrt{|d_k|}} \int_{y^{-1}}^{\infty} (J(yv) - \varrho yv) e^{-\frac{2\pi v}{\sqrt{|d_k|}}} \log v\, dv \ll y^{1/2} \log y.
\end{aligned}$$

This establishes the result. ∎

We shall also need a variation of the previous lemma. Namely,

LEMMA 4. *For $\mu \in \mathcal{O}_k - \{0\}$, let $\mathfrak{f}_\mu$ denote the conductor of χ_μ. Then the series*

$$\sum_{\substack{\mu\in\mathcal{O}\\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y}\log N\mathfrak{f}_\mu = y\log y + O(y)$$

as $y \to \infty$.

Proof. By subtracting the formula above from the one in Lemma 3, we see that it suffices to show that

$$\sum_{\substack{\mu\in\mathcal{O}\\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y}\log\left(\frac{N\mu}{N\mathfrak{f}_\mu}\right) \ll y.$$

To this end, let

$$\sum_{\substack{\mu\in\mathcal{O}\\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y}\log\left(\frac{N\mu}{N\mathfrak{f}_\mu}\right) = \int_{1^-}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}u/y}\,d\alpha(u),$$

where $\alpha(x) = \sum_{N\mu\le x}\log(N\mu/N\mathfrak{f}_\mu)$. We claim $\alpha(x) \ll x$. To see this, first by the conductor-discriminant formula (see [4] or [2]), $\mathfrak{f}_\mu = D_{K/k}$ where $D_{K/k}$ is the relative discriminant, $K = k(\sqrt{\mu})$. But since K/k is a Kummer extension, $\mathfrak{f}_\mu$ can be determined fairly easily; cf. [4], [6], or [2]. In our case, we can see that the fractional ideal $(\mu)/\mathfrak{f}_\mu = \mathfrak{a}^2/\mathfrak{b}$ where $\mathfrak{b}\mid(16)$. Moreover, $\mathfrak{f}_\mu = \mathfrak{c}_0\mathfrak{c}_1$ where $\mathfrak{c}_0$ is square-free and $\mathfrak{c}_1\mid(32)$. Thus we have

$$\alpha(x) \ll \sum_{\substack{\mathfrak{m},\mathfrak{n}\\ N(\mathfrak{m}^2\mathfrak{n})\le x}} \mu^2(\mathfrak{n})\log N\mathfrak{m},$$

where $\mathfrak{m},\mathfrak{n}$ are integral ideals of k and μ is the usual generalization of the Möbius μ-function to the semigroup of integral ideals of k. Now notice that

$$\sum_{N\mathfrak{n}\le x}\mu^2(\mathfrak{n}) \ll \sum_{N\mathfrak{n}\le x} 1 \ll x$$

(see [11], for example). Therefore,

$$\sum_{\substack{\mathfrak{m},\mathfrak{n}\\ N(\mathfrak{m}^2\mathfrak{n})\le x}} \mu^2(\mathfrak{n})\log N\mathfrak{m} = \sum_{N\mathfrak{m}\le\sqrt{x}}\log N\mathfrak{m}\sum_{N\mathfrak{n}\le\frac{x}{N\mathfrak{m}^2}}\mu^2(\mathfrak{n}) \ll \sum_{N\mathfrak{m}\le\sqrt{x}}\log N\mathfrak{m}\,\frac{x}{N\mathfrak{m}^2}$$

$$\ll x\sum_{N\mathfrak{m}\le\sqrt{x}}\frac{\log N\mathfrak{m}}{N\mathfrak{m}^2} \ll_\varepsilon x\zeta_k(2-\varepsilon) \ll x,$$

which establishes the claim.

Now, finally integrating by parts and then by a change of variables we have

$$\int_{1^-}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}u/y}\, d\alpha(u) = e^{-\frac{2\pi}{\sqrt{|d_k|}}u/y}\alpha(u)\Big|_{1^-}^{\infty} + \frac{2\pi}{y\sqrt{|d_k|}}\int_1^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}u/y}\alpha(u)\, du$$
$$\ll \frac{1}{y}\int_1^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}u/y} u\, du \ll \frac{1}{y}\int_1^{\infty} e^{-v}y^2\, dv \ll y.$$

This establishes the lemma. ∎

We assume the "Generalized Riemann Hypothesis (GRH) for k" in the following lemma, i.e. the GRH holds for all abelian L-functions over k.

LEMMA 5. *Let k be any algebraic number field and let $C_{\mathfrak{f}}$ be the ray class group modulo $\mathfrak{f}$, i.e. the group of $\mathfrak{f}$-classes* (*see* [10] *or* [11, *Chapter* VI]). *Denote the order of $C_{\mathfrak{f}}$ by $h_{\mathfrak{f}}$. Let m, n be real numbers with $n \neq 0$. Assuming GRH for k, the sum*

$$\sum_{\substack{\mathfrak{p}\,:\,\mathfrak{p}\in\mathfrak{c}\\ f(\mathfrak{p})=1}} a\left(\frac{N\mathfrak{p}^n}{x}\right) N\mathfrak{p}^m \log N\mathfrak{p} = \frac{1}{h_{\mathfrak{f}}}\frac{1}{n} K\left(\frac{m+1}{n}\right) x^{(m+1)/n}$$
$$+ O(x^{(2m+1)/(2n)}\log^2 x)$$

as $x \to \infty$, where the sum is over all nonzero prime ideals of $\mathcal{O}$ which lie in $\mathfrak{c} \in C_{\mathfrak{f}}$ and have absolute residue degree $f(\mathfrak{p}) = [\mathcal{O}/\mathfrak{p} : \mathbb{Z}/(p)] = 1$.

Proof. By Riemann–Stieltjes integration, we have

$$\sum_{\substack{\mathfrak{p}\,:\,\mathfrak{p}\in\mathfrak{c}\\ f(\mathfrak{p})=1}} a\left(\frac{N\mathfrak{p}^n}{x}\right) N\mathfrak{p}^m \log N\mathfrak{p} = \int_{0^+}^{\infty} a\left(\frac{u^n}{x}\right) u^m\, d\Big(\sum_{\substack{N\mathfrak{p}\le u\\ \mathfrak{p}\in\mathfrak{c}\\ f(\mathfrak{p})=1}} \log N\mathfrak{p}\Big).$$

Hence by the Prime Ideal Theorem, assuming GRH,

$$\sum_{\substack{N\mathfrak{p}\le u\\ \mathfrak{p}\in\mathfrak{c}\\ f(\mathfrak{p})=1}} \log N\mathfrak{p} = \frac{1}{h_{\mathfrak{f}}} u + E(u),$$

with $E(u) \ll u^{1/2}\log^2 u$, where the implied constant depends only on the field k and not on the class $\mathfrak{c}$; see [10]. Thus

$$\int_{0^+}^{\infty} a\left(\frac{u^n}{x}\right) u^m\, d\Big(\sum_{\substack{N\mathfrak{p}\le u\\ \mathfrak{p}\in\mathfrak{c}\\ f(\mathfrak{p})=1}} \log N\mathfrak{p}\Big) = \frac{1}{h_{\mathfrak{f}}}\int_0^{\infty} a\left(\frac{u^n}{x}\right) u^m\, du + \int_{0^+}^{\infty} a\left(\frac{u^n}{x}\right) u^m\, dE(u).$$

In the first integral let $v = u^n/x$; then

$$\int_0^{\infty} a\left(\frac{u^n}{x}\right) u^m\, du = \frac{1}{n} x^{(m+1)/n}\int_0^{\infty} a(v) v^{(m+1)/n}\frac{dv}{v} = \frac{1}{n} K\left(\frac{m+1}{n}\right) x^{(m+1)/n}.$$

