On the distribution of the nontrivial zeros of quadratic $L$-functions of imaginary quadratic number fields close to the real axis

by

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1. Introduction. The distribution of zeros near $s = 1/2$ for various families of $L$-functions has received widespread attention since the appearance of the seminal joint work of Katz and Sarnak, [8] and [9]. One aspect of the basic conjectures which came to light through their research relates the one-level normalized spacings of “low-lying” zeros of certain families of $L$-functions, when ordered by their conductors, to classical symmetry groups associated with each family.

In the case of the family of quadratic Dirichlet $L$-functions (over $\mathbb{Q}$), partial results (cf. [8], [13], [14]) suggest that the symmetry group associated with this family should be symplectic; see [8] for more details. In this case the functional equations for the completed $L$-functions are “self-dual”, i.e. remain invariant under the substitution $s \mapsto 1 - s$. In the function field analogue, where the distribution conjectures become theorems, Katz and Sarnak have shown that for certain families of zeta functions whose functional equations are self-dual, the distribution of the zeros is governed by a symplectic symmetry group (cf. [8], [9]).

In the present note, we consider the family of quadratic $L$-functions over an arbitrary imaginary quadratic base field. We study the distribution of their nontrivial zeros close to the real axis. Here, as above, we find that the completed $L$-functions have self-dual functional equations; and once again we see, as in the case over $\mathbb{Q}$, that the distribution of the zeros indicates a symplectic symmetry group, which is as expected. It should be pointed out, however, that self-duality of the functional equations associated with a family of $L$-functions need not indicate a symplectic distribution of low-lying zeros. Peter Sarnak has pointed out to us that there are families of $L$-functions with associated self-dual functional equations, but for which the

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distribution of the low-lying zeros seems to suggest an orthogonal symmetry group.

The method of proof is similar to that in our previous article, [14], but there were obstacles to overcome. The main problem was figuring out the “main term” in the “form factor” in the case where \( D = o(x) \). This impasse was overcome by using quadratic reciprocity for the 2-power residue symbol (generalized Legendre symbol) over an arbitrary number field as described in Hecke’s text [6].

2. Statements of main results. At this point we state our main theorem and its two corollaries. Before doing so we need to introduce some notation and concepts. Let \( K(s) \) be a function analytic in the strip \(-1 < \Re(s) < 2\) such that \( |K(\sigma + it)| \ll t^{-2} \) as \( t \to \infty \) and such that the function

\[
a(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K(s)x^{-s} \, ds
\]

is absolutely convergent for \(-1 < c < 2\) and all \( x > 0 \), continuous, differentiable for all but finitely many points, of bounded variation, real-valued, non-negative, of compact support on the interval \((0, \infty)\), and such that \( a(1) \neq 0 \). Furthermore, assume \( K(1/2 + it) = K(1/2 - it) \) for all real \( t \). A particular choice of \( K(s) \) is given by

\[
K(s) = \left( \frac{e^{s-1/2} - e^{-s+1/2}}{2s - 1} \right)^2,
\]

in which case

\[
a(t) = \begin{cases} 
\frac{1}{2} t^{-1/2} \left( 1 - \frac{1}{2} \log t \right) & \text{if } e^{-2} < t < e^2, \\
0 & \text{otherwise}
\end{cases}
\]

(see [12]).

Next, let \( k \) be a complex quadratic number field with discriminant \( d_k \) and ring of integers \( \mathcal{O} = \mathcal{O}_k \). If \( \alpha \) is any nonzero integer in \( k \), then denote by \( \chi_\alpha \) the 2-power residue symbol with respect to \( \alpha \). Let \( L(s, \chi_\alpha) \) be the (quadratic) \( L \)-series attached to \( \chi_\alpha \). Also let \( N\alpha \) be the norm of \( \alpha \) from \( k \) to \( \mathbb{Q} \).

The point of our main theorem is to investigate the behavior of the sum

\[
\sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^{\varrho}
\]

as \( D \to \infty \), where the outer sum ranges over the nonzero elements of \( \mathcal{O} \) and the inner sum is over the nontrivial zeros of \( L(s, \chi_\alpha) \).

Main Theorem. Assume all the notation above and suppose that the Generalized Riemann Hypothesis (GRH) holds for all abelian \( L \)-functions.
over $k$. Then as $D \to \infty$ and either $x \to \infty$ or $x = 1$,

$$
\sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{|d_k|} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^\varrho =
$$

$$
\begin{cases}
-\frac{1}{2} K\left(\frac{1}{2}\right) Dx^{1/2} + \frac{1}{2} K(1)\pi(4|d_k|)^{-1/4} xD^{1/2} + a\left(\frac{1}{x}\right) D \log D \\
+ O(D \log D \log x + D x^{1/3} \log x + a\left(\frac{1}{x}\right) D) & \text{if } x = o(D), \\
0 + O(x \log^2 x + Dx^{1/3} \log x) & \text{if } D = o(x).
\end{cases}
$$

All the implied constants depend only on the base field $k$ and the kernel $K$.

Now define, for $y \in \mathbb{R}$,

$$
F_K(y, D) = \left(\frac{1}{2} K\left(\frac{1}{2}\right) D\right)^{-1} \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{|d_k|} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)D^{iy\gamma},
$$

where $\varrho = 1/2 + i\gamma$. Then we have the following corollary, which is just a special case of the Main Theorem.

**First Corollary.** Assuming GRH for all abelian $L$-functions over $k$, as $D \to \infty$,

$$
F_K(y, D) = \begin{cases}
-1 + \left(\frac{1}{2} K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2} a(D^{-y}) \log D + o(1) & \text{if } |y| < 1, \\
0 + o(1) & \text{if } 1 < |y| < 2,
\end{cases}
$$

uniformly on compact subsets of $(-2, -1) \cup (-1, 1) \cup (1, 2)$.

The second corollary concerns the distribution of the nontrivial zeros of our family of $L$-functions near the real axis.

**Second Corollary.** Suppose $r(y)$ is an even continuous function with $r(y)$ and $yr(y)$ in $L^1(\mathbb{R})$, such that its Fourier transform,

$$
\hat{r}(y) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i y u} du,
$$

is also continuous and in $L^1(\mathbb{R})$, and has compact support in $(-2, 2) \setminus \{\pm 1\}$. Then under GRH for all abelian $L$-functions over $k$, an imaginary quadratic number field, as $D \to \infty$,

$$
D^{-1} \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{|d_k|} N\alpha/D} \left(\frac{1}{2} K\left(\frac{1}{2}\right)\right)^{-1} \sum_{\varrho(\alpha)} K(\varrho)r\left(\frac{\gamma \log D}{2\pi}\right)
$$

$$
= 2 \int_{-\infty}^{\infty} \left(1 - \frac{\sin 2\pi y}{2\pi y}\right)r(y) dy + o(1),
$$

where the implied constant depends only on the field $k$ and the kernel $K$. 


3. Preliminaries. Let $k$ be an imaginary quadratic number field. Denote by $\mathcal{O} = \mathcal{O}_k$ the ring of integers of $k$ and by $d_k$ its discriminant. If $\alpha$ is a nonzero algebraic integer of $k$, then let $\chi_\alpha$ be the 2-power residue symbol; i.e. if $p$ is any (nonzero) prime ideal of $\mathcal{O}$ not dividing $(2\alpha)$, then let
\[ \chi_\alpha(p) = \left( \frac{\alpha}{p} \right) = \begin{cases} 1 & \text{if } x^2 \equiv \alpha \pmod{p} \text{ has a solution}, \\ -1 & \text{if } x^2 \equiv \alpha \pmod{p} \text{ has no solutions}. \end{cases} \]
Define $\chi_\alpha(p) = 0$ if $p$ divides $(2\alpha)$, and extend this definition to all nonzero ideals $a$ of $\mathcal{O}$ by multiplicativity. As is well known, $\chi_\alpha$ is induced by a primitive character $\chi_\alpha^*$ which may be identified with the Artin symbol $(\cdot, K/k)$ where $K = k(\sqrt{\alpha})$. Indeed, $(p, K/k)(\sqrt{\alpha}) = (\alpha/p)\sqrt{\alpha}$ for all $p$ relatively prime to $2\alpha$.

Let
\[ L(s, \chi_\alpha) = \sum_{a} \frac{\chi_\alpha(a)}{N_a^s} \quad (\text{Re}(s) > 1), \]
where the sum is over all nonzero integral ideals $a$ of $\mathcal{O}_k$ and $N_a = \#(\mathcal{O}/a)$. The $L$-function associated with $\chi_\alpha^*$ is defined analogously. As is well known, these $L$-series have Euler product expansions by the unique factorization property of ideals in $\mathcal{O}$. Moreover, the Dedekind zeta function of $K$ satisfies
\[ \zeta_K(s) = \zeta_k(s)L(s, \chi_\alpha^*) \quad (\text{Re}(s) > 1). \]
All of the functions above can be analytically continued to the whole complex plane (with simple pole at $s = 1$ for zeta functions), thanks to Hecke. Thus, in particular, $L(s, \chi_\alpha^*)$ and $L(s, \chi_\alpha)$ differ by a finite Euler factor and so have the same set of nontrivial zeros.

Next, we apply Weil’s explicit formula to $L(s, \chi)$ where $\chi = \chi_\alpha^*$. To this end, we refer to the article [1] of Barner. Suppose $F : \mathbb{R} \to \mathbb{C}$ is a function of bounded variation; let
\[ \Phi(s) = \int_{-\infty}^{\infty} F(z)e^{(s-1/2)z} \, dz. \]
By a change of variable, $z = \log t$, we see that
\[ \Phi(s) = \int_{0}^{\infty} F(\log t)t^{s-1/2} \, dt. \]
If $K(s)$ is a function on $\mathbb{C}$ and $x$ is any real number greater than 2, formally set $K(s)x^s = \Phi(s) =: \Phi_x(s)$. Then
\[ K(s) = \int_{0}^{\infty} F(\log t)x^{-s}t^{s-1/2} \, dt = \int_{0}^{\infty} x^{-1/2}F(\log t)(x^{-1}t)^{s-1/2} \, dt. \]
Replacing \( x^{-1}t \) by \( t \) we obtain
\[
K(s) = \int_0^\infty (xt)^{-1/2} F(\log(xt)) t^s \frac{dt}{t} = \int_0^\infty a(t) t^s \frac{dt}{t},
\]
where \( a(t) = (xt)^{-1/2} F(\log(xt)) \). Then notice that \( F(\log z) = z^{1/2} a(z/x) \).

