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The weight of half-integral weight modular forms with few non-vanishing coefficients mod l

by

DOHOON CHOI (Seoul) and TIMOTHY KILBOURN (Urbana, IL)

1. Introduction and statement of results. Much is known about the divisibility of the Fourier coefficients of modular forms of integral weight. When the weight is half-integral, the situation is not as clear. The distribution of the coefficients of half-integral weight modular forms in congruence classes has been studied by, among others, Balog, Darmon, and Ono [4], Ono and Skinner [11], [12], Bruinier [5], Bruinier and Ono [6], and Ahlgren and Boylan [1], [2].

Let $S_{k/2}(\Gamma_0(N),\chi)$ be the space of cusp forms of weight $k/2$ on the congruence subgroup $\Gamma_0(N)$ with Dirichlet character χ. (See Chapter 3 in [10] for the basic definitions of a cusp form.) Recently, Ahlgren and Boylan proved the following theorem:

THEOREM 1 ([2, Theorem 2]). *Suppose that we have the following hypotheses*:

(i) $\lambda \geq 2$ *is an integer*, N *is a positive integer with* $4 \mid N$, *and* χ *is a real Dirichlet character modulo* N.
(ii) $l \geq 5$ *is a prime such that* $l \nmid N$.
(iii) $F(z)$ *is a half-integral weight cusp form such that*

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(N),\chi) \cap \mathbb{Z}[[q]],$$

where $q := e^{2\pi i z}$.
(iv) $F(z) \not\equiv 0 \pmod{l}$, *and there are finitely many square-free integers* $n_1,\ldots,n_t$ *such that*

$$F(z) \equiv \sum_{i=1}^{t}\sum_{m=1}^{\infty} a(n_i m^2) q^{n_i m^2} \pmod{l}.$$

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Write $\lambda = \overline{\lambda} + i_\lambda(l-1)$ *with* $0 \le \overline{\lambda} \le l-2$. *Then the following are true*:

(1) *If* $l \nmid n_i$ *for some* i, *then*

$$\overline{\lambda} \le 2i_\lambda + 1.$$

(2) *If* $l \mid n_i$ *for all* i *and* $\overline{\lambda} \le (l-3)/2$, *then*

$$\overline{\lambda} \le 2i_\lambda - \frac{l-1}{2}.$$

(3) *If* $l \mid n_i$ *for all* i *and* $\overline{\lambda} \ge (l-1)/2$, *then*

$$\overline{\lambda} \le 2i_\lambda + \frac{l+3}{2}.$$

They used these bounds to prove Newman's conjecture [9] for the partition function for prime-power moduli. This has been extended to powers of the generating function for the partition function by the second author [8]. Here we obtain an improvement on these bounds when $l \mid n_i$ for all i.

Theorem 2. *Suppose we have the same notation and hypotheses as in Theorem* 1, *and that* $l \mid n_i$ *for all* i. *Then*:

(1) *If* $\overline{\lambda} \le (l-3)/2$, *then*

$$\overline{\lambda} \le i_\lambda - \frac{l+1}{2}.$$

(2) *If* $\overline{\lambda} \ge (l-1)/2$, *then*

$$\overline{\lambda} \le i_\lambda + \frac{l-1}{2}.$$

These inequalities are sharp. For example, define the quadratic character $\chi_D := \left(\frac{D}{\cdot}\right)$. Let

$$f(z) := \eta^l(24z) = q^l \prod_{n=1}^{\infty}(1-q^{24n})^l \equiv \sum_{n=1}^{\infty} \chi_{12}(n) q^{ln^2} \pmod{l},$$

$$g(z) := \eta^{l^2}(24z) = q^{l^2} \prod_{n=1}^{\infty}(1-q^{24n})^{l^2} \equiv \sum_{n=1}^{\infty} \chi_{12}(n) q^{l^2n^2} \pmod{l}.$$

Using the fact that $\eta(24z) \in S_{1/2}(\Gamma_0(576), \chi_{12})$, we deduce that $f(z) \in S_{(l-1)/2+1/2}(\Gamma_0(576), \chi_{12})$ and $g(z) \in S_{(l^2-1)/2+1/2}(\Gamma_0(576), \chi_{12})$. For $f(z)$, we have $\overline{\lambda} = (l-1)/2$ and $i_\lambda = 0$, and the inequality of Theorem 2 reads $(l-1)/2 \le 0 + (l-1)/2$. Likewise, for $g(z)$, we have $\overline{\lambda} = 0$ and $i_\lambda = (l+1)/2$, and the inequality is $0 \le 0$.

2. Modular forms mod l**.** Throughout this section, we fix an integer $N \ge 1$ and a prime $l \nmid N$. The theory of modular forms modulo l was developed by Serre [13] and Swinnerton-Dyer [14] for forms of level 1. We will recall the definitions and results that will be needed for the next section. For proofs when $N \geq 4$ and $l \nmid N$, see Gross [7].