Using integration by parts on the second integral, we obtain

$$\begin{aligned}
&\int_{0^+}^{\infty} a\Big(\frac{u^n}{x}\Big)u^m\,dE(u)\\
&\qquad = -\frac{n}{x}\int_0^{\infty} E(u)a'\Big(\frac{u^n}{x}\Big)u^{m+n-1}\,du - m\int_0^{\infty} E(u)a\Big(\frac{u^n}{x}\Big)u^{m-1}\,du\\
&\qquad = -\,x^{m/n}\int_0^{\infty} E(x^{1/n}v^{1/n})a'(v)v^{(m+n)/n}\,\frac{dv}{v}\\
&\qquad\quad -\frac{m}{n}\,x^{m/n}\int_0^{\infty} E(x^{1/n}v^{1/n})a'(v)v^{m/n}\,\frac{dv}{v} \ll x^{m/n}x^{1/2n}\log^2 x,
\end{aligned}$$

as desired. ∎

Lemma 6 (Transformation formula). *Let $\mathfrak{p}$ be a nonzero odd prime ideal of $\mathcal{O}$ dividing the rational prime $p>0$, but not dividing the discriminant d_k of k; let $\chi=\chi_{\mathfrak{p}}$, where $\chi_{\mathfrak{p}}(\alpha)=(\alpha/\mathfrak{p})$.*

If $(p)=\mathfrak{p}\overline{\mathfrak{p}}$, then

$$\sum_{\alpha\in\mathcal{O}}\chi_{\mathfrak{p}}(\alpha)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/y}=\frac{y}{\sqrt{N\mathfrak{p}}}\sum_{\nu\in\overline{\mathfrak{p}}/\sqrt{d_k}}\chi_{\mathfrak{p}}(\nu)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu|d_k|y/p^2}.$$

If p is inert, then

$$\sum_{\alpha\in\mathcal{O}}\chi_{\mathfrak{p}}(\alpha)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/y}=\frac{y}{\sqrt{N\mathfrak{p}}}\sum_{\nu\in\mathcal{O}}\chi_{\mathfrak{p}}(\nu)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu y/p^2}.$$

Proof. First notice that if $\chi(W_k)\neq\{1\}$, where W_k denotes the group of roots of unity in k, then $\sum_{\alpha\in W_k}\chi(\alpha)=0$ by a standard argument. But then the assertion of the lemma is obvious as all sides of the equalities are 0.

Now assume $\chi(W_k)=\{1\}$. If $\mathfrak{p}=(p)$, let $\mathfrak{c}=1$; otherwise, let $\mathfrak{c}=\overline{\mathfrak{p}}$. Notice then that $(\mathfrak{c},\mathfrak{p})=1$ since by assumption $\mathfrak{p}/p$ is unramified as $d_k\not\equiv 0 \bmod p$. Furthermore, let $c=2\pi\sqrt{N\mathfrak{c}}/\sqrt{|d_k|}$ and $c'=2\pi|d_k|/(\sqrt{|d_k|}\sqrt{N\mathfrak{c}})$. Then

$$\sum_{\alpha\in\mathcal{O}}\Big(\frac{\alpha}{\mathfrak{p}}\Big)e^{-c(N\alpha)t}=\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\Big(\frac{\varrho}{\mathfrak{p}}\Big)\sum_{\mu\in\mathfrak{p}}e^{-cN(\mu+\varrho)t}=\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\Big(\frac{\varrho}{\mathfrak{p}}\Big)\sum_{\mu\in\mathfrak{p}}e^{-c\frac{N(\mu+\varrho)}{p}pt},$$

where $\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}$ is the sum over the cosets in $\mathcal{O}/\mathfrak{p}$. Then by Hecke [5, pp. 189–190],

$$\sum_{\mu\in\mathfrak{p}}e^{-c\frac{N(\mu+\varrho)}{p}pt}=\frac{1}{pt\sqrt{N\mathfrak{p}}}\sum_{\lambda\in\mathfrak{c}/\sqrt{d_k}}e^{-c'\frac{N\lambda}{p^2t}+2\pi i\,\mathrm{Tr}(\frac{\lambda\varrho}{p})}.$$

Consequently,

$$\sum_{\alpha\in\mathcal{O}}\left(\frac{\alpha}{\mathfrak{p}}\right)e^{-cN\alpha t}=\frac{1}{pt\sqrt{N\mathfrak{p}}}\sum_{\lambda\in\mathfrak{c}/\sqrt{d_k}}e^{-c'\frac{N\lambda}{p^2t}}\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\left(\frac{\varrho}{\mathfrak{p}}\right)e^{2\pi i\,\mathrm{Tr}(\frac{\lambda\varrho}{p})}.$$

Let now $t\sqrt{N\mathfrak{c}}=1/y$, in which case the last equality becomes

$$\sum_{\alpha\in\mathcal{O}}\left(\frac{\alpha}{\mathfrak{p}}\right)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/y}=\frac{y\sqrt{N\mathfrak{c}}}{p\sqrt{N\mathfrak{p}}}\sum_{\lambda\in\mathfrak{c}/\sqrt{d_k}}e^{-\frac{2\pi}{\sqrt{|d_k|}}N\lambda|d_k|y/p^2}\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\left(\frac{\varrho}{\mathfrak{p}}\right)e^{2\pi i\,\mathrm{Tr}(\frac{\lambda\varrho}{p})}.$$

Now we need to evaluate

$$\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\left(\frac{\varrho}{\mathfrak{p}}\right)e^{2\pi i\,\mathrm{Tr}(\frac{\lambda\varrho}{p})}$$

for $\lambda\in\mathfrak{c}/\sqrt{d_k}$. To this end, let $\lambda=\nu/\sqrt{d_k}$ where $\nu\in\mathfrak{c}$. We consider two cases.

CASE 1: Suppose $p\mathcal{O}=\mathfrak{p}$. Notice that if $\nu\in\mathfrak{p}$, then $\mathrm{Tr}(\lambda\varrho/p)=\mathrm{Tr}(\nu\varrho/(p\sqrt{d_k}))\in\mathbb{Z}$. Thus

$$\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\left(\frac{\varrho}{\mathfrak{p}}\right)e^{2\pi i\,\mathrm{Tr}(\frac{\lambda\varrho}{p})}=0$$

in this case. Next notice that $N\mathfrak{p}=p^2$ and $\mathfrak{c}=\mathcal{O}$. Let $\omega\in k-\{0\}$. Then write $(\omega)=\mathfrak{b}\mathfrak{a}^{-1}(\sqrt{d_k})^{-1}$ with $\mathfrak{a}$ and $\mathfrak{b}$ relatively prime integral ideals of $\mathcal{O}$. Let

$$C(\omega)=\sum_{\mu \bmod \mathfrak{a}}e^{2\pi i\,\mathrm{Tr}(\mu^2\omega)},$$

where the sum is over any system of representatives of $\mathcal{O}/\mathfrak{a}$. By formula (171) of [6], we have

$$\sum_{\varrho\in\mathcal{O}/\mathfrak{p}}\left(\frac{\varrho}{\mathfrak{p}}\right)e^{2\pi i\,\mathrm{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})}=C\left(\frac{\nu}{p\sqrt{d_k}}\right).$$

If $\nu\in\mathcal{O}-\mathfrak{p}$, then Satz 155 in [6] implies $C(\nu/(p\sqrt{d_k}))=(\nu/\mathfrak{p})C(1/(p\sqrt{d_k}))$. But then Satz 163 of [6] yields

$$\frac{C(1/(p\sqrt{d_k}))}{\sqrt{N\mathfrak{p}}}=\frac{C(-p/(4\sqrt{d_k}))}{\sqrt{N(8)}}$$

