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Asymptotic formula for sum-free sets in abelian groups

by

R. Balasubramanian (Chennai) and Gyan Prakash (Allahabad)

Let G be a finite abelian group of order n. A subset A of G is said to be *sum-free* if there is no solution of the equation $x+y=z$ with $x,y,z\in A$. Let $\mathrm{SF}(G)$ denote the set of all sum-free subsets of G, and $\sigma(G)$ denote the number $n^{-1}\log_2|\mathrm{SF}(G)|$. In this article we improve the error term in the asymptotic formula for $\sigma(G)$ which was recently obtained by Ben Green and Imre Ruzsa [GR05].

Definition 1.

(I) Let $\mu(G)$ denote the density of a largest sum-free subset of G, so that any such subset has size $\mu(G)n$.

(II) Given a set $B\subset G$ we say that $(x,y,z)\in B^3$ is a *Schur triple* in B if $x+y=z$.

Observing that all subsets of a sum-free set are sum-free we have the obvious inequality

$$|\mathrm{SF}(G)|\ge 2^{\mu(G)n}. \tag{1}$$

From (1) it follows trivially that $\sigma(G)\ge\mu(G)$.

In this article we improve the results of Ben Green and Imre Ruzsa [GR05] and prove Theorems 2 and 3 below. Theorem 2 follows immediately from Theorem 3 and [GR05, Proposition 2.1′]. The methods used to prove Theorem 3 are a slight refinement of the methods in [GR05].

Theorem 2. *When G is a finite abelian group of order n, then*

$$\sigma(G)=\mu(G)+O\bigg(\frac{1}{(\ln n)^{1/27}}\bigg).$$

Theorem 3. *There exists an absolute positive constant δ_0 such that if $F\subset G$ has at most δn^2 Schur triples, where $\delta\le\delta_0$, then*

$$|F|\le(\mu(G)+c\delta^{1/3})n, \tag{2}$$

where c is an absolute positive constant.

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Earlier Ben Green and Ruzsa [GR05] proved the following:

THEOREM 4 ([GR05, Theorem 1.8]). *Let G be a finite abelian group of order n. Then*

$$\sigma(G) = \mu(G) + O\left(\frac{1}{(\ln n)^{1/45}}\right).$$

THEOREM 5 ([GR05, Proposition 2.2]). *Let G be a finite abelian group, and suppose that $F \subseteq G$ has at most δn^2 Schur triples. Then*

$$|F| \le (\mu(G) + 2^{20}\delta^{1/5})n. \tag{3}$$

The following theorem is also proven in [GR05].

THEOREM 6 ([GR05, Corollary 4.3]). *Let G be an abelian group, and suppose that $F \subseteq G$ has at most δn^2 Schur triples. Then*

$$|F| \le (\max(\mu(G), 1/3) + 3\delta^{1/3})n. \tag{4}$$

Theorem 3 follows immediately from Theorem 6 in the case $\mu(G) \ge 1/3$. If $\mu(G) < 1/3$, Theorem 3 again follows from Theorem 6 provided δ is not very small. For δ small we require Lemma 12 with an estimate different from those in [GR05]. For the rest of results required to prove Theorem 3, the methods used are completely identical as in [GR05], but the results used are not identical.

To prove Theorem 2 we use the following result from [GR05].

THEOREM 7 ([GR05, Proposition 2.1′]). *Let G be an abelian group of cardinality n, where n is sufficiently large. Then there is a family $\mathcal{F}$ of subsets of G with the following properties*:

(I) $\log_2 |\mathcal{F}| \le n(\ln n)^{-1/18}$.
(II) *Every $A \in \mathrm{SF}(G)$ is contained in some $F \in \mathcal{F}$.*
(III) *If $F \in \mathcal{F}$ then F has at most $n^2(\ln n)^{-1/9}$ Schur triples.*