Now assume that \( K(s) \) is rapidly decreasing in \( t \) where \( s = \sigma + it \) (i.e. \(|K(\sigma + it)| \ll t^{-2} \) as \( t \to \infty \)) and \( K(1/2 + it) = K(1/2 - it) \) such that \( a(t) \) is nonnegative and has compact support in \((0, \infty)\). In particular, assume that its support lies in \([A, B]\). Then Weil’s explicit formula takes the form
\[
\sum_{\varrho} K(\varrho) x^\varrho = \varepsilon_0 (K(0) + K(1)x) + a \left( \frac{1}{x} \right) \log \left( \frac{Nd_{K/k}|d_k|}{4\pi^2} \right)
- \sum_{p} \sum_{n=1}^\infty (\log Np) a \left( \frac{Np^n}{x} \right) \chi(p^n)
- \sum_{p} \sum_{n=1}^\infty \log Np \frac{Np^n}{Np^n} a \left( \frac{1}{x} \frac{Np^n}{x} \right) \chi(p^n) + W_\infty(a, \chi),
\]
where the \( \sum_{\varrho} \) is over the nontrivial zeros of \( L(s, \chi) \), \( \sum_{\varrho} \) is over the nonzero prime ideals of \( \mathcal{O} \), \( \varepsilon_0 = \varepsilon_0(\chi) = 1 \) if \( \chi \) is the principal character, and \( \varepsilon_0 = 0 \) if not. Finally,
\[
W_\infty(a, \chi) = 2 \frac{I'}{I} \left( \frac{1}{2} \right) a \left( \frac{1}{x} \right)
- \int_1^\infty \left( t^{1/2} a \left( \frac{t}{x} \right) + t^{-1/2} a \left( \frac{t^{-1}}{x} \right) - 2a \left( \frac{1}{x} \right) \right) \frac{t^{-1/2} dt}{1 - t^{-1}}.
\]
This implies that
\[
\sum_{\varrho} K(\varrho) x^\varrho = \varepsilon_0 K(0) x + a \left( \frac{1}{x} \right) \log \left( \frac{Nd_{K/k}|d_k|}{4\pi^2} \right)
- \sum_{p} \sum_{n=1}^\infty (\log Np) a \left( \frac{Np^n}{x} \right) \chi(p^n) + O(1).
\]

We need to extend this result to the (not necessarily primitive) characters \( \chi_\alpha \).

**Proposition 1.** If \( L(s, \chi_\alpha) \) is the \( L \)-function associated with \( \chi_\alpha \), then the following explicit formula holds:
\[
\sum_{\varrho} K(\varrho) x^\varrho = \varepsilon_0 K(1)x + a \left( \frac{1}{x} \right) \log \left( \frac{N\alpha|d_k|}{4\pi^2} \right)
- \sum_{p} \sum_{n=1}^\infty (\log Np) a \left( \frac{Np^n}{x} \right) \left( \frac{\alpha}{p^n} \right) + O((\log N\alpha)(\log x)).
\]
Proof. Suppose \( \chi \) is the primitive character of conductor \( f \) that induces \( \chi_\alpha \). Then as noted earlier \( L(s, \chi) \) and \( L(s, \chi_\alpha) \) share the same set of nontrivial zeros. Then the left-hand side of the equation in the proposition is identical with that in the explicit formula above. Now consider the difference of the right-hand sides of the two formulas:

\[
\sum_{p,n} (\log Np)a\left(\frac{Np^n}{x}\right)(\chi(p^n) - \chi_\alpha(p^n)) \ll \sum_{p(2\alpha)} (\log Np)a\left(\frac{Np}{x}\right)
\]

\[
\ll \sum_{p(2\alpha) \leq x} \log Np \ll (\log N\alpha)(\log x),
\]

since if \((2\alpha) = p_1^{a_1} \cdots p_m^{a_m}\) is the prime ideal factorization of \(2\alpha\), then

\[\log N\alpha = a_1 \log Np_1 + \cdots + a_m \log Np_m \gg m.\]

Now notice that

\[
a\left(\frac{1}{x}\right) \log \left(\frac{N\alpha|d_k|}{4\pi^2}\right) - a\left(\frac{1}{x}\right) \log \left(\frac{Nf|d_k|}{4\pi^2}\right) = O(\log N\alpha).
\]

This establishes the proposition. \(\blacksquare\)

4. Technical lemmas. We start by stating a 2-variable version of Euler–MacLaurin summation. Let \( P_m(x) \) denote the \( m \)th periodic Bernoulli function. Hence \( P_m(x) = B_m(x - \lfloor x \rfloor) \), where as usual \( B_m(x) \) is defined by

\[
\frac{e^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.
\]

**Lemma 1** (Euler–MacLaurin summation). Let \( N \) be a positive integer and \( f(u, v) \) be a function such that any \( 2N \)th-order partial derivative of \( f \) is continuous. Then for integers \( a, b, c, d \) with \( a \leq b \) and \( c \leq d \) we have

\[
\sum_{n=c}^{b} \sum_{m=a}^{d} f(m, n) = \int_{c}^{d} \int_{a}^{b} f(u, v) \, du \, dv
\]

\[
+ \sum_{\mu, \nu = 1}^{N} \frac{(-1)^{\mu+\nu}}{\mu! \nu!} P_{\mu}(u)P_{\nu}(v) \frac{\partial^{\mu+\nu-2} f(u, v)}{\partial u^{\mu-1} \partial v^{\nu-1}} \bigg|_{a}^{b} \bigg|_{c}^{d}
\]

\[
+ \sum_{\mu = 1}^{N} \frac{(-1)^{\mu}}{\mu!} P_{\mu}(u) \int_{c}^{d} \frac{\partial^{\mu-1} f(u, v)}{\partial u^{\mu-1}} \, dv \bigg|_{a}^{b}
\]

\[
+ \sum_{\nu = 1}^{N} \frac{(-1)^{\nu}}{\nu!} P_{\nu}(v) \int_{a}^{d} \frac{\partial^{\nu-1} f(u, v)}{\partial v^{\nu-1}} \, du \bigg|_{c}^{b}
\]

\[
+ \sum_{\nu = 1}^{N} \frac{(-1)^{\nu}}{\nu!} P_{\nu}(v) \int_{a}^{d} \frac{\partial^{\nu-1} f(u, v)}{\partial v^{\nu-1}} \, du \bigg|_{c}^{b}
\]
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\[
+ \sum_{\mu=1}^{N} \frac{(-1)^{N+\mu-1}}{\mu!N!} P_{\mu}(u) \int_{c}^{d} P_N(v) \frac{\partial^{N+\mu-1} f(u,v)}{\partial w^{\mu-1} \partial v^N} dv \bigg|_{a^{-}}^{b^{+}}
\]

\[
+ \sum_{\nu=1}^{N} \frac{(-1)^{N+\nu-1}}{\nu!N!} P_{\nu}(v) \int_{a}^{b} P_N(u) \frac{\partial^{N+\nu-1} f(u,v)}{\partial v^{\nu-1} \partial u^N} du \bigg|_{c^{-}}^{d^{+}}
\]

\[
+ \int_{c}^{d} \int_{a}^{b} \left[ \frac{(-1)^{N-1}}{N!} P_N(u) \frac{\partial^{N} f}{\partial u^N} + \frac{(-1)^{N-1}}{N!} P_N(v) \frac{\partial^{N} f}{\partial v^N} \right] du dv,
\]

where, e.g., \( u = a^{-} \) and \( v = b^{+} \) denote the appropriate one-sided limits.

The proof of the lemma follows by applying the one-variable version twice and is left to the reader. The one-variable version may be found on page 490 of [7], for example.

**Corollary 1.** Let \( m \in \mathbb{Z}^{>0} \), \( \alpha, \beta \in \mathbb{C} \) with \( \alpha \beta - \bar{\alpha} \bar{\beta} \neq 0 \), and \( c > 0 \). Then

\[
\sum_{(m_1, m_2) \in \mathbb{Z}^{2}} e^{-c|m_1 \alpha + m_2 \beta|^2 y^{-2m}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-c|u \alpha + v \beta|^2 y^{-2m}} du dv + O(y^{-M})
\]

as \( y \to \infty \) for any positive integer \( M \).

**Proof.** Let

\[
f(u, v) = e^{-c|u \alpha + v \beta|^2 y^{-2m}}.
\]

Then \( f \) is a rapidly decreasing function and thus we may apply Lemma 1, noting that all the relevant sums and integrals converge, obtaining

\[
\sum_{(m_1, m_2) \in \mathbb{Z}^{2}} f(m_1, m_2)
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u, v) du dv + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{(-1)^{N-1}}{N!} P_N(u) \frac{\partial^{N} f}{\partial u^N} \right] du dv + \frac{(-1)^{N-1}}{N!} P_N(v) \frac{\partial^{N} f}{\partial v^N} + \frac{1}{(N!)^2} P_N(u) P_N(v) \frac{\partial^{2N} f}{\partial u^N \partial v^N} du dv.
\]

We consider the second integral. An easy induction argument shows that

\[
\frac{\partial^{M} f(u, v)}{\partial u^\mu \partial v^\nu} = f(u, v) \sum_{j=1}^{M} y^{-2m_j h_{2m_j-M}(u, v)},
\]

where \( h_n(u, v) \) is a homogeneous polynomial of degree \( n \) and where we set this form equal to 0 if the “degree” is negative. Now let \( u = yu_1 \) and \( v = yv_1 \).
then, in particular,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_N(u) \frac{\partial^N f}{\partial u^N} \, du \, dv
= y^{2-N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_N(yu_1) e^{-|u_1\alpha + v_1\beta|^2 m} \sum_{j=1}^{N} h_{2m_j-N}(u_1, v_1) \, du_1 \, dv_1.
\]
Therefore,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_N(u) \frac{\partial^N f}{\partial u^N} \, du \, dv \ll y^{2-N}.
\]

The other two terms are similarly \( \ll y^{2-N} \). If we let \( N = M + 2 \), the corollary is established.