(by choosing $\gamma=1/\sqrt{d_k}$). Now

$$C\left(-\frac{p}{4\sqrt{d_k}}\right)=\sum_{\mu\in\mathcal{O}/(4)}e^{2\pi i\,\mathrm{Tr}(\frac{-\mu^2p}{4\sqrt{d_k}})}=\sum_{\mu\in\mathcal{O}/(4)}e^{2\pi i\frac{p}{4}\,\mathrm{Tr}(\frac{-\mu^2}{\sqrt{d_k}})}$$

$$= \sum_{\mu\in\mathcal{O}/(4)} e^{(-1)^{(p-1)/2}2\pi i\,\mathrm{Tr}(\frac{-\mu^2}{4\sqrt{d_k}})} = \begin{cases} C\big(\frac{-1}{4\sqrt{d_k}}\big) & \text{if } p\equiv 1 \bmod 4,\\ \overline{C\big(\frac{-1}{4\sqrt{d_k}}\big)} & \text{if } p\equiv 3 \bmod 4.\end{cases}$$

But $C(-p/(4\sqrt{d_k}))$ is real, since $C(1/(p\sqrt{d_k}))$ is real. (Recall that we are assuming that $\chi(W_k)=\{1\}$.) Thus the two cases above coincide. By [6, page 243],

$$C\bigg(\frac{-1}{4\sqrt{d_k}}\bigg) = \sqrt{N(8)},$$

which implies that $C(1/(p\sqrt{d_k})) = \sqrt{N\mathfrak{p}} = p$. Therefore

$$\sum_{\varrho\in\mathcal{O}/\mathfrak{p}} \bigg(\frac{\varrho}{\mathfrak{p}}\bigg) e^{2\pi i\,\mathrm{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} = \bigg(\frac{\nu}{\mathfrak{p}}\bigg)\sqrt{N\mathfrak{p}}.$$

This yields the second part of the lemma.

Case 2: Suppose $(p)=\mathfrak{p}\overline{\mathfrak{p}}$. Then $N\mathfrak{p}=p$, $\mathcal{O}/\mathfrak{p}\simeq\mathbb{Z}/(p)$, and $\mathfrak{c}=\overline{\mathfrak{p}}$. As above, let $\lambda=\nu/\sqrt{d_k}$, but with $\nu\in\overline{\mathfrak{p}}$. Then

$$\begin{aligned}\sum_{\varrho\in\mathcal{O}/\mathfrak{p}} \bigg(\frac{\varrho}{\mathfrak{p}}\bigg) e^{2\pi i\,\mathrm{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} &= \sum_{b\in\mathbb{Z}/(p)} \bigg(\frac{b}{p}\bigg) e^{2\pi i\frac{b}{p}\,\mathrm{Tr}(\frac{\nu}{\sqrt{d_k}})}\\ &= \bigg(\frac{\mathrm{Tr}(\nu/\sqrt{d_k})}{p}\bigg)\sum_{b\in\mathbb{Z}/(p)}\bigg(\frac{b}{p}\bigg)e^{2\pi i b/p} = \bigg(\frac{\mathrm{Tr}(\nu/\sqrt{d_k})}{p}\bigg)\sqrt{p},\end{aligned}$$

by well-known properties of rational Gauss sums (using the assumption $(-1/p)=1$ since $\chi(W_k)=\{1\}$). Now let $\omega=\omega_k=(d_k+\sqrt{d_k})/2$ and thus $\mathcal{O}=\mathbb{Z}[\omega_k]$. Let $p(x)=(x-\omega)(x-\overline{\omega})$; thus $p(x)=x^2-d_kx+(d_k^2-d_k)/4$. Since p splits in $\mathcal{O}$, we know that $p(x)\equiv(x-a)(x-b) \bmod p$ for some $a,b\in\mathbb{Z}$. But then $\mathfrak{p}=(p,\alpha)$ and $\overline{\mathfrak{p}}=(p,\overline{\alpha})$ for some $\alpha=a+\omega_k$ (without loss of generality). Since $\nu\in\overline{\mathfrak{p}}$, there are $x,y\in\mathbb{Z}$ such that

$$\nu = px+\overline{\alpha}y = px+\bigg(a+\frac{d_k-\sqrt{d_k}}{2}\bigg)y.$$

Then $\mathrm{Tr}(\nu/\sqrt{d_k}) = \mathrm{Tr}(-y/2) = -y$. Thus

$$\bigg(\frac{\mathrm{Tr}(\nu/\sqrt{d_k})}{\mathfrak{p}}\bigg) = \bigg(\frac{-y}{p}\bigg) = \bigg(\frac{y}{p}\bigg),$$

again observing that $(-1/p)=1$. Now,

$$\nu = px+\overline{\alpha}y \equiv \overline{\alpha}y \equiv \bigg(a+\frac{d_k+\sqrt{d_k}}{2}\bigg)y - \sqrt{d_k}y \equiv -\sqrt{d_k}y \bmod \mathfrak{p}.$$

Hence

$$\bigg(\frac{\nu}{\mathfrak{p}}\bigg) = \bigg(\frac{-y\sqrt{d_k}}{\mathfrak{p}}\bigg) = \bigg(\frac{y}{p}\bigg)\bigg(\frac{\sqrt{d_k}}{\mathfrak{p}}\bigg).$$

Therefore,

$$\left(\frac{y}{p}\right) = \left(\frac{\nu/\sqrt{d_k}}{\mathfrak{p}}\right),$$

and we see that

$$\sum_{\varrho\in\mathcal{O}/\mathfrak{p}} \left(\frac{\varrho}{\mathfrak{p}}\right) e^{2\pi i\,\mathrm{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} = \left(\frac{\nu/\sqrt{d_k}}{\mathfrak{p}}\right)\sqrt{N\mathfrak{p}}.$$

This establishes the lemma. ■

We continue in this section with some arithmetical results about the 2-power residue symbol. Let $C_{(4)}$ be the ray class group of k modulo (4). Hence, $C_{(4)} = I(4)/P_{(4)}$, where $I(4)$ is the group of fractional ideals prime to 4 and $P_{(4)}$ the group of principal ideals in $I(4)$ with generators $\gamma \equiv 1 \bmod^{\times} 4$. See [11, Chapter VI], for details. Let $H = \{(\gamma) \in P(4) : \gamma \equiv \xi^2 \bmod^{\times} 4\}$. Finally, set $C = I(4)/H$ and $C^{(2)} = \{\mathfrak{a}H : \mathfrak{a}^2 \in H\}$.

LEMMA 7. *Let k be an imaginary quadratic number field. Then the index* $(C : C^{(2)}) = (I(4) : I^2(4)H) = 2^{e+1}$, *where* 2^e *is the index* $(\mathrm{Cl} : \mathrm{Cl}^{(2)})$ (*notice that e is the* 2*-rank of the class group of* k).

Proof. The first equality is obvious. For the second, let $K = k(\sqrt{S})$, where $S = \{\mu \in k^{\times} : (\mu) = \mathfrak{a}^2$ for some fractional ideal $\mathfrak{a}$ of $k^{\times}\}$. Then $[K : k] = 2^{e+1}$ (see [5, paragraph 4 on page 253]). On the other hand, by Satz 171 of [5], the Artin map induces an isomorphism

$$I(4)/I^2(4)H \simeq \mathrm{Gal}(K/k)$$

where $\mathfrak{p} \mapsto (\mathfrak{p}, K/k)$, for any prime ideal $\mathfrak{p} \in I(4)$. Thus the lemma follows. ■

We note for future reference that the prime ideals, $\mathfrak{p}$, in $I^2(4)H$ are characterized by the following reciprocity law (for example see Hecke [5, Satz 171]):

$$\left(\frac{\mu}{\mathfrak{p}}\right) = 1,$$

for all $\mu \in k^{\times}$ for which $(\mu) = \mathfrak{a}^2$ for some fractional ideal $\mathfrak{a}$ prime to $\mathfrak{p}$.

