The weight of half-integral weight modular forms with few non-vanishing coefficients mod $l$

by

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1. Introduction and statement of results. Much is known about the divisibility of the Fourier coefficients of modular forms of integral weight. When the weight is half-integral, the situation is not as clear. The distribution of the coefficients of half-integral weight modular forms in congruence classes has been studied by, among others, Balog, Darmon, and Ono [4], Ono and Skinner [11], [12], Bruinier [5], Bruinier and Ono [6], and Ahlgren and Boylan [1], [2].

Let $S_{k/2}(\Gamma_0(N), \chi)$ be the space of cusp forms of weight $k/2$ on the congruence subgroup $\Gamma_0(N)$ with Dirichlet character $\chi$. (See Chapter 3 in [10] for the basic definitions of a cusp form.) Recently, Ahlgren and Boylan proved the following theorem:

**Theorem 1 ([2, Theorem 2]).** Suppose that we have the following hypotheses:

(i) $\lambda \geq 2$ is an integer, $N$ is a positive integer with $4 \mid N$, and $\chi$ is a real Dirichlet character modulo $N$.

(ii) $l \geq 5$ is a prime such that $l \nmid N$.

(iii) $F(z)$ is a half-integral weight cusp form such that

$$F(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]],$$

where $q := e^{2\pi i z}$.

(iv) $F(z) \not\equiv 0 \pmod{l}$, and there are finitely many square-free integers $n_1, \ldots, n_t$ such that

$$F(z) \equiv \sum_{i=1}^{t} \sum_{m=1}^{\infty} a(n_i m^2)q^{n_i m^2} \pmod{l}.$$
Write $\lambda = \bar{\lambda} + i\lambda(l - 1)$ with $0 \leq \bar{\lambda} \leq l - 2$. Then the following are true:

1. If $l \nmid n_i$ for some $i$, then 
   \[ \lambda \leq 2i\lambda + 1. \]

2. If $l \mid n_i$ for all $i$ and $\bar{\lambda} \leq (l - 3)/2$, then 
   \[ \lambda \leq 2i\lambda - \frac{l - 1}{2}. \]

3. If $l \mid n_i$ for all $i$ and $\bar{\lambda} \geq (l - 1)/2$, then 
   \[ \lambda \leq 2i\lambda + \frac{l + 3}{2}. \]

They used these bounds to prove Newman’s conjecture [9] for the partition function for prime-power moduli. This has been extended to powers of the generating function for the partition function by the second author [8]. Here we obtain an improvement on these bounds when $l \mid n_i$ for all $i$.

**Theorem 2.** Suppose we have the same notation and hypotheses as in Theorem 1, and that $l \mid n_i$ for all $i$. Then:

1. If $\bar{\lambda} \leq (l - 3)/2$, then 
   \[ \lambda \leq i\lambda - \frac{l + 1}{2}. \]

2. If $\bar{\lambda} \geq (l - 1)/2$, then 
   \[ \lambda \leq i\lambda + \frac{l - 1}{2}. \]

These inequalities are sharp. For example, define the quadratic character $\chi_D := (\frac{D}{.})$. Let 

\[
 f(z) := \eta^l(24z) = q^l \prod_{n=1}^{\infty} (1 - q^{24n})^l \equiv \sum_{n=1}^{\infty} \chi_{12}(n)q^{ln^2} \pmod{l},
\]

\[
 g(z) := \eta^l(24z) = (2q^2 \prod_{n=1}^{\infty} (1 - q^{24n})^l) \equiv \sum_{n=1}^{\infty} \chi_{12}(n)q^{ln^2} \pmod{l}.
\]

Using the fact that $\eta(24z) \in S_{1/2}(I_0(576), \chi_{12})$, we deduce that $f(z) \in S_{(l-1)/2+1/2}(I_0(576), \chi_{12})$ and $g(z) \in S_{(l-1)/2+1/2}(I_0(576), \chi_{12})$. For $f(z)$, we have $\bar{\lambda} = (l - 1)/2$ and $i\lambda = 0$, and the inequality of Theorem 2 reads $(l - 1)/2 \leq 0 + (l - 1)/2$. Likewise, for $g(z)$, we have $\bar{\lambda} = 0$ and $i\lambda = (l + 1)/2$, and the inequality is $0 \leq 0$.

**2. Modular forms mod $l$.** Throughout this section, we fix an integer $N \geq 1$ and a prime $l \nmid N$. The theory of modular forms modulo $l$ was developed by Serre [13] and Swinnerton-Dyer [14] for forms of level 1. We will
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recall the definitions and results that will be needed for the next section. For proofs when \( N \geq 4 \) and \( l \nmid N \), see Gross [7].