**Lemma 2.** Let \( a \) be a nonzero fractional ideal of \( k \), and let \( m \in \mathbb{Z}_{>0} \).

Then
\[
\sum_{\mu \in a} e^{-\frac{2\pi}{|d_k|} N(\mu) m y^{-1}} = \frac{y^{1/m} I}{Na \sqrt{|d_k|}^{1-1/m}} + O(y^{-M})
\]
as \( y \to \infty \), for any \( M \in \mathbb{Z}_{>0} \), and where
\[
I = \left( \frac{2\pi}{m} \right)^{1-1/m} \frac{1}{m} \Gamma \left( \frac{1}{m} \right).
\]
The sum is over all elements of \( a \) and \( N\mu \) represents the norm from \( k \) to \( \mathbb{Q} \) of \( \mu \). In particular
\[
\sum_{\mu \in a} e^{-\frac{2\pi}{|d_k|} N(\mu) m y^{-1}} = \begin{cases} y \frac{N\alpha}{Na} + O(y^{-M}) & \text{if } m = 1, \\ y^{1/2} \frac{\pi 2^{-1/2} |d_k|^{-1/4}}{Na} + O(y^{-M}) & \text{if } m = 2. \end{cases}
\]

Proof. Let \( \{\alpha, \beta\} \) be an integral basis of \( a \). Then
\[
\sum_{\mu \in a} e^{-\frac{2\pi}{|d_k|} N\mu m y^{-1}} = \sum_{(m_1, m_2) \in \mathbb{Z}^2} e^{-\frac{2\pi}{|d_k|} |m_1 \alpha + m_2 \beta|^{2m} y^{-1}}.
\]
By the corollary, we need only show that
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{2\pi}{|d_k|} N(x_1\alpha + x_2\beta) m y^{-1}} \, dx_1 \, dx_2 = \frac{y^{1/m} I}{Na \sqrt{|d_k|}^{1-1/m}}.
\]
To this end, let
\[
u + vi = \frac{x_1 \alpha + x_2 \beta}{(y \sqrt{|d_k|})^{1/2m}}, \quad \text{so} \quad u - vi = \frac{x_1 \bar{\alpha} + x_2 \bar{\beta}}{(y \sqrt{|d_k|})^{1/2m}}.
\]
Now
\[
\begin{pmatrix}
  u + vi \\
  u - vi
\end{pmatrix} = \frac{1}{(y\sqrt{|d_k|})^{1/2m}} \begin{pmatrix}
  \alpha & \beta \\
  \overline{\alpha} & \overline{\beta}
\end{pmatrix} \begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix}
\]
and
\[
\left| \det \begin{pmatrix}
  \alpha & \beta \\
  \overline{\alpha} & \overline{\beta}
\end{pmatrix} \right| = N(a)\sqrt{|d_k|}
\]
(see, e.g., [5, p. 188]). By elementary row reduction we get \((u, v)^t = B(x_1, x_2)^t\) where
\[
B = \frac{1}{(y\sqrt{|d_k|})^{1/2m}} \begin{pmatrix}
  (\alpha + \overline{\alpha})/2 & (\beta + \overline{\beta})/2 \\
  (\alpha - \overline{\alpha})/2i & (\overline{\beta} - \beta)/2i
\end{pmatrix}.
\]
But then notice that
\[
\left| \det (B) \right| = \frac{1}{2(y\sqrt{|d_k|})^{1/2m}} \left| \det \begin{pmatrix}
  \alpha & \beta \\
  \overline{\alpha} & \overline{\beta}
\end{pmatrix} \right|.
\]
Hence
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}}|x_1\alpha + x_2\beta|^{2m}y^{-1}} \, dx_1 \, dx_2 = |\det B^{-1}| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi(u^2 + v^2)^m} \, du \, dv.
\]
But
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-2\pi(u^2 + v^2)^m} \, du \, dv = \frac{2\pi}{0 \ 0} e^{-2\pi r^{2m}} \, r \, dr \, d\theta = 2\pi \int_{0}^{\infty} e^{-2\pi r^{2m}} \, r \, dr
\]
\[
= \left( \frac{2\pi}{2m} \right)^{1-1/m} \Gamma \left( \frac{1}{m} \right),
\]
by changing first to polar coordinates and then changing to \(t = 2\pi r^{2m}\).

This establishes the lemma. \(\blacksquare\)

**Lemma 3.** The series
\[
\sum_{\substack{\mu \in \mathcal{O} \\
\mu \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y} \log N\mu = y \log y + O(y)
\]
as \(y \to \infty\).

**Proof.** Write
\[
\sum_{\substack{\mu \in \mathcal{O} \\
\mu \neq 0}} e^{-\frac{2\pi}{\sqrt{|d_k|}}N\mu/y} \log N\mu = S_1 + S_2,
\]
where
\[ S_1 = \sum_{\mu \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mu/y} \log y, \quad S_2 = \sum_{\mu \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mu/y} \log \left( \frac{N\mu}{y} \right). \]

Notice that
\[ S_1 = y \log y + O(\log y) \]

by Lemma 2.

Now consider \( S_2 \). By Riemann–Stieltjes integration,
\[ S_2 = \int_1^\infty e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) dJ(u), \]
where
\[ J(u) = \sum_{\mu \in \mathcal{O}} 1. \]

But \( J(u) = \varrho u + O(u^{1/2}) \) where \( \varrho \) is a constant depending only on the field \( k \); see, e.g., Lang [11, p. 132]. Hence,
\[ S_2 = \int_1^\infty e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) d\varrho u - \int_1^\infty e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) d(J(u) - \varrho u). \]

The first integral is \( \ll y \), by direct computation. In the second integral, we integrate by parts and then make a change of variable, obtaining
\[ \int_1^\infty e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) d(J(u) - \varrho u) \]
\[ = (J(u) - \varrho u)e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) \bigg|_{1}^{\infty} \]
\[ - \int_1^\infty (J(u) - \varrho u) \left( e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) \right) du \]
\[ \ll \log y \int_1^\infty (J(u) - \varrho u) e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \frac{du}{u} \]
\[ + \frac{2\pi}{y\sqrt{|d_k|}} \int_1^\infty (J(u) - \varrho u) e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{u}{y}} \log \left( \frac{u}{y} \right) du \]
\[ \ll \log y \int_1^\frac{y}{y^{-1}} (J(yv) - \varrho yv) e^{-\frac{2\pi}{\sqrt{|d_k|}} \frac{v}{y}} \frac{dv}{v} \]
\[ + \frac{2\pi}{\sqrt{|d_k|}} \int_1^\frac{y}{y^{-1}} (J(yv) - \varrho yv) e^{-\frac{2\pi}{\sqrt{|d_k|}} \log v} dv \ll y^{1/2} \log y. \]

This establishes the result. ■
We shall also need a variation of the previous lemma. Namely,

**Lemma 4.** For \( \mu \in \mathcal{O}_k - \{0\} \), let \( f_\mu \) denote the conductor of \( \chi_\mu \). Then the series

\[
\sum_{\substack{\mu \in \mathcal{O} \\
\mu \neq 0}} e^{-\frac{2\pi}{|d_k|} N\mu/y} \log Nf_\mu = y \log y + O(y)
\]

as \( y \to \infty \).

**Proof.** By subtracting the formula above from the one in Lemma 3, we see that it suffices to show that

\[
\sum_{\substack{\mu \in \mathcal{O} \\
\mu \neq 0}} e^{-\frac{2\pi}{|d_k|} N\mu/y} \log \left( \frac{N\mu}{Nf_\mu} \right) \ll y.
\]

To this end, let

\[
\sum_{\substack{\mu \in \mathcal{O} \\
\mu \neq 0}} e^{-\frac{2\pi}{|d_k|} N\mu/y} \log \left( \frac{N\mu}{Nf_\mu} \right) = \int_1^\infty e^{-\frac{2\pi}{|d_k|} u/y} d\alpha(u),
\]

where \( \alpha(x) = \sum_{N\mu \leq x} \log (N\mu/Nf_\mu) \). We claim \( \alpha(x) \ll x \). To see this, first by the conductor-discriminant formula (see [4] or [2]), \( f_\mu = D_{K/k} \) where \( D_{K/k} \) is the relative discriminant, \( K = k(\sqrt{\mu}) \). But since \( K/k \) is a Kummer extension, \( f_\mu \) can be determined fairly easily; cf. [4], [6], or [2]. In our case, we can see that the fractional ideal \((\mu)/f_\mu = a^2/b\) where \( b \mid (16) \). Moreover, \( f_\mu = c_0c_1 \) where \( c_0 \) is square-free and \( c_1 \mid (32) \). Thus we have

\[
\alpha(x) \ll \sum_{m,n} \mu_2(n) \log Nm,
\]

where \( m, n \) are integral ideals of \( k \) and \( \mu \) is the usual generalization of the Möbius \( \mu \)-function to the semigroup of integral ideals of \( k \). Now notice that

\[
\sum_{Nn \leq x} \mu_2(n) \ll \sum_{Nn \leq x} 1 \ll x
\]

(see [11], for example). Therefore,

\[
\sum_{m,n} \mu_2(n) \log Nm = \sum_{Nm \leq \sqrt{x}} \log Nm \sum_{Nn \leq \frac{x}{Nm^2}} \mu_2(n) \ll \sum_{Nm \leq \sqrt{x}} \log Nm \frac{x}{Nm^2}
\]

\[
\ll x \sum_{Nm \leq \sqrt{x}} \frac{\log Nm}{Nm^2} \ll \varepsilon \zeta_k(2 - \varepsilon) \ll x,
\]

which establishes the claim.

