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Integers with a large friable component

by

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1. Introduction and statement. It is well known (see, e.g., [5, Chap. III.3]) that the logarithms of the prime factors of an integer normally have exponential growth. Therefore, it is expected that the product of the small prime factors of a typical integer remains small—a device which has been employed by Erdős in many different contexts and for which various effective versions appear in the literature. We return here to the problem of finding a quantitative estimate for the number of exceptional integers. Some similar results have been obtained concomitantly, through a more elementary approach, by Banks and Shparlinski [1].

Given an integer n and a real parameter $y \geqslant 1$, we define

$$n_y := \prod_{p^\nu \| n,\, p \leqslant y} p^\nu$$

to be the *y-friable component* of n and we put

$$\varTheta(x, y, z) := \sum_{\substack{n \leqslant x \\ n_y > z}} 1 \qquad (x \geqslant 1,\ y \geqslant 1,\ z \geqslant 1).$$

We also write, for complex s with positive real part,

$$\zeta(s, y) := \sum_{P(n) \leqslant y} 1/n^s = \prod_{p \leqslant y} (1 - p^{-s})^{-1},$$

where $P(n)$ denotes the largest prime factor of n with the convention that $P(1) := 1$. We let ϱ designate Dickman's function and we set

$$S(y, z) := \sum_{\substack{P(m) \leqslant y \\ m > z}} \frac{1}{m}.$$

It has been shown in [6, Corollary 2] that, writing $u := (\log x)/\log y$, we have

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$$\Theta(x,y,z)=\frac{xS(y,z)}{\zeta(1,y)}+O(x\varrho(u)2^{(1+\varepsilon)u}+x^{\varepsilon}) \tag{1·1}$$

uniformly for $x\geqslant 2$, $y\geqslant 2$, $z\geqslant 2$, and also that, denoting Euler's constant by γ, and writing

$$\tau(w):=\int_w^\infty \varrho(t)\,\mathrm{d}t \qquad (w\geqslant 0),$$

we have, with $v:=(\log z)/\log y$,

$$S(y,z)=\Big\{1+O\Big(\frac{\log(v+2)}{\log y}\Big)\Big\}\tau(v)\log y \tag{1·2}$$

for all $\varepsilon>0$ and uniformly for

$$y\geqslant 2,\qquad 1\leqslant z\leqslant \exp\exp\{(\log y)^{3/5-\varepsilon}\}. \tag{1·3}$$

Thus, for all $x\geqslant 2$ and y, z satisfying (1·3) we have

$$\Theta(x,y,z)=\Big\{\mathrm{e}^{-\gamma}+O\Big(\frac{\log(v+2)}{\log y}\Big)\Big\}x\tau(v)+O(x\varrho(u)2^{(1+\varepsilon)u}+x^{\varepsilon}). \tag{1·4}$$

Note that $\Theta(x,y,z)=0$ unless $z\leqslant x$ and that, when the latter holds, (1·3) is implied by Hildebrand's condition

$$x\geqslant 2,\qquad \exp\{(\log_2 x)^{5/3+\eta}\}\leqslant y\leqslant x \tag{H_η}$$

with $\eta=3\varepsilon$.

In particular, if condition (1·3) holds, $u\to\infty$, $y\to\infty$, and, say, $z\leqslant x^{1-\varepsilon}$, then

$$\Theta(x,y,z)\sim \mathrm{e}^{-\gamma}\tau(v)x. \tag{1·5}$$

Formula (1·1) has been derived in [6] as a by-product of a general result on the Kubilius model of probabilistic number theory. However, the estimates established in [6] easily yield an asymptotic formula that is valid also when u is bounded. Let ω denote Buchstab's function. We obtain the following result in which we put

$$\left.\begin{aligned}\sigma(u,v)&:=\int_v^\infty \varrho(t)\omega(u-t)\,\mathrm{d}t\\ \vartheta(u,v)&:=\varrho(u)+\sigma(u,v)\\ \kappa(u,v)&:=(1-\gamma)\varrho(u-1)+\gamma\varrho(v)\omega(u-v)\end{aligned}\right\}\qquad (u\geqslant 1,\ v\geqslant 0).$$