LEMMA 8. *Let k be an imaginary quadratic number field. Suppose that $\mathfrak{p}$ is a prime ideal of absolute degree* 1 *in* $I^2(4)H$, *say* $\mathfrak{p}\mathfrak{b}^2 = (\gamma)$ *where* $\mathfrak{b}$ *is an integral ideal in* $I(4)$ *and* $\gamma \in \mathcal{O}$ *with* $\gamma \equiv \xi^2 \bmod 4$, *and that* $\mathfrak{p}$ *does not divide* $\omega = (d_k + \sqrt{d_k})/2$. *Then*

$$\left(\frac{\overline{\gamma}/\sqrt{d_k}}{\mathfrak{p}}\right) = 1.$$

Proof. It suffices to evaluate $(\overline{\gamma}\sqrt{d_k}/\mathfrak{p})$. To this end, let $\gamma = x + y\omega$. First notice that since $\gamma = x + y\omega$,

$$pb^2 = N\mathfrak{p}N\mathfrak{b}^2 = N\gamma = x^2 + xyd_k + y^2(d_k^2 - d_k)/4,$$

with $b = N\mathfrak{b}$, an odd integer. Then a straightforward computation yields

$$y\overline{\gamma}\sqrt{d_k} = 2(x + yd_k/2)\gamma - 2pb^2 - d_k y^2,$$

and therefore

$$\left(\frac{\overline{\gamma}\sqrt{d_k}}{\mathfrak{p}}\right) = \left(\frac{-d_k y}{p}\right) = \left(\frac{y}{p}\right),$$

since $(d_k/p) = 1$ because $f(\mathfrak{p}) = 1$, and $1 = (-1/\mathfrak{p}) = (-1/p)$ by the reciprocity law for the 2-power residue symbol. Hence we need to evaluate (y/p). Notice that the representation of pb^2 above implies $pb^2 \equiv x^2 \bmod y$. Now we consider the following two cases.

Case 1. Suppose y is odd. Then since $pb^2 \equiv x^2 \bmod y$, $1 = (p/y) = (y/p)$ by an extension of the quadratic law of reciprocity and since $p \equiv 1 \bmod 4$.

Case 2. Suppose y is even, say $y = 2^a y_0$, with y_0 odd. Since $pb^2 \equiv x^2 \bmod y_0$, we have $(y_0/p) = 1$ as above.

If $a \geq 3$, so $8 \mid y$, then $p \equiv 1 \bmod 8$, for $b^2 \equiv 1 \bmod 8$, whence $(2/p) = 1$ and consequently $(y/p) = (2/p)^a (y_0/p) = 1$.

If $a = 2$, then obviously $(y/p) = (4/p)(y_0/p) = 1$.

Now assume $a = 1$, so $y = 2y_0$; then

$$p = x^2 + 2xy_0 d_k + y_0^2(d_k^2 - d_k).$$

If d_k is even, then $d_k \equiv 0 \bmod 4$. If furthermore, $d_k \equiv 0 \bmod 8$, then $p \equiv 1 \bmod 8$, whence $(2/p) = 1$, implying that $(y/p) = 1$. Now, if $d_k \equiv 4 \bmod 8$, then $d_k = 4d$ for some $d \equiv -1 \bmod 4$. Hence, $pb^2 = x^2 + 8xy_0 d + y_0^2(16d^2 - 4d) \equiv 1 - 4d \equiv 5 \bmod 8$. But this cannot happen, since $(2) = \mathfrak{a}^2$ so by the above $(2/\mathfrak{p}) = 1$. But then $1 = (2/\mathfrak{p}) = (2/p)$, implying $p \equiv 1 \bmod 8$.

If d_k is odd, then $d_k \equiv 1 \bmod 4$, in which case $p \equiv x^2 + 2 \bmod 4$. Therefore, $p \equiv pb^2 \equiv 3 \bmod 4$, contrary to our assumptions.

Finally, notice that if $d_k = -4d$ with $d \equiv 1 \bmod 4$, then for $p \equiv 5 \bmod 8$, y must be even. Indeed, since $\gamma \equiv \xi^2 \bmod 4$ with say $\xi = a_1 + b_1\sqrt{-d}$, $a_1, b_1 \in \mathbb{Z}$, we see that $\xi^2 = a_1^2 - db_1^2 + 2a_1 b_1\sqrt{-d}$, implying that y is even. ∎

Lemma 9. *Let χ_1 be a character on $C = C_{\mathfrak{m}} = I(\mathfrak{m})/P_{\mathfrak{m}}$ for some modulus $\mathfrak{m}$ with $\ker \chi_1 = H_1$ and such that the conductor of χ_1 is $\mathfrak{n}$ (notice $\mathfrak{n} \mid \mathfrak{m}$). Suppose H_2 is a subgroup of $C_{\mathfrak{m}}$ such that for $H_0 = H_1 \cap H_2$ we have $H_0 \neq H_i$ for $i = 1, 2$. Then assuming GRH for k, we hve*

$$\sum_{\substack{\mathfrak{p} \in \mathfrak{b}H_2 \\ N\mathfrak{p} \leq x}} \chi_1(\mathfrak{p}) \log N\mathfrak{p} \ll x^{1/2} \log^2 xN\mathfrak{n},$$

as $x \to \infty$, for any $\mathfrak{b} \in I(\mathfrak{m})$.

Proof (sketch). By the usual orthogonality properties of characters and Euler product expansion of L-functions,

$$\sum_{\mathfrak{p}\in\mathfrak{b}H_2}\frac{\chi_1(\mathfrak{p})}{N\mathfrak{p}^s}=\frac{1}{(C:H_0)}\chi_1(\mathfrak{b})\sum_{\mathfrak{c}\in H_2/H_0}\sum_{\chi\in\widehat{C/H_0}}\chi_1\overline{\chi}(\mathfrak{c})\log L(s,\chi)+O(s)$$
$$=\frac{\chi_1(\mathfrak{b})}{(C:H_0)}\sum_{\mathfrak{c}\in H_2/H_0}\sum_{\substack{\chi\in\widehat{C/H_0}\\ \chi\neq 1}}\chi_1\overline{\chi}(\mathfrak{c})\log L(s,\chi)$$
$$+\frac{\chi_1(\mathfrak{b})}{(C:H_0)}\log\zeta_k(s)\sum_{\mathfrak{c}\in H_2/H_0}\chi_1(\mathfrak{c})+O(s)$$

for $\sigma>1/2$. But $\sum_{\mathfrak{c}}\chi_1(\mathfrak{c})=0$ by the assumptions of the lemma, and $\log L(s,\chi)$ is regular for $\sigma>1/2$ since $\chi\neq 1$ and assuming GRH for k. But then the lemma follows from the explicit formula for $\log L(s,\chi)$ following the arguments of Landau [10, Sections 4 and 6], and Davenport [3, Section 20]. ■

5. Main results. Throughout this section assume k is a complex quadratic number field. Denote by d_k its discriminant; by $\mathcal{O}=\mathcal{O}_k$ its ring of integers; by w_k the cardinality of W_k, the group of units in k; by $\mathrm{Cl}=\mathrm{Cl}(k)$ its class group; and by $h=h(k)$ the class number of k.