Now, finally integrating by parts and then by a change of variables we have
\[
\int_1^\infty e^{-\sqrt{|d_k|} u/y} \alpha(u) du = e^{-\sqrt{|d_k|} u/y} \alpha(u) \bigg|_1^\infty + \frac{2\pi}{y \sqrt{|d_k|}} \int_1^\infty e^{-\sqrt{|d_k|} u/y} \alpha(u) du
\]
\[
\ll \frac{1}{y} \int_1^\infty e^{-\sqrt{|d_k|} u/y} u du \ll \frac{1}{y} \int_1^\infty e^{-v^2} dv \ll y.
\]
This establishes the lemma. ■

We assume the “Generalized Riemann Hypothesis (GRH) for \( k \)” in the following lemma, i.e. the GRH holds for all abelian \( L \)-functions over \( k \).

**Lemma 5.** Let \( k \) be any algebraic number field and let \( \mathcal{C}_f \) be the ray class group modulo \( f \), i.e. the group of \( f \)-classes (see [10] or [11, Chapter VI]). Denote the order of \( \mathcal{C}_f \) by \( h_f \). Let \( m, n \) be real numbers with \( n \neq 0 \). Assuming GRH for \( k \), the sum

\[
\sum_{p : p \in \mathcal{C}_f, f(p) = 1} a \left( \frac{Np^n}{x} \right) Np^m \log Np = \frac{1}{h_f} \frac{1}{n} K \left( \frac{m+1}{n} \right) x^{(m+1)/n}
\]

\[
+ O(x^{(2m+1)/(2n)} \log^2 x)
\]
as \( x \to \infty \), where the sum is over all nonzero prime ideals of \( \mathcal{O} \) which lie in \( c \in \mathcal{C}_f \) and have absolute residue degree \( f(p) = [\mathcal{O}/p : \mathbb{Z}/(p)] = 1 \).

**Proof.** By Riemann–Stieltjes integration, we have

\[
\sum_{p : p \in \mathcal{C}_f, f(p) = 1} a \left( \frac{Np^n}{x} \right) Np^m \log Np = \int_0^\infty a \left( \frac{u^n}{x} \right) u^m d \left( \sum_{Np \leq u \atop p \in \mathcal{C}_f, f(p) = 1} \log Np \right).
\]

Hence by the Prime Ideal Theorem, assuming GRH,

\[
\sum_{Np \leq u \atop p \in \mathcal{C}_f, f(p) = 1} \log Np = \frac{1}{h_f} u + E(u),
\]

with \( E(u) \ll u^{1/2} \log^2 u \), where the implied constant depends only on the field \( k \) and not on the class \( c \); see [10]. Thus

\[
\int_0^\infty a \left( \frac{u^n}{x} \right) u^m d \left( \sum_{Np \leq u \atop p \in \mathcal{C}_f, f(p) = 1} \log Np \right) = \frac{1}{h_f} \int_0^\infty a \left( \frac{u^n}{x} \right) u^m du + \int_0^\infty a \left( \frac{u^n}{x} \right) u^m dE(u).
\]

In the first integral let \( v = u^n/x \); then

\[
\int_0^\infty a \left( \frac{u^n}{x} \right) u^m du = \frac{1}{n} x^{(m+1)/n} \int_0^\infty a(v)v^{(m+1)/n} dv \frac{dv}{v} = \frac{1}{n} K \left( \frac{m+1}{n} \right) x^{(m+1)/n}.
\]
Using integration by parts on the second integral, we obtain

$$\int_0^\infty x^{n+m} dE(u)$$

$$= - \frac{n}{x} \int_0^\infty E(u) a(x^{n+1}) u^{m+n-1} du - m \int_0^\infty E(u) u^{m-1} du$$

$$= - x^{m/n} \int_0^\infty E(x^{1/n} v^{1/n}) a(v) v^{(m+n)/n} dv$$

$$- \frac{m}{n} x^{m/n} \int_0^\infty E(x^{1/n} v^{1/n}) a(v) v^{m/n} dv \ll x^{m/n} x^{1/2n} \log^2 x,$$

as desired. □

**Lemma 6 (Transformation formula).** Let \( p \) be a nonzero odd prime ideal of \( \mathcal{O} \) dividing the rational prime \( p > 0 \), but not dividing the discriminant \( d_k \) of \( k \); let \( \chi = \chi_p \), where \( \chi_p(\alpha) = (\alpha/p) \).

If \( (p) = p\mathfrak{p} \), then

$$\sum_{\alpha \in \mathcal{O}} \chi_p(\alpha) e^{-2\pi i \sqrt{|d_k|} N\alpha/y} = \frac{y}{\sqrt{Np}} \sum_{\nu \in \mathfrak{p}/\sqrt{d_k}} \chi_p(\nu) e^{-2\pi i \sqrt{|d_k|} N\nu|y/p^2}.$$

If \( p \) is inert, then

$$\sum_{\alpha \in \mathcal{O}} \chi_p(\alpha) e^{-2\pi i \sqrt{|d_k|} N\alpha/y} = \frac{y}{\sqrt{Np}} \sum_{\nu \in \mathcal{O}} \chi_p(\nu) e^{-2\pi i \sqrt{|d_k|} N\nu y/p^2}.$$

**Proof.** First notice that if \( \chi(W_k) \neq \{1\} \), where \( W_k \) denotes the group of roots of unity in \( k \), then \( \sum_{\alpha \in W_k} \chi_p(\alpha) = 0 \) by a standard argument. But then the assertion of the lemma is obvious as all sides of the equality are 0.

Now assume \( \chi(W_k) = \{1\} \). If \( \mathfrak{c} = (p) \), let \( \mathfrak{c} = 1 \); otherwise, let \( \mathfrak{c} = \mathfrak{p} \). Notice then that \( (\mathfrak{c}, p) = 1 \) since by assumption \( p/p \) is unramified as \( d_k \not\equiv 0 \mod p \). Furthermore, let \( c = 2\pi \sqrt{N\mathfrak{c}/\sqrt{|d_k|}} \) and \( c' = 2\pi |d_k|/(\sqrt{|d_k|} \sqrt{N\mathfrak{c}}) \).

Then

$$\sum_{\alpha \in \mathcal{O}} \left( \frac{\alpha}{\mathfrak{p}} \right) e^{-c(\mathfrak{c}N\alpha)t} = \sum_{\theta \in \mathcal{O}/\mathfrak{p}} \left( \frac{\theta}{\mathfrak{p}} \right) \sum_{\mu \in \mathfrak{p}} e^{-cN(\mu+\mathfrak{p})t} = \sum_{\theta \in \mathcal{O}/\mathfrak{p}} \left( \frac{\theta}{\mathfrak{p}} \right) \sum_{\mu \in \mathfrak{p}} e^{-c \frac{N(\mu+\mathfrak{p})}{p} pt},$$

where \( \sum_{\theta \in \mathcal{O}/\mathfrak{p}} \) is the sum over the cosets in \( \mathcal{O}/\mathfrak{p} \). Then by Hecke [5, pp. 189–190],

$$\sum_{\mu \in \mathfrak{p}} e^{-c \frac{N(\mu+\mathfrak{p})}{p} pt} = \frac{1}{pt \sqrt{N\mathfrak{p}}} \sum_{\lambda \in \mathfrak{c}/\sqrt{d_k}} e^{-c' \frac{N\lambda}{p^2 t} + 2\pi i \text{Tr}(\lambda \mathfrak{p})/p}. $$
Consequently,
\[
\sum_{\alpha \in \mathcal{O}} \left( \frac{\alpha}{p} \right) e^{-cN\alpha t} = \frac{1}{pt\sqrt{Np}} \sum_{\lambda \in \mathcal{O}/\sqrt{d_k}} e^{-c' \frac{N\lambda}{p\sqrt{N}} \sum_{\varrho \in \mathcal{O}/p} \left( \frac{\varrho}{p} \right)^2 e^{2\pi i \text{Tr}(\frac{\lambda\varrho}{p})}}.
\]

Let now \( t\sqrt{Nc} = 1/y \), in which case the last equality becomes
\[
\sum_{\alpha \in \mathcal{O}} \left( \frac{\alpha}{p} \right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/y} = \frac{y\sqrt{Nc}}{p\sqrt{Np}} \sum_{\lambda \in \mathcal{O}/\sqrt{d_k}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\lambda|d_k|y/p^2} \sum_{\varrho \in \mathcal{O}/p} \left( \frac{\varrho}{p} \right) e^{2\pi i \text{Tr}(\frac{\lambda\varrho}{p})}.
\]

Now we need to evaluate
\[
\sum_{\varrho \in \mathcal{O}/p} \left( \frac{\varrho}{p} \right) e^{2\pi i \text{Tr}(\frac{\lambda\varrho}{p})}
\]
for \( \lambda \in \mathcal{O}/\sqrt{d_k} \). To this end, let \( \lambda = \nu/\sqrt{d_k} \) where \( \nu \in \mathcal{O} \). We consider two cases.

