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Mixed sums of squares and triangular numbers

by

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1. Introduction. A classical result of Fermat asserts that any prime $p \equiv 1 \pmod 4$ is a sum of two squares of integers. Fermat also conjectured that each $n \in \mathbb{N}$ can be written as a sum of three triangular numbers, where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ of natural numbers, and triangular numbers are those integers $t_x = x(x+1)/2$ with $x \in \mathbb{Z}$. An equivalent version of this conjecture states that $8n+3$ is a sum of three squares (of odd integers). This follows from the following profound theorem (see, e.g., [G, pp. 38–49] or [N, pp. 17–23]).

Gauss–Legendre Theorem. *$n \in \mathbb{N}$ can be written as a sum of three squares of integers if and only if n is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$.*

Building on some work of Euler, in 1772 Lagrange showed that every natural number is a sum of four squares of integers.

For problems and results on representations of natural numbers by various quadratic forms with coefficients in $\mathbb{N}$, the reader may consult [Du] and [G].

Motivated by Ramanujan's work [Ra], L. Panaitopol [P] proved the following interesting result in 2005.

Theorem A. *Let a, b, c be positive integers with $a \le b \le c$. Then every odd natural number can be written in the form $ax^2+by^2+cz^2$ with $x, y, z \in \mathbb{Z}$ if and only if the vector (a, b, c) is $(1, 1, 2)$ or $(1, 2, 3)$ or $(1, 2, 4)$.*

According to L. E. Dickson [D2, p. 260], Euler already noted that any odd integer $n > 0$ is representable by $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$.

In 1862 J. Liouville (cf. [D2, p. 23]) proved the following result.

Theorem B. *Let a, b, c be positive integers with $a \le b \le c$. Then every $n \in \mathbb{N}$ can be written as $at_x + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$ if and only if (a, b, c)*

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is among the following vectors:

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

Now we turn to representations of natural numbers by mixed sums of squares (of integers) and triangular numbers.

Let $n \in \mathbb{N}$. By the Gauss–Legendre theorem, $8n+1$ is a sum of three squares. It follows that $8n+1 = (2x)^2+(2y)^2+(2z+1)^2$ for some $x,y,z \in \mathbb{Z}$ with $x \equiv y \pmod 2$; this yields the representation

$$n = \frac{x^2+y^2}{2} + t_z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + t_z$$

as observed by Euler. According to Dickson [D2, p. 24], E. Lionnet stated, and V. A. Lebesgue [L] and M. S. Réalis [Re] proved that n can also be written in the form $x^2 + t_y + t_z$ with $x,y,z \in \mathbb{Z}$. Quite recently, this was reproved by H. M. Farkas [F] via the theory of theta functions.

Using the theory of ternary quadratic forms, in 1939 B. W. Jones and G. Pall [JP, Theorem 6] proved that for any $n \in \mathbb{N}$ we have $8n+1 = ax^2+by^2+cz^2$ for some $x,y,z \in \mathbb{Z}$ if the vector (a,b,c) belongs to the set

$$\{(1,1,16),(1,4,16),(1,16,16),(1,2,32),(1,8,32),(1,8,64)\}.$$

As $(2z+1)^2 = 8t_z+1$, the result of Jones and Pall implies that each $n \in \mathbb{N}$ can be written in any of the following three forms with $x,y,z \in \mathbb{Z}$:

$$2x^2+2y^2+t_z = (x+y)^2+(x-y)^2+t_z,\ x^2+4y^2+t_z,\ x^2+8y^2+t_z.$$

In this paper we establish the following result by means of q-series.

Theorem 1.

(i) *Any $n \in \mathbb{N}$ is a sum of an even square and two triangular numbers. Moreover, if $n/2$ is not a triangular number then*

$$(1)\qquad |\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} : x^2+t_y+t_z = n \text{ and } 2\nmid x\}| = |\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} : x^2+t_y+t_z = n \text{ and } 2\mid x\}|.$$

(ii) *If $n \in \mathbb{N}$ is not a triangular number, then*

$$(2)\qquad |\{(x,y,z) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{N} : x^2+y^2+t_z = n \text{ and } x \not\equiv y \pmod 2\}| = |\{(x,y,z) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{N} : x^2+y^2+t_z = n \text{ and } x \equiv y \pmod 2\}| > 0.$$

(iii) *A positive integer n is a sum of an odd square, an even square and a triangular number, unless it is a triangular number t_m $(m>0)$ for which all prime divisors of $2m+1$ are congruent to 1 mod 4 and hence $t_m = x^2+x^2+t_z$ for some integers $x>0$ and z with $x \equiv m/2 \pmod 2$.*

Remark. Note that $t_2 = 1^2+1^2+t_1$ but we cannot write $t_2 = 3$ as a sum of an odd square, an even square and a triangular number.