Theorem 2 follows immediately from Theorems 7 and 3. We shall reproduce the proof given in [GR05]. If n is sufficiently large as required by Theorem 7 then associated to each $A \in \mathrm{SF}(G)$ there is an $F \in \mathcal{F}$ for which $A \subset F$. For a given F, the number of A's which can arise in this way is at most $2^{|F|}$. Thus we have the bound

$$|\mathrm{SF}(G)| \le \sum_{F \in \mathcal{F}} 2^{|F|} \le |\mathcal{F}| \max_{F \in \mathcal{F}} 2^{|F|}.$$

Hence

$$\sigma(G) \le \mu(G) + C\,\frac{1}{(\ln n)^{1/27}} + \frac{1}{(\ln n)^{1/18}}. \tag{5}$$

But from (1) we have $\sigma(G) \ge \mu(G)$. Hence Theorem 2 follows.

In order to prove Theorem 3 we shall need the value of $\mu(G)$, which is now known for all finite abelian groups. In order to explain the results we make the following definition.

Definition 8. Suppose that G is a finite abelian group of order n. If n is divisible by any prime $p \equiv 2 \pmod 3$ then we say that G is of *type I*. We say that G is of *type I*(p) if it is of type I and if p is the *least* prime factor of n of the form $3l+2$. If n is not divisible by any prime $p \equiv 2 \pmod 3$, but $3 \mid n$, then we say that G is of *type II*. Otherwise G is said to be of *type III*. That is, G is of type III if and only if all divisors of n are congruent to 1 modulo 3.

The following theorem is due to P. H. Diananda and H. P. Yap [DY69] for type I and type II groups, and to Green and Ruzsa [GR05] for type III groups.

Theorem 9 ([GR05, Theorem 1.5]). *Let G be a finite abelian group of order n. Then the following hold*:

(I) *If G is of type $I(p)$ then $\mu(G) = 1/3 + 1/3p$.*
(II) *If G is of type II then $\mu(G) = 1/3$.*
(III) *If G is of type III then $\mu(G) = 1/3 - 1/3m$, where m is the exponent of G.*

1. Cardinality of almost sum-free sets. In case the group G is not of type III it follows from Theorem 9 that $\mu(G) \geq 1/3$ and hence Theorem 3 follows immediately using Theorem 6. Therefore we have to prove Theorem 3 for type III groups only.

For the rest of this article G will be a finite abelian group of type III, and m will denote the exponent of G. The following proposition is an immediate corollary of Theorems 9 and 6.

Proposition 10. *Let G be an abelian group of type III with order n and exponent m. If $F \subset G$ has at most δn^2 Schur triples then*:

(I) $|F| \leq (\mu(G) + 1/3m + 3\delta^{1/3})n$.
(II) *If $\delta^{1/3} m \geq 1$ then $|F| \leq (\mu(G) + 4\delta^{1/3})n$, that is, Theorem 3 holds in this case.*

Therefore to prove Theorem 3 we are left with the following case: the group G is an abelian group of type III with order n and exponent m. The subset $F \subset G$ has at most δn^2 Schur triples and $\delta^{1/3} m < 1$.

Let γ be a character of G and let q be the order of γ. For any $j \in \mathbb{Z}/q\mathbb{Z}$, we define $H_j = \gamma^{-1}(e^{2\pi i j/q})$. We also denote the set $H_0 = \ker(\gamma)$ by just H. Notice that H is a subgroup of G, and H_j are cosets of H with cardinality

$|H_j| = |H| = n/q$. For any set $F \subset G$ we also define $F_j = F \cap H_j$ and $\alpha_j = |F_j|/|H_j|$.