We now investigate the sum

$$\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D}\sum_{\varrho(\alpha)}K(\varrho)x^{\varrho}$$

as $D\to\infty$, where the outer sum is over the nonzero elements of $\mathcal{O}$ and the inner sum is over the nontrivial zeros of $L(s,\chi_\alpha)$.

We now state and prove our main result.

THEOREM 1. *Assume GRH for k. Then as $D\to\infty$ and either $x\to\infty$ or $x=1$,*

$$\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D}\sum_{\varrho(\alpha)}K(\varrho)x^{\varrho}$$
$$=\begin{cases}-\frac{1}{2}K\big(\frac{1}{2}\big)Dx^{1/2}+\frac{1}{2}K(1)IxD^{1/2}+a\big(\frac{1}{x}\big)D\log D & \\ \qquad +O\big(D\log D\log x+Dx^{1/3}\log x+a\big(\frac{1}{x}\big)D\big) & \text{if } x=o(D),\\ 0+O(x\log^2 x)+O(Dx^{1/3}\log x) & \text{if } D=o(x),\end{cases}$$

where $I=\pi 2^{-1/2}|d_k|^{-1/4}$ and all the implied constants depend only on the base field k and the kernel K.

Proof. We start by using the explicit formula in Proposition 1 to write the above sum as

$$\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^{\varrho} = A + B + C + O,$$

with

$$A = K(1)x \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} \varepsilon_0(\chi_\alpha) e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D},$$

$$B = -\sum_{\mathfrak{p},n} a\left(\frac{N\mathfrak{p}^n}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \chi_\alpha(\mathfrak{p}^n),$$

$$C = a\left(\frac{1}{x}\right) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \log \frac{N\alpha|d_k|}{4\pi^2},$$

$$O = O\Big(\log x \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \log N\alpha\Big).$$

We now break up B further as

$$B = B_1 + B_2 + B_3 + B_4,$$

where

$$B_1 = -\sum_{\mathfrak{p}} a\left(\frac{N\mathfrak{p}}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \chi_\alpha(\mathfrak{p}),$$

$$B_2 = -\sum_{\mathfrak{p}} a\left(\frac{N\mathfrak{p}^2}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D},$$

$$B_3 = \sum_{\mathfrak{p}} a\left(\frac{N\mathfrak{p}^2}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathfrak{p}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D},$$

$$B_4 = -\sum_{\substack{\mathfrak{p},n\\ n\geq 3}} a\left(\frac{N\mathfrak{p}^n}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \chi_\alpha(\mathfrak{p}^n).$$

By Lemma 3,

$$O \ll D\log D \log x.$$

Next, by Lemma 2,

$$A = K(1)x \sum_{\substack{\alpha=\square\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} = \frac{1}{2}K(1)IxD^{1/2} - \frac{1}{2}K(1)x + O(xD^{-M}),$$

with $I = \pi 2^{-1/2}|d_k|^{-1/4}$, and where $\alpha = \square$ means that α ranges over all squares of elements in $\mathcal{O}$, since χ_α is principal if and only if α is a square.

By Lemma 3 again, we see that

$$C = a\left(\frac{1}{x}\right) D \log D + O(D).$$

Now notice that

$$\begin{aligned} B_3 &= \sum_{\mathfrak{p}} a\left(\frac{N\mathfrak{p}^2}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathfrak{p}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \\ &= \sum_{\mathfrak{p}} a\left(\frac{N\mathfrak{p}^2}{x}\right)(\log N\mathfrak{p}) \left(\frac{D}{N\mathfrak{p}} + O(1)\right), \end{aligned}$$

by Lemma 2. By Lemma 5 we have

$$B_3 \ll D + x^{1/2}.$$

Since $a(N\mathfrak{p}^n/x) = 0$ unless $A \leq N\mathfrak{p}^n/x \leq B$ (see the Preliminaries), we see that

$$\begin{aligned} B_4 &\ll \sum_{\substack{\mathfrak{p},n\\ n\geq 3}} a\left(\frac{N\mathfrak{p}^n}{x}\right)(\log N\mathfrak{p}) \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \\ &\ll D \sum_{\substack{\mathfrak{p},n\\ n\geq 3}} a\left(\frac{N\mathfrak{p}^n}{x}\right) \log N\mathfrak{p} \ll D x^{1/3} \log x, \end{aligned}$$

by Lemmas 2 and 5.

Also by Lemmas 2 and 5, we have

$$B_2 = -\frac{1}{2} K\left(\frac{1}{2}\right) D x^{1/2} + O(x^{1/2}) + O(D x^{1/4} \log^2 x).$$

Now we consider B_1. We decompose B_1 as $B_1 = B_1^{\mathrm{i}} + B_1^{\mathrm{s}}$, where

$$\begin{aligned} B_1^{\mathrm{i}} &= -\sum_{\substack{\mathfrak{p}\\ (d_k/p)=-1}} a\left(\frac{N\mathfrak{p}}{x}\right)(\log N\mathfrak{p}) \sum_{\alpha\in\mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \chi_\alpha(\mathfrak{p}), \\ B_1^{\mathrm{s}} &= -\sum_{\substack{\mathfrak{p}\\ (d_k/p)=1}} a\left(\frac{N\mathfrak{p}}{x}\right)(\log N\mathfrak{p}) \sum_{\alpha\in\mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \chi_\alpha(\mathfrak{p}). \end{aligned}$$

By the transformation formula (Lemma 6),

$$B_1^t = -\sum_{\substack{\mathfrak{p}\\ (d_k/p)=(-1)^\tau}} a\left(\frac{N\mathfrak{p}}{x}\right)(\log N\mathfrak{p}) \frac{D}{\sqrt{N\mathfrak{p}}} \sum_{\nu\in\mathfrak{a}} \left(\frac{\nu}{\mathfrak{p}}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu |d_k|^\varepsilon D/p^2},$$

where $\tau = 0$ or 1, $\mathfrak{a}_\mathfrak{p} = \mathfrak{a} = \overline{\mathfrak{p}}/\sqrt{d_k}$ or $\mathcal{O}$, and $\varepsilon = 1$ or 0, if $t = \mathrm{s}, \mathrm{i}$, respectively.

We now consider two cases.

CASE 1. Assume $x = o(D)$. We first show that $B_1^{\mathrm{i}} \ll x^{1/2}$. To this end notice that

$$\begin{aligned}\frac{D}{\sqrt{N\mathfrak{p}}}\sum_{\nu\in\mathfrak{a}}\left(\frac{\nu}{\mathfrak{p}}\right)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu\,|d_k|^{\varepsilon}D/p^2} &= \frac{D}{p}\sum_{\nu\in\mathcal{O}}\left(\frac{\nu}{\mathfrak{p}}\right)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu\,D/p^2}\\ &\ll \frac{D}{p}\sum_{m=1}^{\infty}e^{-\frac{2\pi}{\sqrt{|d_k|}}mD/p^2}\kappa(m),\end{aligned}$$

where

$$\kappa(m) = \#\{\nu\in\mathfrak{a} : m \le N\nu < m+1\} \ll m;$$

see, for example, Lang [11, Theorem 2, page 128] (and also notice that $\kappa(0) = 0$). Thus

$$\sum_{m=1}^{\infty}e^{-\frac{2\pi}{\sqrt{|d_k|}}mD/p^2}\kappa(m) \ll \sum_{m=1}^{\infty}e^{-\frac{2\pi}{\sqrt{|d_k|}}mD/p^2}m = o(1).$$

From this we see by Lemma 5 that $B_1^{\mathrm{i}} \ll \sum_p a(p^2/x)\log p \ll x^{1/2}$.