Here are two more theorems of this paper.

THEOREM 2. *Let a, b, c be positive integers with $a \le b$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + by^2 + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a, b, c) is among the following vectors*:

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

THEOREM 3. *Let a, b, c be positive integers with $b \ge c$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a, b, c) is among the following vectors*:

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

Theorem 1 and Theorems 2–3 will be proved in Sections 2 and 3 respectively. In Section 4, we will pose three conjectures and discuss the converses of Theorems 2 and 3.

2. Proof of Theorem 1. Given two integer-valued quadratic polynomials $f(x, y, z)$ and $g(x, y, z)$, by writing $f(x, y, z) \sim g(x, y, z)$ we mean that

$$\{f(x, y, z) : x, y, z \in \mathbb{Z}\} = \{g(x, y, z) : x, y, z \in \mathbb{Z}\}.$$

Clearly $\sim$ is an equivalence relation on the set of all integer-valued ternary quadratic polynomials.

The following lemma is a refinement of Euler's observation $t_y + t_z \sim y^2 + 2t_z$ (cf. [D2, p. 11]).

LEMMA 1. *For any $n \in \mathbb{N}$ we have*

$$|\{(y, z) \in \mathbb{N}^2 : t_y + t_z = n\}| = |\{(y, z) \in \mathbb{Z} \times \mathbb{N} : y^2 + 2t_z = n\}|. \tag{3}$$

Proof. Note that $t_{-y-1} = t_y$. Thus

$$\begin{aligned}
|\{(y, z) \in \mathbb{N}^2 : t_y + t_z = n\}| &= \frac{1}{4}\,|\{(y, z) \in \mathbb{Z}^2 : t_y + t_z = n\}| \\
&= \frac{1}{4}\,|\{(y, z) \in \mathbb{Z}^2 : 4n + 1 = (y + z + 1)^2 + (y - z)^2\}| \\
&= \frac{1}{4}\,|\{(x_1, x_2) \in \mathbb{Z}^2 : 4n + 1 = x_1^2 + x_2^2 \text{ and } x_1 \not\equiv x_2 \pmod 2\}| \\
&= \frac{2}{4}\,|\{(y, z) \in \mathbb{Z}^2 : 4n + 1 = (2y)^2 + (2z + 1)^2\}| \\
&= \frac{1}{2}\,|\{(y, z) \in \mathbb{Z}^2 : n = y^2 + 2t_z\}| = |\{(y, z) \in \mathbb{Z} \times \mathbb{N} : n = y^2 + 2t_z\}|. \quad \blacksquare
\end{aligned}$$

Lemma 1 is actually equivalent to the following observation of Ramanujan (cf. Entry 25(iv) of [B, p. 40]): $\psi(q)^2 = \varphi(q)\psi(q^2)$ for $|q| < 1$, where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{t_n}. \tag{4}$$

Let $n \in \mathbb{N}$ and define

$$r(n) = |\{(x, y, z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = n\}|, \tag{5}$$

$$r_0(n) = |\{(x, y, z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = n \text{ and } 2 \mid x\}|, \tag{6}$$

$$r_1(n) = |\{(x, y, z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = n \text{ and } 2 \nmid x\}|. \tag{7}$$

Clearly $r_0(n) + r_1(n) = r(n)$. In the following lemma we investigate the difference $r_0(n) - r_1(n)$.

LEMMA 2. *For $m = 0, 1, 2, \ldots$ we have*

$$r_0(2t_m) - r_1(2t_m) = (-1)^m(2m + 1). \tag{8}$$

Also, $r_0(n) = r_1(n)$ *if $n \in \mathbb{N}$ is not a triangular number times* 2.