PROPOSITION 11. *Let G be a finite abelian group of order n. Let F be a subset of G having at most δn^2 Schur triples where $\delta \geq 0$. Let γ be any character of G and q be its order. Also let F_i and α_i be as defined above. Then the following holds*:

(I) *If $x \in F_i$ and $y \in F_j$ then $x + y$ belongs to H_{i+j}.*

(II) *The number of Schur triples $\{x, y, z\}$ in F with $x \in F_l$, $y \in F_j$ and $z \in F_{j+l}$ is at least $|F_l|(|F_j| + |F_{j+l}| - |H|)$. In other words, there are at least $\alpha_l(\alpha_j + \alpha_{j+l} - 1)(n/q)^2$ Schur triples $\{x, y, z\}$ in F with $x \in F_l$.*

(III) *Given any $l \in \mathbb{Z}/q\mathbb{Z}$ such that $\alpha_l > 0$, we have*

$$\alpha_j + \alpha_{j+l} \leq 1 + \delta q^2/\alpha_l \tag{6}$$

for any $j \in \mathbb{Z}/q\mathbb{Z}$.

(IV) *Given any $t \in \mathbb{R}$ we define*

$$L(t) = \{i \in \mathbb{Z}/q\mathbb{Z} : \alpha_i + \alpha_{2i} \geq 1 + t\}.$$

Then

$$\sum_{i \in L(t)} \alpha_i \leq \delta q^2/t. \tag{7}$$

Proof. (I) This follows immediately from the fact that γ is a homomorphism.

(II) If $|F_l|(|F_j| + |F_{j+l}| - |H|) \leq 0$, there is nothing to prove. Hence we can assume that $F_l \neq \emptyset$. Then for any $x \in F_l$, we have $x + F_j \subset H_{j+l}$. Since also $F_{j+l} \subset H_{j+l}$ and $|F_j| + |F_{j+l}| - |H| > 0$, it follows that

$$|(x + F_j) \cap F_{j+l}| = |F_j| + |F_{j+l}| - |(x + F_j) \cup F_{j+l}| \geq |F_j| + |F_{j+l}| - |H|.$$

Now for any $z \in (x + F_j) \cap F_{j+l}$ there exists $y \in F_j$ such that $x + y = z$. Hence the claim follows.

(III) From (II) there are at least $\alpha_l(\alpha_j + \alpha_{j+l} - 1)(n/q)^2$ Schur triples in F. Hence the claim follows by the assumed upper bound on the number of those triples.

(IV) For any fixed $i \in L(t)$, taking $j = l = i$ in (II), we see that there are at least $\alpha_i(n/q)^2 t$ Schur triples $\{x, y, z\}$ in F with $x \in F_i$. Now for any $i_1, i_2 \in L(t)$ such that $i_1 \neq i_2$, the sets F_{i_1} and F_{i_2} are disjoint. Therefore there are at least $(n/q)^2 t \sum_{i \in L(t)} \alpha_i$ Schur triples in F. Hence the claim follows. ∎

Since the order of any character of an abelian group G divides the order of the group and G is of type III, the order q of any character γ of G is odd and congruent to 1 modulo 3. Therefore $q = 6k + 1$ for some $k \in \mathbb{N}$. Let

$I, H, M, T \subset \mathbb{Z}/q\mathbb{Z}$ denote the images of the intervals $\{k+1, k+2, \ldots, 5k-1, 5k\}, \{k+1, k+2, \ldots, 2k-1, 2k\}, \{2k+1, 2k+2, \ldots, 4k-1, 4k\}, \{4k+1, 4k+2, \ldots, 5k-1, 5k\}$ in $\mathbb{Z}/q\mathbb{Z}$. Then the set I is divided into $2k$ disjoint pairs of the form $(i, 2i)$ where $i \in H \cup T$.