Next we show that $B_1^{\mathrm{s}} \ll x$. To this end, let $\pi_1 \in \overline{\mathfrak{p}} - \overline{\mathfrak{p}}^2$; then $N(\pi_1) = bp$ for some positive integer b. Also set $\nu = \mu\pi_1/\sqrt{d_k}$. We then see that

$$\frac{D}{\sqrt{N\mathfrak{p}}}\sum_{\nu\in\mathfrak{a}}\left(\frac{\nu}{\mathfrak{p}}\right)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu|d_k|^{\varepsilon}D/p^2} \ll \frac{D}{\sqrt{p}}\sum_{\mu\in\overline{\mathfrak{p}}/(\pi_1)}e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu bD/p} = o(1).$$

Thus $B_1^{\mathrm{s}} \ll \sum_p a(p/x)\log p \ll x$.

CASE 2. Assume $D = o(x)$. First we consider B_1^{i} and claim that $B_1^{\mathrm{i}} \ll x$. Since $D = o(x)$,

$$B_1^{\mathrm{i}} \ll \sum_p a\left(\frac{p^2}{x}\right)(\log p)\frac{D}{p}\sum_{\nu\in\mathcal{O}}e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu D/p^2} \ll \sum_p a\left(\frac{p^2}{x}\right)p\log p \ll x,$$

by Lemma 5.

Now consider B_1^{s}. From the above we know that

$$B_1^{\mathrm{s}} = -\sum_{\substack{\mathfrak{p}\\(d_k/p)=1}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}}\sum_{\nu\in\overline{\mathfrak{p}}/\sqrt{d_k}}\left(\frac{\nu}{\mathfrak{p}}\right)e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu|d_k|D/p^2}.$$

We decompose B_1^{s} in the following way. By Lemma 7, $C/C^2 \simeq I(4)/I^2(4)H$ has order 2^{e+1}, where e is the 2-rank of Cl. Hence

$$I(4) = \bigcup_{i=1}^{2^{e+1}}\mathfrak{a}_i I^2(4)H$$

for some ideals $\mathfrak{a}_i$ and where $\mathfrak{a}_1 = (1)$. Now we write

$$B_1^{\mathrm{s}} = \sum_{i=1}^{2^{e+1}} B(i),$$

where

$$B(i) = -\sum_{\substack{\mathfrak{p}\in\mathfrak{a}_i I^2(4)H \\ (d_k/p)=1}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \sum_{\nu\in\overline{\mathfrak{p}}/\sqrt{d_k}} \left(\frac{\nu}{\mathfrak{p}}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}}N\nu|d_k|D/p^2}.$$

Now, consider $B(1)$. To simplify the notation slightly, we denote by $\mathcal{P}$ the set of prime ideals of absolute degree 1 contained in $I^2(4)H$. Let $\mathfrak{p}\in\mathcal{P}$, say $\mathfrak{p}\overline{\mathfrak{b}}_{\mathfrak{p}}^2 = (\gamma_{\mathfrak{p}})$ for some $(\gamma_{\mathfrak{p}}) \in H$. Hence, $\overline{\mathfrak{p}}\mathfrak{b}_{\mathfrak{p}}^2 = (\overline{\gamma}_{\mathfrak{p}})$. Notice that then $pN\mathfrak{b}_{\mathfrak{p}}^2 = N\gamma_{\mathfrak{p}}$. We change the summation variable by letting $\nu = \mu\overline{\gamma}_{\mathfrak{p}}/\sqrt{d_k}$ and thus obtain

$$B(1) = -\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}}\left(\frac{\overline{\gamma}_{\mathfrak{p}}/\sqrt{d_k}}{\mathfrak{p}}\right) \sum_{\mu\in\mathfrak{b}_{\mathfrak{p}}^{-2}} \left(\frac{\mu}{\mathfrak{p}}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}}(N\mu)N\mathfrak{b}_{\mathfrak{p}}^2 D/p}.$$

Now we break up $B(1)$ further. To this end, notice that $\mathrm{Cl}/\mathrm{Cl}^2 \simeq \mathrm{Cl}^{(2)}$ where $\mathrm{Cl}^{(2)} = \{\mathfrak{c}\in\mathrm{Cl} : \mathfrak{c}^2 = 1\}$. Then we write

$$B(1) = B(\square) + B(\not\square)$$

where

$$B(\square) = -\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}}\left(\frac{\overline{\gamma}_{\mathfrak{p}}/\sqrt{d_k}}{\mathfrak{p}}\right) \sum_{\mathfrak{c}\in\mathrm{Cl}^{(2)}} \sum_{\substack{\mu\in\mathfrak{b}_{\mathfrak{p}}^{-2} \\ (\mu)=\mathfrak{a}^2 \\ \mathfrak{a}\in\mathfrak{c}}} \left(\frac{\mu}{\mathfrak{p}}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}}(N\mu)N\mathfrak{b}_{\mathfrak{p}}^2 D/p}.$$

Finally, we split $B(\square)$ as

$$B(\square) = B'(\square) + B''(\square),$$

with

$$B'(\square) = -w_k\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \sum_{\mathfrak{c}\in\mathrm{Cl}^{(2)}} \sum_{\substack{\mathfrak{a}\in\mathfrak{c} \\ \mathfrak{b}_{\mathfrak{p}}^{-1}|\mathfrak{a}}} e^{-\frac{2\pi}{\sqrt{|d_k|}}(N\mathfrak{a}^2)N\mathfrak{b}_{\mathfrak{p}}^2 D/p}.$$

Now, we evaluate $B'(\square)$ asymptotically. For each $\mathfrak{c}\in\mathrm{Cl}^{(2)}$ let $\mathfrak{b}_{\mathfrak{c}}\in\mathfrak{c}$ be such that if $\mathfrak{a}\in\mathfrak{c}$, then $\mathfrak{a}\mathfrak{b}_{\mathfrak{c}} = (\gamma)$ for some $\gamma\in\mathfrak{b}_{\mathfrak{c}}$. Notice that then there is a bijection between the set $\{\mathfrak{a}\in\mathfrak{c} : \mathfrak{b}_{\mathfrak{p}}^{-1} \mid \mathfrak{a}\}$ and the set (of equivalence classes) $(\mathfrak{b}_{\mathfrak{c}}\mathfrak{b}_{\mathfrak{p}}^{-1} - \{0\})/W_k$ given by

$$\mathfrak{a} \mapsto (\gamma) = \mathfrak{a}\mathfrak{b}_{\mathfrak{c}}.$$

In addition, notice that $N\gamma = N\mathfrak{a}N\mathfrak{b}_{\mathfrak{c}}$. From this and Lemma 2, it follows that

$$\begin{aligned}
B'(\square) &= -\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \sum_{\mathfrak{c}\in \mathrm{Cl}^{(2)}} \sum_{\substack{\gamma\in\mathfrak{b}_{\mathfrak{c}}\mathfrak{b}_{\mathfrak{p}}^{-1}\\ \gamma\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}(D/p)N\gamma^2 N\mathfrak{b}_{\mathfrak{p}}^2/N\mathfrak{b}_{\mathfrak{c}}^2} \\
&= -\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \sum_{\mathfrak{c}\in \mathrm{Cl}^{(2)}} \left(\frac{\sqrt{p}}{\sqrt{D}}\, I - 1 + O((p/D)^{-M})\right) \\
&= -|\mathrm{Cl}^{(2)}| \sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \left(\frac{\sqrt{p}}{\sqrt{D}}\, I - 1 + O((p/D)^{-M})\right) \\
&\quad - 2^e I\sqrt{D}\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)\log p + 2^e D\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}} \\
&\quad + O\left(\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)(p/D)^{-M'}\right).
\end{aligned}$$