Proof. Let $|q| < 1$. Recall the following three known identities implied by Jacobi's triple product identity (cf. [AAR, pp. 496–501]):

$$\varphi(-q) = \prod_{n=1}^{\infty}(1 - q^{2n-1})^2(1 - q^{2n}) \quad \text{(Gauss)},$$

$$\psi(q) = \prod_{n=1}^{\infty}\frac{1 - q^{2n}}{1 - q^{2n-1}} \quad \text{(Gauss)},$$

$$\prod_{n=1}^{\infty}(1 - q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n + 1)q^{t_n} \quad \text{(Jacobi)}.$$

Observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}(r_0(n) - r_1(n))q^n \\
&= \Big(\sum_{x=-\infty}^{\infty}(-1)^x q^{x^2}\Big)\Big(\sum_{y=0}^{\infty} q^{t_y}\Big)\Big(\sum_{z=0}^{\infty} q^{t_z}\Big) = \varphi(-q)\psi(q)^2 \\
&= \Big(\prod_{n=1}^{\infty}(1 - q^{2n-1})^2(1 - q^{2n})\Big)\Big(\prod_{n=1}^{\infty}\frac{1 - q^{2n}}{1 - q^{2n-1}}\Big)^2 \\
&= \prod_{n=1}^{\infty}(1 - q^{2n})^3 = \sum_{m=0}^{\infty}(-1)^m(2m + 1)(q^2)^{t_m}.
\end{aligned}$$

Comparing the coefficients of q^n on both sides, we obtain the desired result. ■

The following result was discovered by Hurwitz in 1907 (cf. [D2, p. 271]); an extension was established in [HS] via the theory of modular forms of half integer weight.

LEMMA 3. *Let $n > 0$ be an odd integer, and let $p_1, \ldots, p_r$ be all the distinct prime divisors of n congruent to* 3 *mod* 4. *Write $n = n_0 \prod_{0<i\le r} p_i^{\alpha_i}$, where $n_0, \alpha_1, \ldots, \alpha_r$ are positive integers and n_0 has no prime divisors congruent to* 3 *mod* 4. *Then*

$$|\{(x,y,z)\in\mathbb{Z}^3 : x^2+y^2+z^2=n^2\}| = 6n_0 \prod_{0<i\le r}\left(p_i^{\alpha_i} + 2\,\frac{p_i^{\alpha_i}-1}{p_i-1}\right). \tag{9}$$

Proof. We deduce (9) in a new way and use some standard notations in number theory.

By (4.8) and (4.10) of [G],

$$\begin{aligned}
&|\{(x,y,z)\in\mathbb{Z}^3 : x^2+y^2+z^2=n^2\}| \\
&\quad = \sum_{d\mid n}\frac{24}{\pi}\,d\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{-4d^2}{m}\right) \\
&\quad = \frac{24}{\pi}\sum_{d\mid n} d\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\sum_{c\mid \gcd(2k-1,d)}\mu(c) \\
&\quad = \frac{24}{\pi}\sum_{d\mid n} d\sum_{c\mid d}\mu(c)\sum_{k=1}^{\infty}\frac{(-1)^{((2k-1)c-1)/2}}{(2k-1)c} \\
&\quad = \frac{24}{\pi}\sum_{d\mid n} d\sum_{c\mid d}\frac{\mu(c)}{c}(-1)^{(c-1)/2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1} \\
&\quad = 6\sum_{c\mid n}(-1)^{(c-1)/2}\,\frac{\mu(c)}{c}\sum_{q\mid \frac{n}{c}} cq \\
&\quad = 6\sum_{c\mid n}(-1)^{(c-1)/2}\mu(c)\sigma\left(\frac{n}{c}\right)
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{1}{6}\,|\{(x,y,z)\in\mathbb{Z}^3 : x^2+y^2+z^2=n^2\}| \\
&\quad = \sum_{d_0\mid n_0}\ \sum_{d_1\mid p_1^{\alpha_1},\ldots,d_r\mid p_r^{\alpha_r}}\left(\frac{-1}{d_0d_1\cdots d_r}\right)\mu(d_0d_1\cdots d_r)\sigma\left(\frac{n_0}{d_0}\prod_{0<i\le r}\frac{p_i^{\alpha_i}}{d_i}\right) \\
&\quad = \sum_{d_0\mid n_0}\left(\frac{-1}{d_0}\right)\mu(d_0)\sigma\left(\frac{n_0}{d_0}\right)\prod_{0<i\le r}\sum_{d_i\mid p_i^{\alpha_i}}\left(\frac{-1}{d_i}\right)\mu(d_i)\sigma\left(\frac{p_i^{\alpha_i}}{d_i}\right)
\end{aligned}$$