**CASE 1:** Suppose \( p\mathcal{O} = p \). Notice that if \( \nu \in p \), then \( \text{Tr}(\lambda\varrho/p) = \text{Tr}(\nu\varrho/(p\sqrt{d_k})) \in \mathbb{Z} \). Thus
\[
\sum_{\varrho \in \mathcal{O}/p} \left( \frac{\varrho}{p} \right) e^{2\pi i \text{Tr}(\frac{\lambda\varrho}{p})} = 0
\]
in this case. Next notice that \( Np = p^2 \) and \( \mathcal{O} = \mathcal{O}/\mathcal{O} \). Let \( \omega \in k - \{0\} \). Then write \( (\omega) = ba^{-1}(\sqrt{d_k})^{-1} \) with \( a \) and \( b \) relatively prime integral ideals of \( \mathcal{O} \). Let
\[
C(\omega) = \sum_{\mu \mod a} e^{2\pi i \text{Tr}(\mu^2\omega)},
\]
where the sum is over any system of representatives of \( \mathcal{O}/a \). By formula (171) of [6], we have
\[
\sum_{\varrho \in \mathcal{O}/p} \left( \frac{\varrho}{p} \right) e^{2\pi i \text{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} = C\left( \frac{\nu}{p\sqrt{d_k}} \right).
\]
If \( \nu \in \mathcal{O} - p \), then Satz 155 in [6] implies \( C(\nu/(p\sqrt{d_k})) = (\nu/p)C(1/(p\sqrt{d_k})) \). But then Satz 163 of [6] yields
\[
\frac{C(1/(p\sqrt{d_k}))}{\sqrt{Np}} = \frac{C(-p/(4\sqrt{d_k}))}{\sqrt{N(8)}}
\]
(by choosing \( \gamma = 1/\sqrt{d_k} \). Now
\[
C\left( -\frac{p}{4\sqrt{d_k}} \right) = \sum_{\mu \in \mathcal{O}/(4)} e^{2\pi i \text{Tr}(\frac{-\mu^2p}{\sqrt{d_k}})} = \sum_{\mu \in \mathcal{O}/(4)} e^{2\pi i \frac{p}{4} \text{Tr}(\frac{-\mu^2}{\sqrt{d_k}})}
\]
\[
\sum_{\mu \in O/(4)} e^{(-1)^{(p-1)/2}2\pi i \text{Tr}(\frac{\mu^2}{4\sqrt{d_k}})} = \begin{cases} 
C\left(\frac{-1}{4\sqrt{d_k}}\right) & \text{if } p \equiv 1 \mod 4, \\
C\left(\frac{-1}{4\sqrt{d_k}}\right) & \text{if } p \equiv 3 \mod 4.
\end{cases}
\]

But \(C(-p/(4\sqrt{d_k}))\) is real, since \(C(1/(p\sqrt{d_k}))\) is real. (Recall that we are assuming that \(\chi(W_k) = \{1\}\).) Thus the two cases above coincide. By [6, page 243],

\[
C\left(\frac{-1}{4\sqrt{d_k}}\right) = \sqrt{N(8)},
\]

which implies that \(C(1/(p\sqrt{d_k})) = \sqrt{Np} = p\). Therefore

\[
\sum_{\varrho \in O/p} \left(\frac{\varrho}{p}\right) e^{2\pi i \text{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} = \left(\frac{\nu}{p}\right) \sqrt{Np}.
\]

This yields the second part of the lemma.

**Case 2:** Suppose \((p) = pp\). Then \(Np = p, O/p \simeq \mathbb{Z}/(p)\), and \(c = \bar{p}\). As above, let \(\lambda = \nu/\sqrt{d_k}\), but with \(\nu \in \bar{p}\). Then

\[
\sum_{\varrho \in O/p} \left(\frac{\varrho}{p}\right) e^{2\pi i \text{Tr}(\frac{\nu\varrho}{p\sqrt{d_k}})} = \sum_{b \in \mathbb{Z}/(p)} \left(\frac{b}{p}\right) e^{2\pi i b/p} \text{Tr}(\frac{\nu}{\sqrt{d_k}})
\]

by well-known properties of rational Gauss sums (using the assumption \((-1/p) = 1\) since \(\chi(W_k) = \{1\}\)). Now let \(\omega = \omega_k = (d_k + \sqrt{d_k})/2\) and thus \(O = \mathbb{Z}[\omega_k]\). Let \(p(x) = (x - \omega)(x - \bar{\omega})\); thus \(p(x) = x^2 - d_kx + (d_k^2 - d_k)/4\).

Since \(p\) splits in \(O\), we know that \(p(x) \equiv (x - a)(x - b) \mod p\) for some \(a, b \in \mathbb{Z}\). But then \(p = (p, \alpha)\) and \(\bar{p} = (p, \bar{\alpha})\) for some \(\alpha = a + \omega_k\) (without loss of generality). Since \(\nu \in \bar{p}\), there are \(x, y \in \mathbb{Z}\) such that

\[
\nu = px + \bar{\alpha}y = px + \left(a + \frac{d_k - \sqrt{d_k}}{2}\right)y.
\]

Then \(\text{Tr}(\nu/\sqrt{d_k}) = \text{Tr}(-y/2) = -y\). Thus

\[
\left(\frac{\text{Tr}(\nu/\sqrt{d_k})}{p}\right) = \left(\frac{-y}{p}\right) = \left(\frac{y}{p}\right),
\]

again observing that \((-1/p) = 1\). Now,

\[
\nu = px + \bar{\alpha}y \equiv \bar{\alpha}y \equiv \left(a + \frac{d_k + \sqrt{d_k}}{2}\right)y - \sqrt{d_k}y \equiv -\sqrt{d_k}y \mod p.
\]

Hence

\[
\left(\frac{\nu}{p}\right) = \left(\frac{-y\sqrt{d_k}}{p}\right) = \left(\frac{y}{p}\right) \left(\frac{\sqrt{d_k}}{p}\right).
\]
Therefore, \[
\left(\frac{y}{p}\right) = \left(\frac{\nu/\sqrt{dk}}{p}\right),
\]
and we see that
\[
\sum_{\rho \in \mathcal{O}/p} \left(\frac{\rho}{p}\right) e^{2\pi i \text{Tr}(\frac{\nu^p}{p\sqrt{dk}})} = \left(\frac{\nu/\sqrt{dk}}{p}\right) \sqrt{N_p}.
\]
This establishes the lemma. ■

We continue in this section with some arithmetical results about the 2-power residue symbol. Let \(C_{(4)}\) be the ray class group of \(k\) modulo \((4)\). Hence, \(C_{(4)} = I(4)/P_{(4)}\), where \(I(4)\) is the group of fractional ideals prime to \(4\) and \(P_{(4)}\) the group of principal ideals in \(I(4)\) with generators \(\gamma \equiv 1 \mod 4\). See [11, Chapter VI], for details. Let \(H = \{(\gamma) \in P(4) : \gamma \equiv \xi^2 \mod 4\}\). Finally, set \(C = I(4)/H\) and \(C_{(2)} = \{aH : a^2 \in H\}\).

**Lemma 7.** Let \(k\) be an imaginary quadratic number field. Then the index 
\[(C : C_{(2)}) = (I(4) : I^2(4)H) = 2^{e+1},\]
where \(2^e\) is the index \((\text{Cl} : \text{Cl}_{(2)})\) (notice that \(e\) is the \(2\)-rank of the class group of \(k\)).

**Proof.** The first equality is obvious. For the second, let \(K = k(\sqrt{S})\), where \(S = \{\mu \in k^\times : (\mu) = a^2\ \text{for some fractional ideal } a \text{ of } k^\times\}\). Then \([K : k] = 2^{e+1}\) (see [5, paragraph 4 on page 253]). On the other hand, by Satz 171 of [5], the Artin map induces an isomorphism
\[I(4)/I^2(4)H \simeq \text{Gal}(K/k)\]
where \(p \mapsto (p, K/k)\), for any prime ideal \(p \in I(4)\). Thus the lemma follows. ■

We note for future reference that the prime ideals, \(p\), in \(I^2(4)H\) are characterized by the following reciprocity law (for example see Hecke [5, Satz 171]):
\[
\left(\frac{\mu}{p}\right) = 1,
\]
for all \(\mu \in k^\times\) for which \((\mu) = a^2\) for some fractional ideal \(a\) prime to \(p\).

**Lemma 8.** Let \(k\) be an imaginary quadratic number field. Suppose that \(p\) is a prime ideal of absolute degree \(1\) in \(I^2(4)H\), say \(pb^2 = (\gamma)\) where \(b\) is an integral ideal in \(I(4)\) and \(\gamma \in \mathcal{O}\) with \(\gamma \equiv \xi^2 \mod 4\), and that \(p\) does not divide \(\omega = (d_k + \sqrt{d_k})/2\). Then
\[
\left(\frac{\gamma/\sqrt{dk}}{p}\right) = 1.
\]

**Proof.** It suffices to evaluate \((\gamma\sqrt{d_k}/p)\). To this end, let \(\gamma = x + y\omega\). First notice that since \(\gamma = x + y\omega\),
\[
pb^2 = NpNb^2 = N\gamma = x^2 + xyd_k + y^2(d_k^2 - d_k)/4,
\]
with $b = Nb$, an odd integer. Then a straightforward computation yields
\[ y\sqrt{d_k} = 2(x + yd_k/2)\gamma - 2pb^2 - d_k y^2, \]
and therefore
\[ \left( \frac{y\sqrt{d_k}}{p} \right) = \left( \frac{-d_k y}{p} \right) = \left( \frac{y}{p} \right), \]
since $(d_k/p) = 1$ because $f(p) = 1$, and $1 = (-1/p) = (-1/p)$ by the reciprocity law for the 2-power residue symbol. Hence we need to evaluate $(y/p)$. Notice that the representation of $pb^2$ above implies $pb^2 \equiv x^2 \mod y$.

Now we consider the following two cases.

**Case 1.** Suppose $y$ is odd. Then since $pb^2 \equiv x^2 \mod y$, $1 = (p/y) = (y/p)$ by an extension of the quadratic law of reciprocity and since $p \equiv 1 \mod 4$.

**Case 2.** Suppose $y$ is even, say $y = 2^a y_0$, with $y_0$ odd. Since $pb^2 \equiv x^2 \mod y_0$, we have $(y_0/p) = 1$ as above.

If $a \geq 3$, so $8 | y$, then $p \equiv 1 \mod 8$, for $b^2 \equiv 1 \mod 8$, whence $(2/p) = 1$ and consequently $(y/p) = (2/p)^a (y_0/p) = 1$.

If $a = 2$, then obviously $(y/p) = (4/p) (y_0/p) = 1$.

Now assume $a = 1$, so $y = 2y_0$; then
\[ p = x^2 + 2xy_0 d_k + y_0^2 (d_k^2 - d_k). \]

If $d_k$ is even, then $d_k \equiv 0 \mod 4$. If furthermore, $d_k \equiv 0 \mod 8$, then $p \equiv 1 \mod 8$, whence $(2/p) = 1$, implying that $(y/p) = 1$. Now, if $d_k \equiv 4 \mod 8$, then $d_k = 4d$ for some $d \equiv -1 \mod 4$. Hence, $pb^2 = x^2 + 8xy_0d + y_0^2(16d^2 - 4d) \equiv 1 - 4d \equiv 5 \mod 8$. But this cannot happen, since $(2) = a^2$ so by the above $(2/p) = 1$. But then $1 = (2/p) = (2/p)$, implying $p \equiv 1 \mod 8$.