Denote by \( M_k \) the space of modular forms of weight \( k \) on \( \Gamma_1(N) \) with integer coefficients. If \( f \in M_k \), let \( \tilde{f} \in (\mathbb{Z}/l\mathbb{Z})[[q]] \) be the (coefficient-wise) reduction of \( f \mod l \). We write \( \tilde{M}_k := \{ \tilde{f} \mid f \in M_k \} \) for the space of weight \( k \) modular forms modulo \( l \) with level \( N \).

The filtration \( \omega(\tilde{f}) \) of a modular form \( \tilde{f} \in \tilde{M}_k \) is defined to be

\[
\omega(\tilde{f}) := \inf \{ k' \mid \tilde{f} \in \tilde{M}_{k'} \}.
\]

If \( f \in M_k \), we sometimes write \( \omega(f) \) instead of \( \omega(\tilde{f}) \) if no confusion can arise. Note that \( \omega(\tilde{f}) = -\infty \) if and only if \( f \equiv 0 \pmod{l} \). If \( f \in M_k \), \( g \in M_{k'} \), and \( f \equiv g \neq 0 \pmod{l} \), then we must have \( k \equiv k' \pmod{l-1} \). From this it follows that \( \omega(\tilde{f}) \equiv k \pmod{l-1} \).

The operator \( U_l \) acts on \( q \)-expansions by

\[
\sum_n a(n)q^n|U_l = \sum_n a(ln)nq^n.
\]

Note that if \( T_l \) denotes the \( l \)th Hecke operator on \( M_k \), then \( f|T_l \equiv f|U_l \pmod{l} \), so \( U_l \) acts on the spaces \( \tilde{M}_k \).

We record some basic properties of the filtration of a modular form modulo \( l \).

**Lemma 1** ([13], [14], [7]). Let \( f \in \tilde{M}_k \) be such that \( f \not\equiv 0 \pmod{l} \). Write \( k = \overline{k} + i_k(l-1) \) with \( 0 \leq \overline{k} \leq l-2 \). Then:

1. \( \omega(f) \geq 0 \).
2. If \( f \) is not a constant modulo \( l \), then \( \omega(f) > 0 \).
3. \( \omega(f) \equiv \omega(f|U_l) \pmod{l-1} \).
4. \( \omega(f^l) = l \cdot \omega(f) \).
5. \( \overline{k} \leq \omega(f) \leq k \).

**3. Proof of Theorem 2.** Let \( F \) be as in the hypotheses. Recall the definition of the usual theta function

\[
\theta(z) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4), \chi_{-4}).
\]

We define an integral weight modular form \( G(z) \) by

\[
G(z) := F(z)\theta^l(z) \in S_{\lambda+(l+1)/2}(\Gamma_0(N), \chi\chi_{-4}).
\]

By combining hypothesis (iv) with the fact that \( l \mid n_i \) for all \( i \), we see that

\[
F(z) \equiv \sum_{i=1}^{t} \sum_{m=1}^{\infty} a(ln_i|m^2)q^{ln_i|m^2} \pmod{l}
\]
and hence
\[(G(z)|U_l)^l \equiv G(z) \pmod{l} \].

This implies that the two forms have the same filtration, and by part (4) of Lemma 1,
\[\omega(G) = l \cdot \omega(G|U_l)\].

First suppose \(\bar{\lambda} \leq (l - 3)/2\). By hypothesis (iv) and the fact that \(G(z)\) is a cusp form, we have \(\omega(G) > 0\) by Lemma 1(1),(2). Combining assertions (3) and (5) of Lemma 1, we get the following inequality:
\[l \cdot \left(\bar{\lambda} + \frac{l + 1}{2}\right) \leq l \cdot \omega(G|U_l) = \omega(G) \leq i\lambda(l - 1) + \bar{\lambda} + \frac{l + 1}{2}\].

Solving for \(\bar{\lambda}\) yields
\[\bar{\lambda} \leq i\lambda - \frac{l + 1}{2}\],
proving (1).

Next suppose \(\bar{\lambda} \geq (l-1)/2\). By the same arguments we find the inequality
\[l \cdot \left(\bar{\lambda} + \frac{l + 1}{2} - (l - 1)\right) \leq l \cdot \omega(G|U_l) = \omega(G) \leq i\lambda(l - 1) + \bar{\lambda} + \frac{l + 1}{2}\],
and hence
\[\bar{\lambda} \leq i\lambda + \frac{l - 1}{2}\].
This proves (2), and hence Theorem 2. 

We can also improve Corollaries 3 and 4 in [2].

**Corollary.** If \(F(z)\) has the form
\[F(z) \equiv \sum_{i=1}^{t} \sum_{m=1}^{\infty} a(n_i m^2) q^{n_i m^2} \pmod{l},\]
then we must have \(\bar{\lambda} \leq 2i\lambda + 1\). In particular, if \(\lambda \leq l - 2\), then \(\lambda \in \{0, 1\}\).

Note that under the additional restriction \(\lambda \leq l - 2\), this Corollary demonstrates that Conjecture B in [2] is true. Ahlgren and Boylan have shown [3] that this Conjecture is false in general.

**References**


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