LEMMA 12. *Let G be a finite abelian group of type III and order n. Suppose that $F \subset G$ has at most δn^2 Schur triples. Let γ be a character of G. Let the order of γ be $q = 6k + 1$. Then*

$$\sum_{i=k+1}^{5k} \alpha_i \leq 2k + 2\delta^{1/2} q^{3/2}. \tag{8}$$

Proof. The set $I = \{k+1, k+2, \ldots, 5k\}$ is divided into $2k$ disjoint pairs of the form $(i, 2i)$ where $i \in H \cup T$. Therefore

$$\sum_{i=k+1}^{5k} \alpha_i = \sum_{i \in H \cup T} (\alpha_i + \alpha_{2i}). \tag{9}$$

Given a $t > 0$ we divide $H \cup T$ into two disjoint sets,

$$S = \{i \in H \cup T : \alpha_i + \alpha_{2i} \leq 1 + t\},$$
$$L = \{i \in H \cup T : \alpha_i + \alpha_{2i} > 1 + t\}.$$

Therefore

$$\sum_{i \in H \cup T} (\alpha_i + \alpha_{2i}) = \sum_{i \in S} (\alpha_i + \alpha_{2i}) + \sum_{i \in L} (\alpha_i + \alpha_{2i}). \tag{10}$$

From (7) we have

$$\sum_{i \in L} \alpha_i \leq \delta q^2 / t.$$

Since for any $l \in \mathbb{Z}/q\mathbb{Z}$, the inequality $\alpha_l \leq 1$ holds trivially, it follows that

$$\sum_{i \in L} (\alpha_i + \alpha_{2i}) \leq |L| + \delta q^2 / t. \tag{11}$$

Also

$$\sum_{i \in S} (\alpha_i + \alpha_{2i}) \leq |S| + |S| t \tag{12}$$

just by the definition of the set S. Now from (9), it follows that

$$\sum_{i=k+1}^{5k} \alpha_i \leq |L| + \delta q^2/t + |S| + |S| t \leq 2k + qt + \delta q^2 / t. \tag{13}$$

Choosing $t = (\delta q)^{1/2}$ completes the proof of the lemma. ■

REMARK. The sum appearing in the last lemma was estimated by $2k + \delta^{1/2} q^2$ in [GR05]. There the estimate $\alpha_i + \alpha_{2i} \leq \delta^{1/2} q$ was used to estimate the right hand side of (9).

Notice that Lemma 12 holds for any character γ of a group G of type III. We would like to show that given $F \subset G$ having at most δn^2 Schur triples and also assuming that $\delta^{1/3} m < 1$ where m is the exponent of G, there is a character γ such that $\alpha_i \le c(\delta q)^{1/2}$ for $i \in \{0, 1, \ldots, k\} \cup \{5k+1, \ldots, 6k\}$ where c is an absolute positive constant, q is the order of γ and $k = (q-1)/6$. To do this we recall the concept of special direction as defined in [GR05]. The method of proof of this part is identical as in [GR05], though the results are not.

Given any set $B \subset G$ and a character γ of G we define $\widehat{B}(\gamma) = \sum_{b \in B} \gamma(b)$. Fix a character γ_s such that $\operatorname{Re} \widehat{B}(\gamma)$ is minimal. We follow the terminology in [GR05] and call γ_s a *special direction* of B.

The following lemma is proven in [GR05], but we shall reproduce the proof here for the sake of completeness.

LEMMA 13 ([GR05, Lemmas 7.1 and 7.3(iv)]). *Let G be an abelian group of type III. Suppose $F \subset G$ has at most δn^2 Schur triples. Let γ_s be a special direction of F. Set $\alpha = |F|/|G|$. Then the following hold*:

(I) $\operatorname{Re} \widehat{F}(\gamma_s) \le \left(\dfrac{\delta}{\alpha(1-\alpha)} - \dfrac{\alpha^2}{\alpha(1-\alpha)} \right) n.$

(II) *If $\delta \le \eta/5$, then either $|F| \le \mu(G)n$ or*

$$q^{-1} \sum_{j=0}^{q-1} \alpha_j \cos\left(\frac{2\pi j}{q}\right) + \frac{(\mu(\mathbb{Z}/q\mathbb{Z}))^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})} < 6\delta. \tag{14}$$