$$= \sum_{d_0 \mid n_0} \mu(d_0)\sigma\left(\frac{n_0}{d_0}\right) \prod_{0<i\le r} \left(\sigma(p_i^{\alpha_i}) + \left(\frac{-1}{p_i}\right)\mu(p_i)\sigma(p_i^{\alpha_i-1})\right)$$

$$= n_0 \prod_{0<i\le r} (p_i^{\alpha_i} + 2\sigma(p_i^{\alpha_i-1})) = n_0 \prod_{0<i\le r} \left(p_i^{\alpha_i} + 2\,\frac{p_i^{\alpha_i}-1}{p_i-1}\right). \ \blacksquare$$

Proof of Theorem 1. (i) By the Gauss–Legendre theorem, $4n+1$ is a sum of three squares and hence $4n+1 = (2x)^2 + (2y)^2 + (2z+1)^2$ (i.e., $n = x^2 + y^2 + 2t_z$) for some $x, y, z \in \mathbb{Z}$. Combining this with Lemma 1 we obtain a simple proof of the known result

$$r(n) = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = n\}| > 0.$$

Recall that $r_0(n) + r_1(n) = r(n)$. If $n/2$ is not a triangular number, then $r_0(n) = r_1(n) = r(n)/2 > 0$ by Lemma 2. If $n = 2t_m$ for some $m \in \mathbb{N}$, then we also have $r_0(n) > 0$ since $n = 0^2 + t_m + t_m$.

(ii) Note that

$$n = x^2 + y^2 + t_z \ \Leftrightarrow\ 2n = 2(x^2+y^2) + 2t_z = (x+y)^2 + (x-y)^2 + 2t_z.$$

From this and Lemma 1, we get

$$\begin{aligned}
&|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + t_z = n\}| \\
&\qquad = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + 2t_z = 2n\}| \\
&\qquad = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = 2n\}| \\
&\qquad = r(2n) > 0;
\end{aligned}$$

in the language of generating functions, it says that

$$\varphi(q)\psi(q)^2 + \varphi(-q)\psi(-q)^2 = 2\varphi(q^2)^2\psi(q^2).$$

Similarly,

$$\begin{aligned}
&|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + t_z = n \text{ and } x \equiv y \pmod 2\}| \\
&\quad = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + 2t_z = 2n \text{ and } 2 \mid x\}| \\
&\quad = |\{(x,y,z) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} : x^2 + t_y + t_z = 2n \text{ and } 2 \mid x\}| = r_0(2n) > 0
\end{aligned}$$

and

$$(10) \quad |\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + t_z = n \text{ and } x \not\equiv y \pmod 2\}| = r_1(2n).$$

If n is not a triangular number, then $r_0(2n) = r_1(2n) = r(2n)/2 > 0$ by Lemma 2, and hence (2) follows from the above.

(iii) By Theorem 1(ii), if n is not a triangular number then $n = x^2+y^2+t_z$ for some $x, y, z \in \mathbb{Z}$ with $2 \mid x$ and $2 \nmid y$.

Now assume that $n = t_m$ $(m > 0)$ is not a sum of an odd square, an even square and a triangular number. Then $r_1(2t_m) = 0$ by (10). In view of (ii) and (8),

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{N} : x^2 + y^2 + t_z = t_m\}|\\ &\qquad = r_0(2t_m) + r_1(2t_m) = r_0(2t_m) - r_1(2t_m) = (-1)^m(2m+1).\end{aligned}$$