Now, by Lemmas 5 and 7,

$$\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)\log p = \frac{1}{2^{e+1}}\, K(1)x + O(x^{1/2}\log^2 x),$$
$$\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}} = \frac{1}{2^{e+1}}\, K\left(\frac{1}{2}\right)x^{1/2} + O(\log^2 x),$$

and moreover,

$$\sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)(p/D)^{M'} \ll 1.$$

Therefore,

$$B'(\square) = -\frac{1}{2}\, IK(1)xD^{1/2} + \frac{1}{2}\, K\left(\frac{1}{2}\right)x^{1/2}D + O(D^{1/2}x^{1/2}\log^2 x).$$

On the other hand, notice that

$$\begin{aligned}
B''(\square) &\ll \sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)(\log p)\frac{D}{\sqrt{p}} \sum_{\mu\in\mathfrak{b}_{\mathfrak{p}}^{-2}\mathfrak{p}} e^{-\frac{2\pi}{\sqrt{|d_k|}}(N\mu N\mathfrak{b}^2)D/p} \\
&\ll \sum_{\mathfrak{p}\in\mathcal{P}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}} \ll x^{1/2},
\end{aligned}$$

by Lemmas 2 and 5.

Now consider $B(\not\square)$. For this expression we partition the sums differently. Let $\mathfrak{c}\in I^2(4)H/P_{(4)}$; pick an integral ideal $\mathfrak{a}_{\mathfrak{c}}\in\mathfrak{c}$. Then if $\mathfrak{p}\in\mathfrak{c}$ is a prime ideal of absolute degree 1, we have $\bar{\mathfrak{p}}\in\mathfrak{c}^{-1}$, so $\bar{\mathfrak{p}}\mathfrak{a}_{\mathfrak{c}} = (\alpha_{\mathfrak{p}})$ for $(\alpha_{\mathfrak{p}})\in P_{(4)}$.

Therefore by changing summation variable $\nu = \mu\alpha_{\mathfrak{p}}/\sqrt{d_k}$, we get

$$B(\not\square) = -D \sum_{\mathfrak{c}\in I^2(4)H/P_{(4)}} \sum_{\mathfrak{p}\in\mathfrak{c}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{\alpha_{\mathfrak{p}}/\sqrt{d_k}}{\mathfrak{p}}\right) \times \sum_{\substack{\mu\in\mathfrak{a}_{\mathfrak{c}}^{-1}\\ (\mu)\neq\mathfrak{a}^2}} \left(\frac{\mu}{\mathfrak{p}}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mathfrak{a}_{\mathfrak{c}} \frac{D}{p} N\mu}.$$

But since, by Lemma 8, $\left(\frac{\alpha_{\mathfrak{c}}/\sqrt{d_k}}{\mathfrak{p}}\right) = 1$, and since $a(p/x) \neq 0$ implies $p/x < B$ (see the Preliminaries), we obtain

$$B(\not\square) \ll D \sum_{(\mu)\neq\mathfrak{a}^2} \sum_{\substack{\mathfrak{c}\in I^2(4)H/P_{(4)}\\ \mu\in\mathfrak{a}_{\mathfrak{c}}^{-1}}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mathfrak{a}_{\mathfrak{c}} \frac{D}{Bx} N\mu} \left|\sum_{\mathfrak{p}\in\mathfrak{c}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{\mu}{\mathfrak{p}}\right)\right|.$$

Now, Lemma 9 implies that

$$\sum_{\mathfrak{p}\in\mathfrak{c}} \left(\frac{\mu}{\mathfrak{p}}\right) \log N\mathfrak{p} \ll x^{1/2} \log^2(xN\mu).$$

Hence the argument (using Riemann–Stieltjes integration) applied to B_{13} in [14] implies

$$\sum_{\mathfrak{p}\in\mathfrak{c}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{\mu}{\mathfrak{p}}\right) \ll \log^2(xN\mu).$$

Next notice that $\mu \in \bigcup_{\mathfrak{c}} \mathfrak{a}_{\mathfrak{c}}^{-1}$ so $\mu \in \mathfrak{b}^{-1}$ for some integral ideal $\mathfrak{b}$, e.g. $\mathfrak{b} = \prod_{\mathfrak{c}} \mathfrak{a}_{\mathfrak{c}}$. Hence

$$B(\not\square) \ll D \sum_{\substack{\mu\in\mathfrak{b}^{-1}\\ \mu\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{D}{Bx} N\mu} \log^2(xN\mu) \ll x \log^2\left(\frac{x}{D}\right).$$

Next, consider $\sum_{i=2}^{2^{e+1}} B(i)$. We partition the sum over $\mathfrak{p}$ analogously to the previous case. If $\mathfrak{c} \in C_{(4)} = I(4)/P_{(4)}$, let $\mathfrak{a}_{\mathfrak{c}} \in \mathfrak{c}$, so if $\mathfrak{p} \in \mathfrak{c}$, then $\overline{\mathfrak{p}}\mathfrak{a}_{\mathfrak{c}} = (\alpha_{\mathfrak{p}})$ for some $(\alpha_{\mathfrak{p}}) \in P_{(4)}$. Then as above

$$\begin{aligned}\sum_{i=1}^{2^{e+1}} B(i) &\ll D \sum_{\substack{\mu\in\mathfrak{b}^{-1}\\ \mu\neq 0}} \sum_{\substack{\mathfrak{c}\in C_{(4)}\\ \mathfrak{c}\notin I^2(4)H/P_{(4)}}} e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{D}{Bx} N\mu} \left|\sum_{\mathfrak{p}\in\mathfrak{c}} a\left(\frac{p}{x}\right)\frac{\log p}{\sqrt{p}}\left(\frac{\mu}{\mathfrak{p}}\right)\right| \\ &\ll x \log^2\left(\frac{x}{D}\right),\end{aligned}$$

arguing as above using Lemma 9 when μ is not a square in k, since then $\{(\mu/\mathfrak{p}) : \mathfrak{p} \in \mathfrak{c}\} = \{\pm 1\}$. On the other hand, the sum over square μ contributes $\ll D^{1/2}x^{1/2}\log^2(x/D) \ll x\log^2(x/D)$, again.

Now suppose $x = 1$. Then by the explicit formula for primitive characters (just above Proposition 1) and Lemma 4 we obtain

$$\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)$$
$$= \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D}(a(1)\log N\mathfrak{f}_\alpha + O(1)) = a(1)D\log D + O(D).$$

This completes the proof of the main theorem. ∎

Now we come to the first main corollary, which is a special case of the theorem, but will be useful in studying the distribution of the nontrivial zeros of the quadratic L-series. Define, for $y \in \mathbb{R}$,

$$F_K(y,D) = \left(\frac{1}{2}K\left(\frac{1}{2}\right)D\right)^{-1} \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)D^{iy\gamma},$$

where $\varrho = 1/2 + i\gamma$. Then we have

Corollary 1. *Assuming GRH for all abelian L-functions over k, as $D \to \infty$,*

$$F_K(y,D) = \begin{cases} -1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2}a(D^{-y})\log D + o(1) & \text{if } |y| < 1,\\ 0 + o(1) & \text{if } 1 < |y| < 2,\end{cases}$$

uniformly on compact subsets of $(-2,-1)\cup(-1,1)\cup(1,2)$.