Proof. (I) There are exactly $n^{-1} \sum_\gamma (\widehat{F}(\gamma))^2 \widehat{F}(\gamma)$ Schur triples in the set F. This follows after straightforward calculation, using the fact that

$$\sum_\gamma \gamma(b) = \begin{cases} 0 & \text{if } b \neq 0, \\ n & \text{if } b = 0, \end{cases} \tag{15}$$

where 0 denotes the identity element of the group G. Therefore using the assumed upper bound on the number of Schur triples in F it follows that

$$n^{-1} \sum_\gamma (\widehat{F}(\gamma))^2 \widehat{F}(\gamma) = n^{-1} \sum_{\gamma \neq 1} (\widehat{F}(\gamma))^2 \widehat{F}(\gamma) + n^{-1} (\widehat{F}(1))^2 \widehat{F}(1) \le \delta n^2,$$

where $\gamma = 1$ is the trivial character of G. Since $n^{-1}(\widehat{F}(1))^2 \widehat{F}(1) = \alpha^3 n^2$, it follows that

$$\operatorname{Re} \widehat{F}(\gamma_s) \sum_{\gamma \neq 1} (\widehat{F}(\gamma))^2 \le n^{-1} \sum_{\gamma \neq 1} (\widehat{F}(\gamma))^2 \widehat{F}(\gamma) \le (\delta - \alpha^3) n^2.$$

Since from (15) it follows that $\sum_{\gamma \neq 1} (\widehat{F}(\gamma))^2 = \alpha(1-\alpha^2) n^2$, the claim follows.

(II) We have $\operatorname{Re}\widehat{F}(\gamma_s) = |H|\sum_j \alpha_j \cos(2\pi j/q)$. Therefore in the case $|F| \ge \mu(G)$, from (I) it follows that

$$q^{-1}\sum_{j=0}^{q-1}\alpha_j \cos\left(\frac{2\pi j}{q}\right) \le \frac{\delta}{\alpha(1-\alpha)} - \frac{\alpha^2}{\alpha(1-\alpha)}, \tag{16}$$

$$q^{-1}\sum_{j=0}^{q-1}\alpha_j \cos\left(\frac{2\pi j}{q}\right) + \frac{(\mu(G))^2}{1-\mu(G)} \le \frac{\delta}{\alpha(1-\alpha)}. \tag{17}$$

Since from Theorem 9 we know that $\mu(G) \ge \mu(\mathbb{Z}/q\mathbb{Z})$ it follows that

$$\frac{(\mu(G))^2}{1-\mu(G)} \ge \frac{(\mu(\mathbb{Z}/q\mathbb{Z}))^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})}.$$

The claim follows from this and the fact that $1/2 \ge \mu(G) \ge 1/4$, which implies that $\delta/\alpha(1-\alpha) \le 6\delta$. ∎

LEMMA 14. *Let G be an abelian group of type III with order n and exponent m. Suppose $F \subset G$ has at most δn^2 Schur triples and $\delta^{1/3}m \le 1$. Let $|F| \ge \mu(G)n$. Let γ_s be a special direction of F and q the order of γ_s. Let $q = 6k+1$ and α_i be as defined above. There exist absolute positive constants q_0 and δ_1 such that if $q \ge q_0$ and $\delta \le \delta_1$, then*

$$\alpha_i \le c(\delta q)^{1/2} \quad \textit{for all } i \in \{0,1,\dots,k\} \cup \{5k+1,\dots,6k-1\}, \tag{18}$$

where c is an absolute positive constant.

Proof. If $F \subset G$ is as in the statement, then so is $-F \subset G$. Moreover $|F_j| = |(-F)_{-j}|$. Therefore to prove the proposition it is sufficient to show that

$$\alpha_i \le c(\delta q)^{1/2} \quad \text{for all } i \in \{0,1,\dots,k\}$$

for some absolute positive constant c.