If $d_k$ is odd, then $d_k \equiv 1 \mod 4$, in which case $p \equiv x^2 + 2 \mod 4$. Therefore, $p \equiv pb^2 \equiv 3 \mod 4$, contrary to our assumptions.

Finally, notice that if $d_k = -4d$ with $d \equiv 1 \mod 4$, then for $p \equiv 5 \mod 8$, $y$ must be even. Indeed, since $\gamma \equiv \xi^2 \mod 4$ with say $\xi = a_1 + b_1 \sqrt{-d}$, $a_1, b_1 \in \mathbb{Z}$, we see that $\xi^2 = a_1^2 - db_1^2 + 2a_1b_1 \sqrt{-d}$, implying that $y$ is even.

**Lemma 9.** Let $\chi_1$ be a character on $C = C_m = I(m)/P_m$ for some modulus $m$ with $\ker \chi_1 = H_1$ and such that the conductor of $\chi_1$ is $n$ (notice $n \mid m$). Suppose $H_2$ is a subgroup of $C_m$ such that for $H_0 = H_1 \cap H_2$ we have $H_0 \neq H_i$ for $i = 1, 2$. Then assuming GRH for $k$, we have
\[ \sum_{p \in \mathcal{B} H_2 \atop Np \leq x} \chi_1(p) \log Np \ll x^{1/2} \log^2 x \lambda \mathcal{N}, \]
as $x \to \infty$, for any $b \in I(m)$. 


Proof (sketch). By the usual orthogonality properties of characters and Euler product expansion of $L$-functions,

$$
\sum_{p \in \mathcal{H}_2} \frac{\chi_1(p)}{Np^s} = \frac{1}{(C : H_0)} \chi_1(b) \sum_{c \in \mathcal{H}_2/H_0} \sum_{\chi \in \tilde{C}/H_0} \chi_1(\chi) \log L(s, \chi) + O(s)
$$

$$
= \frac{\chi_1(b)}{(C : H_0)} \sum_{c \in \mathcal{H}_2/H_0} \sum_{\chi \in \tilde{C}/H_0} \chi_1(\chi) \log L(s, \chi)
$$

$$
+ \frac{\chi_1(b)}{(C : H_0)} \log \zeta_k(s) \sum_{c \in \mathcal{H}_2/H_0} \chi_1(c) + O(s)
$$

for $\sigma > 1/2$. But $\sum_c \chi_1(c) = 0$ by the assumptions of the lemma, and $\log L(s, \chi)$ is regular for $\sigma > 1/2$ since $\chi \neq 1$ and assuming GRH for $k$. But then the lemma follows from the explicit formula for $\log L(s, \chi)$ following the arguments of Landau [10, Sections 4 and 6], and Davenport [3, Section 20].

5. Main results. Throughout this section assume $k$ is a complex quadratic number field. Denote by $d_k$ its discriminant; by $\mathcal{O} = \mathcal{O}_k$ its ring of integers; by $w_k$ the cardinality of $W_k$, the group of units in $k$; by $\text{Cl} = \text{Cl}(k)$ its class group; and by $h = h(k)$ the class number of $k$.

We now investigate the sum

$$
\sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^\varrho
$$

as $D \to \infty$, where the outer sum is over the nonzero elements of $\mathcal{O}$ and the inner sum is over the nontrivial zeros of $L(s, \chi_\alpha)$.

We now state and prove our main result.

Theorem 1. Assume GRH for $k$. Then as $D \to \infty$ and either $x \to \infty$ or $x = 1$,

$$
\sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^\varrho
$$

$$
= \begin{cases} 
-\frac{1}{2} K\left(\frac{1}{2}\right) D x^{1/2} + \frac{1}{2} K(1) I x D^{1/2} + a\left(\frac{1}{2}\right) D \log D \\
+ O(D \log D \log x + D x^{1/3} \log x + a\left(\frac{1}{2}\right) D) & \text{if } x = o(D), \\
0 + O(x \log^2 x) + O(D x^{1/3} \log x) & \text{if } D = o(x),
\end{cases}
$$

where $I = \pi 2^{-1/2}|d_k|^{-1/4}$ and all the implied constants depend only on the base field $k$ and the kernel $K$. 

Proof. We start by using the explicit formula in Proposition 1 to write the above sum as

\[
\sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \sum_{\varrho(\alpha)} K(\varrho)x^\varrho = A + B + C + O,
\]

with

\[
A = K(1)x \sum_{\alpha \in \mathcal{O}} \varepsilon_0(\chi\alpha)e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D},
\]

\[
B = -\sum_{p,n} a\left(\frac{Np^n}{x}\right)(\log Np) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \chi\alpha(p^n),
\]

\[
C = a\left(\frac{1}{x}\right) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \log \frac{N\alpha|d_k|}{4\pi^2},
\]

\[
O = O\left(\log x \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \log N\alpha\right).
\]

We now break up \(B\) further as

\[
B = B_1 + B_2 + B_3 + B_4,
\]

where

\[
B_1 = -\sum_{p} a\left(\frac{Np}{x}\right)(\log Np) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \chi\alpha(p),
\]

\[
B_2 = -\sum_{p} a\left(\frac{Np^2}{x}\right)(\log Np) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D},
\]

\[
B_3 = \sum_{p} a\left(\frac{Np^2}{x}\right)(\log Np) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D},
\]

\[
B_4 = -\sum_{p,n, n \geq 3} a\left(\frac{Np^n}{x}\right)(\log Np) \sum_{\alpha \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} \chi\alpha(p^n).
\]

By Lemma 3,

\[
O \ll D \log D \log x.
\]

Next, by Lemma 2,

\[
A = K(1)x \sum_{\alpha = 0} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha/D} = \frac{1}{2} K(1)x D^{1/2} - \frac{1}{2} K(1)x + O(xD^{-M}),
\]
with \( I = \pi 2^{-1/2}|d_k|^{-1/4} \), and where \( \alpha = \square \) means that \( \alpha \) ranges over all squares of elements in \( \mathcal{O} \), since \( \chi_\alpha \) is principal if and only if \( \alpha \) is a square.

By Lemma 3 again, we see that

\[
C = a\left(\frac{1}{x}\right) D \log D + O(D).
\]

Now notice that

\[
B_3 = \sum_p a\left(\frac{N \mathbf{p}^2}{x}\right) (\log N \mathbf{p}) \sum_{\alpha \in \mathcal{O}} \sum_{\alpha \neq 0} e^{-\frac{2\pi}{|d_k|} N \alpha / D}
\]

\[
= \sum_p a\left(\frac{N \mathbf{p}^2}{x}\right) (\log N \mathbf{p}) \left(\frac{D}{N \mathbf{p}} + O(1)\right),
\]

by Lemma 2. By Lemma 5 we have

\[
B_3 \ll D + x^{1/2}.
\]

Since \( a(N \mathbf{p}^n / x) = 0 \) unless \( A \leq N \mathbf{p}^n / x \leq B \) (see the Preliminaries), we see that

\[
B_4 \ll \sum_{\mathbf{p}, n \geq 3} a\left(\frac{N \mathbf{p}^n}{x}\right) (\log N \mathbf{p}) \sum_{\alpha \in \mathcal{O}} \sum_{\alpha \neq 0} e^{-\frac{2\pi}{|d_k|} N \alpha / D}
\]

\[
\ll D \sum_{\mathbf{p}, n \geq 3} a\left(\frac{N \mathbf{p}^n}{x}\right) \log N \mathbf{p} \ll D x^{1/3} \log x,
\]

by Lemmas 2 and 5.

Also by Lemmas 2 and 5, we have

\[
B_2 = -\frac{1}{2} K \left(\frac{1}{2}\right) D x^{1/2} + O(x^{1/2}) + O(D x^{1/4} \log^2 x).
\]

Now we consider \( B_1 \). We decompose \( B_1 \) as \( B_1 = B_1^i + B_1^s \), where

\[
B_1^i = -\sum_{(d_k / p) = -1} a\left(\frac{N \mathbf{p}}{x}\right) (\log N \mathbf{p}) \sum_{\alpha \in \mathcal{O}} \sum_{\alpha \neq 0} e^{-\frac{2\pi}{|d_k|} N \alpha / D} \chi_\alpha (p),
\]

\[
B_1^s = -\sum_{(d_k / p) = 1} a\left(\frac{N \mathbf{p}}{x}\right) (\log N \mathbf{p}) \sum_{\alpha \in \mathcal{O}} \sum_{\alpha \neq 0} e^{-\frac{2\pi}{|d_k|} N \alpha / D} \chi_\alpha (p).
\]

By the transformation formula (Lemma 6),

\[
B_1^t = -\sum_{(d_k / p) = (-1)^\tau} a\left(\frac{N \mathbf{p}}{x}\right) (\log N \mathbf{p}) \frac{D}{\sqrt{N \mathbf{p}}} \sum_{\nu \in \mathcal{A}} \left(\frac{\nu}{p}\right) e^{-\frac{2\pi}{|d_k|} \nu |d_k|^\tau D / p^2},
\]
where $\tau = 0$ or $1$, $a_p = a = \bar{p}/\sqrt{d_k}$ or $\mathcal{O}$, and $\varepsilon = 1$ or $0$, if $t = s, i$, respectively.

We now consider two cases.

**Case 1.** Assume $x = o(D)$. We first show that $B^i_1 \ll x^{1/2}$. To this end notice that

$$\frac{D}{\sqrt{Np}} \sum_{\nu \in a} \left( \frac{\nu}{p} \right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu |d_k|^\varepsilon D/p^2} = \frac{D}{p} \sum_{\nu \in \mathcal{O}} \left( \frac{\nu}{p} \right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu D/p^2} \ll \frac{D}{p} \sum_{m=1}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} mD/p^2} \kappa(m),$$

where

$$\kappa(m) = \#\{ \nu \in a : N\nu \leq m+1 \} \ll m;$$

see, for example, Lang [11, Theorem 2, page 128] (and also notice that $\kappa(0) = 0$). Thus

$$\sum_{m=1}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} mD/p^2} \kappa(m) \ll \sum_{m=1}^{\infty} e^{-\frac{2\pi}{\sqrt{|d_k|}} mD/p^2} m = o(1).$$

From this we see by Lemma 5 that $B^i_1 \ll \sum_p a(p^2/x) \log p \ll x^{1/2}$.