Therefore

$$\begin{aligned}(-1)^m(2m+1) &= \frac{1}{2}\,|\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + t_z = t_m\}|\\ &= \frac{1}{2}\,|\{(x,y,z) \in \mathbb{Z}^3 : 2(2x)^2 + 2(2y)^2 + (2z+1)^2 = 8t_m + 1\}|\\ &= \frac{1}{2}\,|\{(x,y,z) \in \mathbb{Z}^3 : (2x+2y)^2 + (2x-2y)^2 + (2z+1)^2 = 8t_m + 1\}|\\ &= \frac{1}{2}\,|\{(x_1,y_1,z) \in \mathbb{Z}^3 : 4(x_1^2 + y_1^2) + (2z+1)^2 = (2m+1)^2\}|\\ &= \frac{1}{6}\,|\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = (2m+1)^2\}|.\end{aligned}$$

Since

$$(-1)^m 6(2m+1) = |\{(x,y,z) \in \mathbb{Z}^3 : x^2 + y^2 + z^2 = (2m+1)^2\}| \not> 6(2m+1),$$

by Lemma 3 the odd number $2m+1$ cannot have a prime divisor congruent to 3 mod 4. So all the prime divisors of $2m+1$ are congruent to 1 mod 4, and hence

$$|\{(x,y) \in \mathbb{N}^2 : x > 0 \text{ and } x^2 + y^2 = 2m+1\}| = \sum_{d \mid 2m+1} 1 > 1$$

by Proposition 17.6.1 of [IR, p. 279]. Thus $2m+1$ is a sum of two squares of positive integers. Choose positive integers x and y such that $x^2+y^2 = 2m+1$ with $2 \mid x$ and $2 \nmid y$. Then

$$8t_m + 1 = (2m+1)^2 = (x^2 - y^2)^2 + 4x^2y^2 = 8t_{(x^2-y^2-1)/2} + 1 + 16\left(\frac{x}{2}\right)^2 y^2$$

and hence

$$t_m = t_{(x^2-y^2-1)/2} + \left(\frac{x}{2}\,y\right)^2 + \left(\frac{x}{2}\,y\right)^2.$$

As $x^2 = 2m + 1 - y^2 \equiv 2m \pmod 8$, m is even and

$$\frac{m}{2} \equiv \left(\frac{x}{2}\right)^2 \equiv \frac{x}{2} \equiv \frac{x}{2}\,y \pmod 2.$$

We are done. ■

3. Proofs of Theorems 2 and 3

Proof of Theorem 2. We distinguish four cases.

CASE 1: $a = c = 1$. Write $8 = x_0^2 + by_0^2 + t_{z_0}$ with $x_0, y_0, z_0 \in \mathbb{Z}$. Then $y_0 \neq 0$ and hence $8 \geq b$. Since $x^2 + 5y^2 + t_z \neq 13$, $x^2 + 6y^2 + t_z \neq 47$ and $x^2 + 7y^2 + t_z \neq 20$, we must have $b \in \{1, 2, 3, 4, 8\}$.

CASE 2: $a = 1$ and $c = 2$. Write $5 = x_0^2 + by_0^2 + 2t_{z_0}$ with $x_0, y_0, z_0 \in \mathbb{Z}$. Then $y_0 \neq 0$ and hence $5 \geq b$. Observe that $x^2 + 3y^2 + 2t_z \neq 8$ and $x^2 + 5y^2 + 2t_z \neq 19$. Therefore $b \in \{1, 2, 4\}$.

CASE 3: $a = 1$ and $c \geq 3$. Since $2 = x^2 + by^2 + ct_z$ for some $x, y, z \in \mathbb{Z}$, we must have $b \leq 2$. If $b = 1$, then there are $x_0, y_0, z_0 \in \mathbb{Z}$ such that $3 = x_0^2 + y_0^2 + ct_{z_0} \geq c$ and hence $c = 3$. But $x^2 + y^2 + 3t_z \neq 6$, therefore $b = 2$. For some $x, y, z \in \mathbb{Z}$ we have $5 = x^2 + 2y^2 + ct_z \geq c$. Since $x^2 + 2y^2 + 3t_z \neq 23$ and $x^2 + 2y^2 + 5t_z \neq 10$, c must be 4.

CASE 4: $a > 1$. As $b \geq a \geq 2$ and $ax^2 + by^2 + ct_z = 1$ for some $x, y, z \in \mathbb{Z}$, we must have $c = 1$. If $a > 2$, then $ax^2 + by^2 + t_z \neq 2$. Thus $a = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $4 = 2x_0^2 + by_0^2 + t_{z_0} \geq b$. Note that $2x^2 + 3y^2 + t_z \neq 7$ and $2x^2 + 4y^2 + t_z \neq 20$. Therefore $b = 2$.