Let

$$S = q^{-1}\sum_{j=0}^{q-1}\alpha_j \cos\left(\frac{2\pi j}{q}\right) + \frac{(\mu(\mathbb{Z}/q\mathbb{Z}))^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})}.$$

Then from Lemma 13 we have

$$S \le 6\delta. \tag{19}$$

Now suppose that $\alpha_l > c(\delta q)^{1/2}$ for some $l \in \{0,1,\dots,k\}$ (where c is a positive number to be chosen later). We shall show that this violates (19), provided q and c are sufficiently large and δ is sufficiently small. For this we shall find the lower bound of $M = q^{-1}\sum_{j=0}^{q-1}\alpha_j\cos(2\pi j/q)$.

Set $\gamma_j = (\alpha_j + \alpha_{j+l})/2$. Then we have

$$M = \frac{1}{2q\cos(\pi l/q)}\sum_{j=0}^{q-1}\alpha_j\left(\cos\left(\frac{(2j+l)\pi}{q}\right) + \cos\left(\frac{(2j-l)\pi}{q}\right)\right).$$

That is,

$$M = \frac{1}{q\cos(\pi l/q)} \sum_{j=0}^{q-1} \gamma_j \cos\left(\frac{(2j+l)\pi}{q}\right). \tag{20}$$

Notice that $\cos(\pi l/q)$ is not well defined if we consider l as an element of $\mathbb{Z}/q\mathbb{Z}$. This is because the function $\cos(\pi t/q)$ as a function of t is periodic, but with period $2q$ and not q. But we have assumed that $l \in \{0, 1, \ldots, k\}$, so the above computation is valid.

Since $\delta^{1/2}q^{3/2} \leq \delta^{1/2}m^{3/2} < 1$ by assumption, recalling Lemma 11 it follows that

$$2\gamma_j = \alpha_j + \alpha_{j+l} \leq 1 + \frac{1}{c}\delta^{1/2}q^{3/2} \leq 1 + \frac{1}{c} \quad \text{for any } j \in \mathbb{Z}/q\mathbb{Z} \tag{21}$$

and

$$\sum_j \gamma_j = \sum_j \alpha_j \geq \mu(G)n \geq 2k. \tag{22}$$

The inequality (22) follows from the assumption that $|F| \geq \mu(G)n$.

Set $t_c = 1 + 1/c$. Let $E(c, q)$ denote the minimum of the expression $\sum_{j=0}^{q-1} \gamma_j \cos((2j+l)\pi/q)$ subject to the constraints $0 \leq \gamma_j \leq t_c/2$ and $\sum_j \gamma_j \geq 2k$.

The function $f : \mathbb{Z} \to \mathbb{R}$ given by $f(x) = \cos((q+x)\pi/q)$ is even with period $2q$ and

$$f(0) < f(1) < \cdots < f(q). \tag{23}$$

Now to determine $E(c, q)$, we should choose γ_j to be as large as we can when $\cos((2j+l)\pi/q)$ is small. We have two cases to discuss: when l is even and when l is odd. The image of the function $g : \mathbb{Z}/q\mathbb{Z} \to \mathbb{R}$ defined by $g(j) = \cos((2j+l)\pi/q)$ is equal to $\{f(x) : x \text{ is even}\}$ if l is odd, and to $\{f(x) : x \text{ is odd}\}$ if l is even. From this it is also easy to observe that the number of $j \in \mathbb{Z}/q\mathbb{Z}$ such that $\cos((2j+l)\pi/q)$ is negative is at most $(q+1)/2$. Now let

$$-\frac{q-1}{2} - l \leq j \leq \frac{q-1}{2} - l$$

so that $-q \leq 2j + l \leq q$. For l odd consider the case when $\gamma_j = t_c/2$ if

$$2j + l = q - \left[\frac{k}{t_c} - \frac{1}{2}\right], \ldots, q-2, q, q+1, \ldots, q + \left[\frac{k}{t_c} - \frac{1}{2}\right]$$

and $\gamma_j = 0$ otherwise. The condition $2[k/t_c - 1/2] + 1 \geq (q+1)/2$ ensures that in the above configuration for all possible negative values of $\cos((2j+l)\pi/q)$ the maximum possible weight $t_c/2$ is chosen. This condition can be ensured if $q \geq 11$ by choosing $c \geq c_1$ where c_1 is a sufficiently large absolute positive constant. Therefore a small calculation shows that for $c \geq c_1$,

$$E(c,q) \geq -t_c \frac{\sin(2\pi[k/t_c - 1/2]/q)}{2q\sin(\pi/q)\cos(\pi l/q)} - \frac{1}{q}. \tag{24}$$