Next we show that $B^s_1 \ll x$. To this end, let $\pi_1 \in \bar{p} - \overline{p}^2$; then $N(\pi_1) = bp$ for some positive integer $b$. Also set $\nu = \mu \pi_1/\sqrt{d_k}$. We then see that

$$\frac{D}{\sqrt{Np}} \sum_{\nu \in a} \left( \frac{\nu}{p} \right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu |d_k|^\varepsilon D/p^2} \ll \frac{D}{\sqrt{p}} \sum_{\mu \in \overline{\mathcal{O}}/(\pi_1)} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\mu bD/p} = o(1).$$

Thus $B^s_1 \ll \sum_p a(p/x) \log p \ll x$.

**Case 2.** Assume $D = o(x)$. First we consider $B^i_1$ and claim that $B^i_1 \ll x$.

Since $D = o(x)$,

$$B^i_1 \ll \sum_p a \left( \frac{p^2}{x} \right) (\log p) \frac{D}{p} \sum_{\nu \in \mathcal{O}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu D/p^2} \ll \sum_p a \left( \frac{p^2}{x} \right) p \log p \ll x,$$

by Lemma 5.

Now consider $B^s_1$. From the above we know that

$$B^s_1 = - \sum_{p} a \left( \frac{p}{x} \right) (\log p) \frac{D}{\sqrt{p}} \sum_{\nu \in \mathcal{B}/\sqrt{d_k}} \left( \frac{\nu}{p} \right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\nu |d_k| D/p^2}.$$

We decompose $B^s_1$ in the following way. By Lemma 7, $C/C^2 \simeq I(4)/I^2(4)H$ has order $2^{e+1}$, where $e$ is the 2-rank of Cl. Hence

$$I(4) = \bigcup_{i=1}^{2^{e+1}} a_i I^2(4)H.$$
for some ideals \(a_i\) and where \(a_1 = (1)\). Now we write

\[
B_1^s = \sum_{i=1}^{2s+1} B(i),
\]

where

\[
B(i) = - \sum_{p \in a_i I^2(4) H} a\left(\frac{p}{x}\right) \left(\log p\right) \frac{D}{\sqrt{p}} \sum_{\nu \in \mathbb{P}/\sqrt{d_k}} \left(\frac{\nu}{p}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N \nu |d_k| D/p^2}. \tag{4}
\]

Now, consider \(B(1)\). To simplify the notation slightly, we denote by \(P\) the set of prime ideals of absolute degree 1 contained in \(I^2(4) H\). Let \(p \in P\), say \(p b_p^2 = (\gamma_p)\) for some \((\gamma_p) \in H\). Hence, \(\mathbb{P} b_p^2 = (\overline{\gamma}_p)\). Notice that then \(p N b_p^2 = N \gamma_p\). We change the summation variable by letting \(\nu = \mu \overline{\gamma}_p / \sqrt{d_k}\) and thus obtain

\[
B(1) = - \sum_{p \in P} a\left(\frac{p}{x}\right) \left(\log p\right) \frac{D}{\sqrt{p}} \left(\frac{\gamma_p / \sqrt{d_k}}{p}\right) \sum_{\mu \in b_p^{-2}} \left(\frac{\mu}{p}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} (N \mu) N b_p^2 D/p}. \tag{1}
\]

Now we break up \(B(1)\) further. To this end, notice that \(Cl/Cl^2 \simeq Cl^{(2)}\) where \(Cl^{(2)} = \{c \in Cl : c^2 = 1\}\). Then we write

\[
B(1) = B(\square) + B(\square),
\]

where

\[
B(\square) = - \sum_{p \in P} a\left(\frac{p}{x}\right) \left(\log p\right) \frac{D}{\sqrt{p}} \left(\frac{\gamma_p / \sqrt{d_k}}{p}\right) \sum_{\mu \in b_p^{-2}} \left(\frac{\mu}{p}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} (N \mu) N b_p^2 D/p}.
\]

Finally, we split \(B(\square)\) as

\[
B(\square) = B'(\square) + B''(\square),
\]

with

\[
B'(\square) = - w_{k} \sum_{p \in P} a\left(\frac{p}{x}\right) \left(\log p\right) \frac{D}{\sqrt{p}} \sum_{c \in Cl^{(2)}} \sum_{\mu \in b_c^{-2}} \left(\frac{\mu}{p}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} (N \mu) N b_c^2 D/p}.
\]

Now, we evaluate \(B'(\square)\) asymptotically. For each \(c \in Cl^{(2)}\) let \(b_c \in c\) be such that if \(a \in c\), then \(a b_c = (\gamma)\) for some \(\gamma \in b_c\). Notice that then there is a bijection between the set \(\{a \in c : b_c^{-1} | a\}\) and the set (of equivalence classes) \((b_c b_c^{-1} - \{0\})/W_k\) given by

\[
a \mapsto (\gamma) = a b_c.
\]
In addition, notice that \( N\gamma = N\alpha Nb_c \). From this and Lemma 2, it follows that

\[
B'(\square) = -\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p) \frac{D}{\sqrt{p}} \sum_{\mu \in \mathcal{C}} \sum_{\gamma \in b_{\mathfrak{p}}^{-1}} N_{\mathfrak{b}} N_{\mathfrak{b}} \sum_{\gamma \neq 0} e^{-\frac{2\pi}{|d_{\mathfrak{k}}|}(D/p)N\gamma^2N\mathfrak{b}_\mathfrak{p}^2/N\mathfrak{b}_\mathfrak{c}^2}.
\]

\[
= -\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p) \frac{D}{\sqrt{p}} \sum_{\mu \in \mathcal{C}} \left(\frac{\sqrt{p}}{\sqrt{D}} I - 1 + O((p/D)^{-M})\right)
\]

\[
= -|\mathcal{C}| \sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p) \frac{D}{\sqrt{p}} \left(\frac{\sqrt{p}}{\sqrt{D}} I - 1 + O((p/D)^{-M})\right)
\]

\[
- 2e I\sqrt{D} \sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) \log p + 2e D \sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}}
\]

\[
+ O\left(\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p)(p/D)^{-M'}\right).
\]

Now, by Lemmas 5 and 7,

\[
\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) \log p = \frac{1}{2e+1} K(1)x + O(x^{1/2}\log^2 x),
\]

\[
\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} = \frac{1}{2e+1} K\left(\frac{1}{2}\right) x^{1/2} + O(\log^2 x),
\]

and moreover,

\[
\sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p)(p/D)^{M'} \ll 1.
\]

Therefore,

\[
B'(\square) = -\frac{1}{2} IK(1)x D^{1/2} + \frac{1}{2} K\left(\frac{1}{2}\right) x^{1/2} D + O(D^{1/2} x^{1/2}\log^2 x).
\]

On the other hand, notice that

\[
B''(\square) \ll \sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) (\log p) \frac{D}{\sqrt{p}} \sum_{\mu \in b_{\mathfrak{p}}^{-2}} e^{-\frac{2\pi}{|d_{\mathfrak{k}}|}(N\muNb_c^2)D/p}
\]

\[
\ll \sum_{p \in \mathcal{P}} a\left(\frac{p}{x}\right) \frac{\log p}{\sqrt{p}} \ll x^{1/2},
\]

by Lemmas 2 and 5.

Now consider \( B(\mathfrak{p}) \). For this expression we partition the sums differently. Let \( c \in I^2(4)H/P(4) \); pick an integral ideal \( a_c \in c \). Then if \( \mathfrak{p} \in c \) is a prime ideal of absolute degree 1, we have \( \mathfrak{p} \in c^{-1} \), so \( \mathfrak{p}a_c = (\alpha_p) \) for \( (\alpha_p) \in P(4) \).
Therefore by changing summation variable \( \nu = \mu \alpha_\nu / \sqrt{d_k} \), we get

\[
B(\nu) = -D \sum_{c \in I^2(4)H/P(4)} \sum_{p \in c} a\left(\frac{p}{x}\right) \log p \left(\frac{\alpha_\nu / \sqrt{d_k}}{p}\right) \times \sum_{\mu \in \alpha_\nu^{-1}} \left(\frac{\mu}{p}\right) e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha_\nu D p N\mu}.
\]

But since, by Lemma 8, \( (\alpha_\nu / \sqrt{d_k}) = 1 \), and since \( a(p/x) \neq 0 \) implies \( p/x < B \) (see the Preliminaries), we obtain

\[
B(\nu) \ll D \sum_{(\mu) \neq a^2} \sum_{c \in I^2(4)H/P(4)} \left| \sum_{p \in c} a\left(\frac{p}{x}\right) \log p \left(\frac{\mu}{p}\right) \right|.
\]

Now, Lemma 9 implies that

\[
\sum_{p \in c} \left(\frac{\mu}{p}\right) \log Np \ll x^{1/2} \log^2 (xN\mu).
\]

Hence the argument (using Riemann–Stieltjes integration) applied to \( B_{13} \) in [14] implies

\[
\sum_{p \in c} a\left(\frac{p}{x}\right) \log p \left(\frac{\mu}{p}\right) \ll \log^2 (xN\mu).
\]

Next notice that \( \mu \in \bigcup_c \alpha_c^{-1} \) so \( \mu \in b^{-1} \) for some integral ideal \( b \), e.g. \( b = \prod_c \alpha_c \). Hence

\[
B(\nu) \ll D \sum_{\mu \in b^{-1}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha_\nu D p N\mu} \log^2 (xN\mu) \ll x \log^2 \left(\frac{x}{D}\right).
\]