In view of the above, Theorem 2 has been proven. ∎

Proof of Theorem 3. Let us first consider the case $c > 1$. Since $1 = ax^2 + bt_y + ct_z$ for some $x, y, z \in \mathbb{Z}$, we must have $a = 1$. Clearly $x^2 + bt_y + ct_z \neq 2$ if $c \geq 3$. So $c = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $5 = x_0^2 + bt_{y_0} + 2t_{z_0} \geq b$. It is easy to check that $x^2 + 3t_y + 2t_z \neq 8$. Therefore $b \in \{2, 4, 5\}$.

Below we assume that $c = 1$. If a and b are both greater than 2, then $ax^2 + bt_y + t_z \neq 2$. So $a \leq 2$ or $b \leq 2$.

CASE 1: $a = 1$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $8 = x_0^2 + bt_{y_0} + t_{z_0} \geq b$. Note that $x^2 + 5t_y + t_z \neq 13$ and $x^2 + 7t_y + t_z \neq 20$. So $b \in \{1, 2, 3, 4, 6, 8\}$.

CASE 2: $a = 2$. For some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $4 = 2x_0^2 + bt_{y_0} + t_{z_0} \geq b$. Thus $b \in \{1, 2, 4\}$ since $2x^2 + 3t_y + t_z \neq 7$.

CASE 3: $a > 2$. In this case $b \leq 2$. If $b = 1$, then for some $x_0, y_0, z_0 \in \mathbb{Z}$ we have $5 = ax_0^2 + t_{y_0} + t_{z_0} \geq a$, and hence $a = 4$ since $3x^2 + t_y + t_z \neq 8$ and $5x^2 + t_y + t_z \neq 19$. If $b = 2$, then for some $x, y, z \in \mathbb{Z}$ we have $4 = ax^2 + 2t_y + t_z \geq a$ and so $a \in \{3, 4\}$.

The proof of Theorem 3 is now complete. ∎

4. Some conjectures and related discussion. In this section we raise three related conjectures.

CONJECTURE 1. *Any positive integer n is a sum of a square, an odd square and a triangular number. In other words, each natural number can be written in the form $x^2 + 8t_y + t_z$ with $x, y, z \in \mathbb{Z}$.*

We have verified Conjecture 1 for $n \le 15\,000$. By Theorem 1(iii), Conjecture 1 is valid when $n \ne t_4, t_8, t_{12}, \ldots$.

CONJECTURE 2. *Each $n \in \mathbb{N}$ can be written in any of the following forms with $x, y, z \in \mathbb{Z}$:*

$$x^2 + 3y^2 + t_z,\ x^2 + 3t_y + t_z,\ x^2 + 6t_y + t_z,\ 3x^2 + 2t_y + t_z,\ 4x^2 + 2t_y + t_z.$$

CONJECTURE 3. *Every $n \in \mathbb{N}$ can be written in the form $x^2 + 2y^2 + 3t_z$ (with $x, y, z \in \mathbb{Z}$) except $n = 23$, in the form $x^2 + 5y^2 + 2t_z$ (or the equivalent form $5x^2 + t_y + t_z$) except $n = 19$, in the form $x^2 + 6y^2 + t_z$ except $n = 47$, and in the form $2x^2 + 4y^2 + t_z$ except $n = 20$.*

Both Conjectures 2 and 3 have been verified for $n \le 10\,000$.

The second statement in Conjecture 3 is related to an assertion of Ramanujan confirmed by Dickson [D1] which states that even natural numbers not of the form $4^k(16l + 6)$ (with $k, l \in \mathbb{N}$) can be written as $x^2 + y^2 + 10z^2$ with $x, y, z \in \mathbb{Z}$. Observe that

$$\begin{aligned}
&n = x^2 + 5y^2 + 2t_z \text{ for some } x, y, z \in \mathbb{Z}\\
&\qquad \Leftrightarrow 4n + 1 = x^2 + 5y^2 + z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid z\\
&\qquad \Leftrightarrow 8n + 2 = 2(x^2 + y^2) + 10z^2 = (x + y)^2 + (x - y)^2 + 10z^2\\
&\qquad\qquad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid y\\
&\qquad \Leftrightarrow 8n + 2 = x^2 + y^2 + 10z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } x \not\equiv y \pmod 4.
\end{aligned}$$