For l even and $c \geq c_1$, choosing $\gamma_j = t_c/2$ if

$$2j + l = q - \left[\frac{k}{t_c}\right], \ldots, q-1, q+1, \ldots, q + \left[\frac{k}{t_c}\right]$$

and $\gamma_j = 0$ otherwise, we get

$$E(c) \geq -t_c \frac{\sin((2\pi[k/t_c]+1)/q)}{2q\sin(\pi/q)} \cos\pi q - \frac{t_c}{q}. \tag{25}$$

Using this we get

$$S \geq -t_c \frac{\sin(2\pi[k/t_c]/q)}{2q\sin(\pi/q)\cos(\pi l/q)} + \frac{(\mu(\mathbb{Z}/q\mathbb{Z}))^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})} \quad \text{when } l \text{ is even}, \tag{26}$$

$$S \geq t_c \frac{\sin(2\pi[k/t_c - 1/2]/q)}{2q\sin(\pi/q)\cos(\pi l/q)} - \frac{1}{q} + \frac{(\mu(\mathbb{Z}/q\mathbb{Z}))^2}{1-\mu(\mathbb{Z}/q\mathbb{Z})} \quad \text{when } l \text{ is odd}. \tag{27}$$

Now as $q \to \infty$ the right hand side of (26) as well as (27) converges to

$$-t_c \frac{\sin(2\pi/3t_c)}{2\pi\cos(\pi l/q)} + \frac{1}{6}.$$

Let $\eta = 2^{-20}$. Then choosing $c \geq c_2$ and $q \geq q_0$ and noticing that $l \leq q/6$ we get

$$S \geq -\frac{1}{2\pi} + \frac{1}{6} - \eta = 8\delta_1 \quad \text{say}. \tag{28}$$

When $\delta \leq \delta_1$, the lower bound on S given by (28) is in contradiction to the upper bound on S given by (19). Hence the lemma follows. ∎

To complete the proof of Theorem 3, we require the following result from [GR05].

LEMMA 15 ([GR05, Proposition 7.2]). *Let G be an abelian group of type III and n, m be its order and exponent respectively. Suppose $F \subset G$ has at most δn^2 Schur triples, with $\delta^{1/3} m < 1$. Let q be the order of a special direction such that $q \leq q_0$, where q_0 is an absolute positive constant as in Lemma 14. Also assume that $\delta \leq \eta/q^5 = \delta_2$, where $\eta = 2^{-50}$. Then either $|F| \leq \mu(G)n$ or $\alpha_i \leq 64\delta^{1/3}q^{2/3}$ for any $i \in \{0,1,\ldots,k\} \cup \{5k+1,\ldots,6k\}$.*

Let δ_1 and δ_2 be as in Lemmas 14 and 15 respectively. Then we take $\delta_0 = \min(\delta_1, \delta_2)$ in Theorem 3. Combining Lemmas 12, 14 and 15 yields Theorem 3 in case $\delta^{1/3} m < 1$. In case $\delta^{1/3} m > 1$, Theorem 3 follows from Proposition 10.

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The Institute of Mathematical Sciences
CIT Campus, Taramani
Chennai 600113, India
E-mail: balu@imsc.res.in

Harish-Chandra Research Institute
Chhatnag Road, Jhunsi
Allahabad 211019, India
E-mail: gyan@mri.ernet.in

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