Next, consider \( \sum_{i=2}^{2e+1} B(i) \). We partition the sum over \( p \) analogously to the previous case. If \( c \in C(4) = I(4)/P(4) \), let \( \alpha_c \in c \), so if \( p \in c \), then \( \overline{p} \alpha_c = (\alpha_p) \) for some \( (\alpha_p) \in P(4) \). Then as above

\[
\sum_{i=1}^{2e+1} B(i) \ll D \sum_{\mu \in b^{-1}} e^{-\frac{2\pi}{\sqrt{|d_k|}} N\alpha_\nu D p N\mu} \left| \sum_{p \in c} a\left(\frac{p}{x}\right) \log p \left(\frac{\mu}{p}\right) \right| \ll x \log^2 \left(\frac{x}{D}\right),
\]

arguing as above using Lemma 9 when \( \mu \) is not a square in \( k \), since then \( \{(\mu/p) : p \in c\} = \{-1\} \). On the other hand, the sum over square \( \mu \) contributes \( \ll D^{1/2} x^{1/2} \log^2 (x/D) \ll x \log^2 (x/D) \), again.
Now suppose \( x = 1 \). Then by the explicit formula for primitive characters (just above Proposition 1) and Lemma 4 we obtain

\[
\sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|\Delta_k|}} N\alpha / D} \sum_{\varrho(\alpha)} K(\varrho) = \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|\Delta_k|}} N\alpha / D} (a(1) \log Nf_\alpha + O(1)) = a(1)D \log D + O(D).
\]

This completes the proof of the main theorem. \( \blacksquare \)

Now we come to the first main corollary, which is a special case of the theorem, but will be useful in studying the distribution of the nontrivial zeros of the quadratic \( L \)-series. Define, for \( y \in \mathbb{R} \),

\[
F_K(y, D) = \left( \frac{1}{2} K\left( \frac{1}{2} \right) D \right)^{-1} \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|\Delta_k|}} N\alpha / D} \sum_{\varrho(\alpha)} K(\varrho) D^{iy\gamma},
\]

where \( \varrho = 1/2 + i\gamma \). Then we have

\textbf{Corollary 1.} Assuming GRH for all abelian \( L \)-functions over \( k \), as \( D \to \infty \),

\[
F_K(y, D) = \begin{cases} 
-1 + \left( \frac{1}{2} K\left( \frac{1}{2} \right) \right)^{-1} D^{-y/2} a(D^{-y}) \log D + o(1) & \text{if } |y| < 1, \\
0 + o(1) & \text{if } 1 < |y| < 2,
\end{cases}
\]

uniformly on compact subsets of \((-2, -1) \cup (-1, 1) \cup (1, 2)\).

\textbf{Proof.} First in the main theorem dividing both sides by \( \frac{1}{2} K\left( \frac{1}{2} \right) D x^{1/2} \), absorbing one term into the error, and combining two error terms we get

\[
\left( \frac{1}{2} K\left( \frac{1}{2} \right) D \right)^{-1} \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{\sqrt{|\Delta_k|}} N\alpha / D} \sum_{\varrho(\alpha)} K(\varrho) x^{\varrho - 1/2} = \begin{cases} 
-1 + \left( \frac{1}{2} K\left( \frac{1}{2} \right) \right)^{-1} x^{-1/2} a\left( \frac{1}{2} \right) (\log D) \left( 1 + O\left( \frac{1}{\log D} \right) \right) & \text{if } x = o(D), \\
+ O\left( \frac{x^{1/2}}{D^{1/2}} + x^{-1/6} \log D \log x \right) & \text{if } D = o(x). 
\end{cases}
\]

Now plugging in \( x = D^y \) yields, for \( y \geq 0 \),

\[
F_K(y, D) = \begin{cases} 
-1 + \left( \frac{1}{2} K\left( \frac{1}{2} \right) \right)^{-1} D^{-y/2} a(D^{-y}) (\log D) \left( 1 + O\left( \frac{1}{\log D} \right) \right) & \text{if } y < 1, \\
+ O(D^{(y-1)/2} + D^{-y/6} \log^2 D) & \text{if } y > 1.
\end{cases}
\]
From this the corollary follows for $y \geq 0$. (Take the compact set to be $[0, 1 - \varepsilon] \cup [1 + \varepsilon, 2 - \varepsilon].$) Finally, notice that $F_K(-y, D) = F_K(y, D)$ since the nontrivial zeros of our $L$-functions are symmetric about the real axis.

Finally, we come to the second main corollary, which concerns the distribution of the nontrivial zeros close to the real axis.

**Corollary 2.** Suppose $r(y)$ is an even continuous function with $r(y)$ and $yr(y)$ in $L^1(\mathbb{R})$, such that its Fourier transform, 

$$\hat{r}(y) = \int_{-\infty}^{\infty} r(u)e^{-2\pi i y u} du,$$

is also continuous and in $L^1(\mathbb{R})$, and has compact support in $(-2, 2) \setminus \{\pm 1\}$. Then under GRH for all abelian $L$-functions over $k$, an imaginary quadratic number field, as $D \to \infty$,

$$D^{-1} \sum_{\alpha \in \mathcal{O}, \alpha \neq 0} e^{-\frac{2\pi}{\sqrt{\Delta_k}} N\alpha / D} \left( \frac{1}{2} K \left( \frac{1}{2} \right) \right)^{-1} \sum_{\varrho(\alpha)} K(\varrho) r \left( \frac{\gamma \log D}{2\pi} \right)$$

$$= 2 \int_{-\infty}^{\infty} \left( 1 - \frac{\sin 2\pi y}{2\pi y} \right) r(y) dy + o(1),$$

where the implied constant depends only on the field $k$ and the kernel $K$.

**Proof.** By Corollary 1 and since $\hat{r}(y)$ has compact support in $(-2, 2) \setminus \{\pm 1\}$,

$$\int_{-\infty}^{\infty} F_K(y, D)\hat{r}(y) dy$$

$$= \int_{-\infty}^{\infty} \left( -\xi_{[-1,1]}(y) + \left( \frac{1}{2} K \left( \frac{1}{2} \right) \right)^{-1} D^{-y/2} a(D^{-y}) \log D \right) \hat{r}(y) dy + o(1),$$

where $\xi_{[-1,1]}$ is the characteristic function of $[-1, 1]$. But

$$\int_{-\infty}^{\infty} \xi_{[-1,1]}(y)\hat{r}(y) dy = \int_{-\infty}^{\infty} \hat{\xi}_{[-1,1]}(y) r(y) dy$$

$$= 2 \int_{-\infty}^{\infty} \frac{\sin 2\pi y}{2\pi y} r(y) dy.$$

On the other hand,

$$\int_{-\infty}^{\infty} D^{-y/2} a(D^{-y})\hat{r}(y) dy = \int_{-\infty}^{\infty} (D^{-y/2} a(D^{-y}))^\wedge r(y) dy.$$
Distribution of zeros of L-functions

But
\[(D^{-y/2} a(D^{-y}))^\wedge = \int_{-\infty}^{\infty} D^{-\beta/2} a(D^{-\beta}) e^{-2\pi i y \beta} \, d\beta.\]

By the change of variable \(t = D^{-\beta},\) this last integral equals
\[
\frac{1}{\log D} \int_0^\infty a(t) t^{1/2} + 2\pi i \frac{y}{\log D} \frac{dt}{t}.
\]

Notice that
\[t^{2\pi i \frac{y}{\log D} - 1} = \exp\left(2\pi i \frac{\log t}{\log D} y\right) - 1 = 2\pi i y \frac{\log t}{\log D} \exp\left(2\pi i \frac{\log t}{\log D} \theta_y\right)\]
for some \(\theta_y\) between \(0\) and \(y.\) Therefore
\[t^{2\pi i \frac{y}{\log D} - 1} \ll \frac{\log t}{\log D} y,\]
so that
\[\frac{1}{\log D} \int_0^\infty a(t) t^{1/2} (t^{2\pi i \frac{y}{\log D} - 1}) \frac{dt}{t} \ll \frac{y}{\log^2 D}\]
where the implied constant is independent of \(y\) and \(D.\) Therefore,
\[
\int_{-\infty}^{\infty} (D^{-y/2} a(D^{-y}))^\wedge r(y) \, dy
= \frac{1}{\log D} K\left(\frac{1}{2}\right) \int_{-\infty}^{\infty} r(y) \, dy + O\left(\frac{1}{\log^2 D} \int_{-\infty}^{\infty} y r(y) \, dy\right)
= \frac{1}{\log D} K\left(\frac{1}{2}\right) \int_{-\infty}^{\infty} r(y) \, dy + O\left(\frac{1}{\log^2 D}\right).
\]

Thus,
\[
\int_{-\infty}^{\infty} \left( -\xi_{[-1,1]}(y) + \left(\frac{1}{2} K\left(\frac{1}{2}\right)\right)^{-1} D^{-y/2} a(D^{-y}) \log D \right) \hat{r}(y) \, dy
= 2 \int_{-\infty}^{\infty} \left(1 - \frac{\sin 2\pi y}{2\pi y}\right) r(y) \, dy + O\left(\frac{1}{\log D}\right).
\]

Consequently,
\[
\int_{-\infty}^{\infty} F_K(y, D) \hat{r}(y) \, dy = 2 \int_{-\infty}^{\infty} \left(1 - \frac{\sin 2\pi y}{2\pi y}\right) r(y) \, dy + o(1).
\]
On the other hand,
\[
\int_{-\infty}^{\infty} F_K(y, D) \hat{r}(y) \, dy = D^{-1} \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{|d_K|} N\alpha/D} \left( \frac{1}{2} K \left( \frac{1}{2} \right) \right)^{-1} \sum_{\varrho(\alpha)} \int_{-\infty}^{\infty} e^{i\alpha \gamma \log D} \hat{r}(y) \, dy
\]

\[
= D^{-1} \sum_{\substack{\alpha \in \mathcal{O} \\ \alpha \neq 0}} e^{-\frac{2\pi}{|d_K|} N\alpha/D} \left( \frac{1}{2} K \left( \frac{1}{2} \right) \right)^{-1} \sum_{\varrho(\alpha)} K(\varrho) r \left( \frac{\gamma \log D}{2\pi} \right),
\]

since by Fourier duality
\[
\hat{r}(y) = r(-y) = r(y).
\]

This establishes the corollary. ■

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References


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