Below we reduce the converses of Theorems 2 and 3 to Conjectures 1 and 2. For convenience, we call a ternary quadratic polynomial $f(x, y, z)$ *essential* if $\{f(x, y, z) : x, y, z \in \mathbb{Z}\} = \mathbb{N}$. (Actually, in 1748 Goldbach (cf. [D2, p. 11]) already stated that $x^2 + y^2 + 2t_z$, $x^2 + 2y^2 + t_z$, $x^2 + 2y^2 + 2t_z$ and $2x^2 + 2t_y + t_z$ are essential.)

STEP I. We show that the 10 quadratic polynomials listed in Theorem 2 are essential except for the form $x^2 + 3y^2 + t_z$ appearing in Conjecture 2.

As $4x^2 + y^2 + 2t_z \sim 4x^2 + t_y + t_z$, the form $x^2 + (2y)^2 + 2t_z$ is essential by Theorem 1(i). Both $x^2 + (2y)^2 + t_z$ and $2x^2 + 2y^2 + t_z = (x+y)^2 + (x-y)^2 + t_z$ are essential by Theorem 1(ii) and the trivial fact $t_z = 0^2 + 0^2 + t_z$. We have pointed out in Section 1 that $x^2 + 2(2y)^2 + t_z$ is essential by [JP, Theorem 6], and we do not have an easy proof of this deep result.

Since

$$x^2 + 2y^2 + 4t_z \sim x^2 + 2(t_y + t_z) \sim t_x + t_y + 2t_z,$$

the form $x^2 + 2y^2 + 4t_z$ is essential by Theorem B (of Liouville). By the Gauss–Legendre theorem, for each $n \in \mathbb{N}$ we can write $8n + 2 = (4x)^2 + (2y + 1)^2 + (2z + 1)^2$ (i.e., $n = 2x^2 + t_y + t_z$) with $x, y, z \in \mathbb{Z}$. Thus the form $x^2 + 2y^2 + 2t_z$ is essential since $2x^2 + y^2 + 2t_z \sim 2x^2 + t_y + t_z$.

Step II. We analyze the 15 quadratic polynomials listed in Theorem 3. By Theorem 1(i), $(2x)^2 + t_y + t_z$ and $x^2 + t_y + t_z$ are essential. Since

$$\begin{aligned} x^2 + 2t_y + t_z &\sim t_x + t_y + t_z, \\ x^2 + 2t_y + 2t_z &\sim t_x + t_y + 2t_z, \\ x^2 + 4t_y + 2t_z &\sim t_x + 4t_y + t_z, \\ x^2 + 5t_y + 2t_z &\sim t_x + 5t_y + t_z, \\ 2x^2 + 4t_y + t_z &\sim 2t_x + 2t_y + t_z, \end{aligned}$$

the forms

$$x^2 + 2t_y + t_z,\ x^2 + 2t_y + 2t_z,\ x^2 + 4t_y + 2t_z,\ x^2 + 5t_y + 2t_z,\ 2x^2 + 4t_y + t_z$$

are all essential by Liouville's theorem. For $n \in \mathbb{N}$ we can write $2n = x^2 + 4t_y + 2t_z$ with $x, y, z \in \mathbb{Z}$, and hence $n = 2x_0^2 + 2t_y + t_z$ with $x_0 = x/2 \in \mathbb{Z}$. So the form $2x^2 + 2t_y + t_z$ is also essential.

Recall that $2x^2 + t_y + t_z$ and $2x^2 + y^2 + 2t_z$ are essential by the last two sentences of Step I. For each $n \in \mathbb{N}$ we can choose $x, y, z \in \mathbb{Z}$ such that $2n + 1 = 2x^2 + (2y + 1)^2 + 2t_z$ and hence $n = x^2 + 4t_y + t_z$. So the form $x^2 + 4t_y + t_z$ is essential.

The remaining forms listed in Theorem 3 are $x^2 + 8t_y + t_z$ and four other forms, which appear in Conjectures 1 and 2 respectively. We are done.

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Added in proof (February 2007)**.** The second conjecture in Section 4 has been confirmed by Song Guo, Hao Pan and the author.

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