THEOREM 1.1. *Let $\varepsilon>0$. Under conditions* (1·3) *and $1\leqslant z\leqslant x/y$, we have uniformly*

$$\Theta(x,y,z)=x\Big\{\vartheta(u,v)-\frac{\kappa(u,v)}{\log y}+O\Big(\frac{\tau(v)}{\log y}+\frac{\varrho(v)\log(v+2)}{(\log y)^2}+\frac{1}{z}\Big)\Big\}. \tag{1·6}$$

In particular, if $(\log y)^2 \leqslant z \leqslant \min(x/y^{1+\varepsilon/\log(u+1)}, \mathrm{e}^{\sqrt{y}})$, *we have*

$$\Theta(x,y,z) = \left\{1 + O\left(\frac{\log(v+2)}{\log y}\right)\right\}\vartheta(u,v)x. \tag{1·7}$$

Remarks. (i) The condition $z \leqslant x/y$ is not restrictive since $\Theta(x,y,z) = \Psi(x,y) - \Psi(z,y)$ otherwise.

(ii) It will be clear from the proof that, under suitable assumptions, more precise estimates may be derived by the same method.

(iii) It follows from classical estimates on Dickman's function (see e.g. [5, Chap. III.5]) that

$$\tau(w) = \frac{\varrho(w)}{\log(w+1)}\left\{1 + O\left(\frac{1}{\log(w+1)}\right)\right\} \qquad (w \geqslant 1). \tag{1·8}$$

Since $1/2 \leqslant \omega(t) \leqslant 1$ for $t \geqslant 1$ and $\varrho(v+h) \gg \varrho(v)\{v\log(v+1)\}^{-h}$ for $0 \leqslant h \leqslant 1 \leqslant v$ (see e.g. [5, Chaps. III.5 and III.6]), this implies, for instance, that $\vartheta(u,v) \asymp \tau(v)$ whenever $u - 1 - v \gg 1/\log(v+1)$.

2. Proof of Theorem 1.1. We start with an improvement of (1·2) established by the same method. As usual, we define the derivatives of the Dickman function at integer points by right continuity. It is known (see e.g. [5, Cor. III.5.8.3]) that $\varrho^{(k)}(w) \sim (-1)^k(\log w)^k\varrho(w)$ as $w \to \infty$. We also denote by $\{a_j\}_{j=0}^{\infty}$ the Taylor coefficients of $s\zeta(s+1)/(s+1)$ at $s=0$.

Lemma 2.1. *Let* $\varepsilon > 0$ *and* $k \in \mathbb{N}$ *be given. Then, uniformly under condition* (1·3), *we have*

$$\begin{aligned} S(y,z) = \tau(v)\log y - \sum_{0\leqslant j\leqslant k} a_{j+1}\,\frac{\varrho^{(j)}(v)}{(\log y)^j} - \frac{\Psi(z,y)}{z} \\ + O\left(\frac{\varrho(v)\{\log(v+2)\}^{k+1}}{(\log y)^{k+1}}\right). \end{aligned} \tag{2·1}$$

In particular, we have

$$S(y,z) = \tau(v)\log y - \gamma\varrho(v) + O\left(\frac{\varrho(v)\log(v+2)}{\log y} + \frac{1}{z}\right). \tag{2·2}$$

Proof. We have

$$S(y,z) = (\log y)\int_v^\infty \frac{\Psi(y^w,y)}{y^w}\,\mathrm{d}w - \frac{\Psi(z,y)}{z}.$$

Inserting Saias' estimate for $\Psi(y^w,y)$ (see [4] or [5, Th. III.5.9]) in its range of validity and estimating the contribution of large w as in [6], we obtain

$$\int_v^\infty \frac{\Psi(y^w,y)}{y^w}\,\mathrm{d}w = \{1 + O(\mathrm{e}^{-(\log y)^{3/5-\varepsilon}})\}\int_{0-}^\infty \tau(v-t)\,\mathrm{d}\left(\frac{[y^t]}{y^t}\right).$$

Formula (2·1) follows by integrating by parts and inserting the generalized Taylor expansion for the Dickman function established in [2, Lemma 4.2]. We omit the details, which are very similar to those in [2]. Then we derive (2·2) by appealing to Hildebrand's formula [3]

$$\Psi(z,y) = z\varrho(v)\Big\{1+O\Big(\frac{\log(v+1)}{\log y}\Big)\Big\}+O(1) \qquad (z\geqslant 2,\ y\geqslant \mathrm{e}^{(\log_2 z)^{5/3+\varepsilon}}). \tag{2·3}$$ ∎

For $y\geqslant 2$, $t\geqslant 1$, we write $u_t := (\log t)/\log y$.