Proof. First in the main theorem dividing both sides by $\frac{1}{2}K\left(\frac{1}{2}\right)Dx^{1/2}$, absorbing one term into the error, and combining two error terms we get

$$\left(\frac{1}{2}K\left(\frac{1}{2}\right)D\right)^{-1} \sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^{\varrho-1/2}$$
$$= \begin{cases} -1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} x^{-1/2}a\left(\frac{1}{x}\right)(\log D)\left(1 + O\left(\frac{1}{\log D}\right)\right) & \\ \quad + O\left(\frac{x^{1/2}}{D^{1/2}} + x^{-1/6}\log D\log x\right) & \text{if } x = o(D),\\ 0 + O(x^{1/2}D^{-1}\log^2 x + x^{-1/6}\log x) & \text{if } D = o(x).\end{cases}$$

Now plugging in $x = D^y$ yields, for $y \geq 0$,

$$F_K(y,D) = \begin{cases} -1 + \left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2}a(D^{-y})(\log D)\left(1 + O\left(\frac{1}{\log D}\right)\right) & \\ \quad + O(D^{(y-1)/2} + D^{-y/6}\log^2 D) & \text{if } y < 1,\\ 0 + O(D^{y/2-1}\log^2 D + D^{-y/6}\log D) & \text{if } y > 1.\end{cases}$$

From this the corollary follows for $y \geq 0$. (Take the compact set to be $[0, 1-\varepsilon] \cup [1+\varepsilon, 2-\varepsilon]$.) Finally, notice that $F_K(-y, D) = F_K(y, D)$ since the nontrivial zeros of our L-functions are symmetric about the real axis. ∎

Finally, we come to the second main corollary, which concerns the distribution of the nontrivial zeros close to the real axis.

Corollary 2. *Suppose $r(y)$ is an even continuous function with $r(y)$ and $yr(y)$ in $L^1(\mathbb{R})$, such that its Fourier transform,*

$$\widehat{r}(y) = \int_{-\infty}^{\infty} r(u) e^{-2\pi i y u}\, du,$$

is also continuous and in $L^1(\mathbb{R})$, and has compact support in $(-2,2) \setminus \{\pm 1\}$. Then under GRH for all abelian L-functions over k, an imaginary quadratic number field, as $D \to \infty$,

$$D^{-1} \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha / D} \left(\frac{1}{2} K\left(\frac{1}{2}\right)\right)^{-1} \sum_{\varrho(\alpha)} K(\varrho) r\left(\frac{\gamma \log D}{2\pi}\right)$$
$$= 2 \int_{-\infty}^{\infty} \left(1 - \frac{\sin 2\pi y}{2\pi y}\right) r(y)\, dy + o(1),$$

where the implied constant depends only on the field k and the kernel K.

Proof. By Corollary 1 and since $\widehat{r}(y)$ has compact support in $(-2,2) \setminus \{\pm 1\}$,

$$\int_{-\infty}^{\infty} F_K(y, D) \widehat{r}(y)\, dy$$
$$= \int_{-\infty}^{\infty} \left(-\xi_{[-1,1]}(y) + \left(\frac{1}{2} K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2} a(D^{-y}) \log D\right) \widehat{r}(y)\, dy + o(1),$$

where $\xi_{[-1,1]}$ is the characteristic function of $[-1,1]$. But

$$\int_{-\infty}^{\infty} \xi_{[-1,1]}(y) \widehat{r}(y)\, dy = \int_{-\infty}^{\infty} \widehat{\xi}_{[-1,1]}(y) r(y)\, dy$$
$$= 2 \int_{-\infty}^{\infty} \frac{\sin 2\pi y}{2\pi y} r(y)\, dy.$$

On the other hand,

$$\int_{-\infty}^{\infty} D^{-y/2} a(D^{-y}) \widehat{r}(y)\, dy = \int_{-\infty}^{\infty} (D^{-y/2} a(D^{-y}))^{\wedge} r(y)\, dy.$$

But

$$(D^{-y/2}a(D^{-y}))^\wedge = \int_{-\infty}^{\infty} D^{-\beta/2}a(D^{-\beta})e^{-2\pi i y\beta}\,d\beta.$$

By the change of variable $t = D^{-\beta}$, this last integral equals

$$\begin{aligned}\frac{1}{\log D}&\int_0^\infty a(t)t^{1/2+2\pi i\frac{y}{\log D}}\,\frac{dt}{t}\\ &= \frac{1}{\log D}\int_0^\infty a(t)t^{1/2}\,\frac{dt}{t} + \frac{1}{\log D}\int_0^\infty a(t)t^{1/2}(t^{2\pi i\frac{y}{\log D}}-1)\,\frac{dt}{t}.\end{aligned}$$

Notice that

$$t^{2\pi i\frac{y}{\log D}} - 1 = \exp\biggl(2\pi i\,\frac{\log t}{\log D}\,y\biggr) - 1 = 2\pi i y\,\frac{\log t}{\log D}\exp\biggl(2\pi i\,\frac{\log t}{\log D}\,\theta_y\biggr)$$

for some θ_y between 0 and y. Therefore

$$t^{2\pi i\frac{y}{\log D}} - 1 \ll \frac{\log t}{\log D}\,y,$$

so that

$$\frac{1}{\log D}\int_0^\infty a(t)t^{1/2}(t^{2\pi i\frac{y}{\log D}}-1)\,\frac{dt}{t} \ll \frac{y}{\log^2 D}$$

where the implied constant is independent of y and D. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty}&(D^{-y/2}a(D^{-y}))^\wedge r(y)\,dy\\ &= \frac{1}{\log D}\,K\biggl(\frac{1}{2}\biggr)\int_{-\infty}^{\infty} r(y)\,dy + O\biggl(\frac{1}{\log^2 D}\int_{-\infty}^{\infty} yr(y)\,dy\biggr)\\ &= \frac{1}{\log D}\,K\biggl(\frac{1}{2}\biggr)\int_{-\infty}^{\infty} r(y)\,dy + O\biggl(\frac{1}{\log^2 D}\biggr).\end{aligned}$$

Thus,

$$\begin{aligned}\int_{-\infty}^{\infty}&\biggl(-\xi_{[-1,1]}(y) + \biggl(\frac{1}{2}\,K\biggl(\frac{1}{2}\biggr)\biggr)^{-1} D^{-y/2}a(D^{-y})\log D\biggr)\widehat{r}(y)\,dy\\ &= 2\int_{-\infty}^{\infty}\biggl(1-\frac{\sin 2\pi y}{2\pi y}\biggr)r(y)\,dy + O\biggl(\frac{1}{\log D}\biggr).\end{aligned}$$

Consequently,

$$\int_{-\infty}^{\infty} F_K(y,D)\widehat{r}(y)\,dy = 2\int_{-\infty}^{\infty}\biggl(1-\frac{\sin 2\pi y}{2\pi y}\biggr)r(y)\,dy + o(1).$$

On the other hand,

$$\int_{-\infty}^{\infty} F_K(y,D)\widehat{r}(y)\,dy$$

$$= D^{-1}\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D}\left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1}\sum_{\varrho(\alpha)} K(\varrho)\int_{-\infty}^{\infty} e^{i\alpha\gamma\log D}\widehat{r}(y)\,dy$$

$$= D^{-1}\sum_{\substack{\alpha\in\mathcal{O}\\ \alpha\neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\alpha/D}\left(\frac{1}{2}K\left(\frac{1}{2}\right)\right)^{-1}\sum_{\varrho(\alpha)} K(\varrho)r\left(\frac{\gamma\log D}{2\pi}\right),$$

since by Fourier duality

$$\widehat{\widehat{r}}(y) = r(-y) = r(y).$$

This establishes the corollary. ∎

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