Denote by M_k the space of modular forms of weight k on $\Gamma_1(N)$ with integer coefficients. If $f \in M_k$, let $\widetilde{f} \in (\mathbb{Z}/l\mathbb{Z})[[q]]$ be the (coefficient-wise) reduction of f mod l. We write $\widetilde{M_k} := \{\widetilde{f} \mid f \in M_k\}$ for the space of weight k modular forms modulo l with level N.

The filtration $\omega(\widetilde{f})$ of a modular form $\widetilde{f} \in \widetilde{M_k}$ is defined to be

$$\omega(\widetilde{f}) := \inf\{k' \mid \widetilde{f} \in \widetilde{M_{k'}}\}.$$

If $f \in M_k$, we sometimes write $\omega(f)$ instead of $\omega(\widetilde{f})$ if no confusion can arise. Note that $\omega(\widetilde{f}) = -\infty$ if and only if $f \equiv 0 \pmod{l}$. If $f \in M_k$, $g \in M_{k'}$, and $f \equiv g \not\equiv 0 \pmod{l}$, then we must have $k \equiv k' \pmod{l-1}$. From this it follows that $\omega(\widetilde{f}) \equiv k \pmod{l-1}$.

The operator U_l acts on q-expansions by

$$\sum_n a(n)q^n | U_l = \sum_n a(ln)q^n.$$

Note that if T_l denotes the lth Hecke operator on M_k, then $f|T_l \equiv f|U_l \pmod{l}$, so U_l acts on the spaces $\widetilde{M_k}$.

We record some basic properties of the filtration of a modular form modulo l.

Lemma 1 ([13], [14], [7]). *Let $f \in \widetilde{M_k}$ be such that $f \not\equiv 0 \pmod{l}$. Write $k = \overline{k} + i_k(l-1)$ with $0 \leq \overline{k} \leq l-2$. Then*:

(1) $\omega(f) \geq 0$.
(2) *If f is not a constant modulo l, then $\omega(f) > 0$.*
(3) $\omega(f) \equiv \omega(f|U_l) \pmod{l-1}$.
(4) $\omega(f^l) = l \cdot \omega(f)$.
(5) $\overline{k} \leq \omega(f) \leq k$.

3. Proof of Theorem 2. Let F be as in the hypotheses. Recall the definition of the usual theta function

$$\theta(z) := 1 + 2\sum_{n=1}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4), \chi_{-4}).$$

We define an integral weight modular form $G(z)$ by

$$G(z) := F(z)\theta^l(z) \in S_{\lambda+(l+1)/2}(\Gamma_0(N), \chi\chi_{-4}).$$

By combining hypothesis (iv) with the fact that $l \mid n_i$ for all i, we see that

$$F(z) \equiv \sum_{i=1}^{t}\sum_{m=1}^{\infty} a(ln_i'm^2)q^{ln_i'm^2} \pmod{l}$$

and hence
$$(G(z)|U_l)^l \equiv G(z) \pmod{l}.$$
This implies that the two forms have the same filtration, and by part (4) of Lemma 1,
$$\omega(G) = l \cdot \omega(G|U_l).$$

First suppose $\overline{\lambda} \leq (l-3)/2$. By hypothesis (iv) and the fact that $G(z)$ is a cusp form, we have $\omega(G) > 0$ by Lemma 1(1),(2). Combining assertions (3) and (5) of Lemma 1, we get the following inequality:
$$l \cdot \left(\overline{\lambda} + \frac{l+1}{2}\right) \leq l \cdot \omega(G|U_l) = \omega(G) \leq i_\lambda(l-1) + \overline{\lambda} + \frac{l+1}{2}.$$
Solving for $\overline{\lambda}$ yields
$$\overline{\lambda} \leq i_\lambda - \frac{l+1}{2},$$
proving (1).

Next suppose $\overline{\lambda} \geq (l-1)/2$. By the same arguments we find the inequality
$$l \cdot \left(\overline{\lambda} + \frac{l+1}{2} - (l-1)\right) \leq l \cdot \omega(G|U_l) = \omega(G) \leq i_\lambda(l-1) + \overline{\lambda} + \frac{l+1}{2},$$
and hence
$$\overline{\lambda} \leq i_\lambda + \frac{l-1}{2}.$$
This proves (2), and hence Theorem 2. ■

We can also improve Corollaries 3 and 4 in [2].

Corollary. *If $F(z)$ has the form*
$$F(z) \equiv \sum_{i=1}^{t} \sum_{m=1}^{\infty} a(n_i m^2) q^{n_i m^2} \pmod{l},$$
then we must have $\overline{\lambda} \leq 2i_\lambda + 1$. In particular, if $\lambda \leq l-2$, then $\lambda \in \{0,1\}$.

Note that under the additional restriction $\lambda \leq l-2$, this Corollary demonstrates that Conjecture B in [2] is true. Ahlgren and Boylan have shown [3] that this Conjecture is false in general.

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School of Mathematics, KIAS
207-43 Cheongnyangni 2-dong
Seoul 130-722, Korea
E-mail: choija@postech.ac.kr

Department of Mathematics
University of Illinois
Urbana, IL 61801, U.S.A.
E-mail: tkilbour@math.uiuc.edu

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