Lemma 2.2. *Under conditions* (1·3) *and* $1\leqslant z\leqslant x/y$, *we have uniformly*

$$\sum_{\substack{P(m)\leqslant y\\ m>z}} \frac{\omega(u-u_m)}{m} = \sigma(u,v)\log y-\kappa(u,v)+\varrho(u-1) + O\Big(\tau(v)+\frac{\varrho(v)\log(v+2)}{\log y}+\frac1z\Big). \tag{2·4}$$

Proof. Since $\omega(s)$ is continuous for $s\geqslant 1$ and differentiable for $s>1$, we have

$$\begin{aligned}\sum_{\substack{P(m)\leqslant y\\ m>z}} \frac{\omega(u-u_m)}{m} &= \sum_{\substack{P(m)\leqslant y\\ z<m\leqslant x/y}} \frac1m\Big\{1+\int_{u_m}^{u-1}\omega'(u-t)\,\mathrm{d}t\Big\}\\ &= S(y,z)-S(y,x/y)+\int_v^{u-1}\omega'(u-t)\{S(y,z)-S(y,y^t)\}\,\mathrm{d}t\\ &= \omega(u-v)S(y,z)-S(y,x/y)-\int_v^{u-1}\omega'(u-t)S(y,y^t)\,\mathrm{d}t.\end{aligned}$$

The required formula then follows by inserting (2·2), and using the estimate (1·8). We omit the details which only involve standard partial integration and the fact that $\omega'\in L^1(\mathbb{R})$. ∎

We are now in a position to complete the proof of Theorem 1.1.

We first consider the case when (x,y) lies outside the region H_ε. Appealing to the bound $\omega(t)-\mathrm{e}^{-\gamma}\ll\varrho(t)$ $(t\geqslant 0)$ established, in a more precise form, in Lemma 4 of [6] and noting that, say, $u>3v$ provided y is large enough, we see that

$$\begin{aligned}\sigma(u,v) &= \mathrm{e}^{-\gamma}\tau(v)\{1+O(1/(\log y)^2)\},\\ \kappa(u,v) &= \gamma\mathrm{e}^{-\gamma}\varrho(v)\{1+O(1/(\log y)^2)\}.\end{aligned}$$

Thus (1·6) is in this case an immediate consequence of (1·1) and (2·2).

When $(x,y)\in H_\varepsilon$, $z\leqslant x/y$, we apply the formula

$$\varTheta(x,y,z)=\sum_{\substack{P(m)\leqslant y\\ z<m\leqslant x/y}}\varPhi\Big(\frac{x}{m},y\Big)+\varPsi(x,y)-\varPsi(x/y,y),$$

where $\varPhi(t,y)$ denotes the number of positive integers not exceeding t and all of whose prime factors exceed y. The last two terms may be evaluated by (2·3). Note that $\varPsi(x/y,y)$ may be regarded as an error term since it is $\ll x\varrho(v)/y$. Thus, we may restrict our attention to evaluating the m-sum. To this end, we apply Corollary 3 of [6] in the form

$$\varPhi\Big(\frac{x}{m},y\Big)=\frac{\mathrm{e}^\gamma x\omega(u-u_m)}{m\zeta(1,y)}-\frac{\mathrm{e}^\gamma y}{\zeta(1,y)}+O\Big(\frac{x\varrho(u-u_m)}{m(\log y)^2}\Big)\qquad(z<m\leqslant x/y).$$

By formula (1·2), the contribution of the remainder term of the left-hand side is dominated by that of (1·6) and that of the second term equals $-x\varrho(u-1)/\log y$ to within an acceptable error. Since the remaining sum depends on (2·4), this completes the proof of (1·6). Formula (1·7) then follows from the bound $\tau(v)\gg v^{-2v}$.

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