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A constructive approach to p-adic deformations of arithmetic homology

by

Avner Ash (Chestnut Hill, MA)

1. Introduction. Let p be a prime number, K a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, and ϖ a uniformizer in $\mathcal{O}$. Set $\mathbb{F} = \mathcal{O}/\varpi$.

We study the homology of an arithmetic group Γ with coefficients in a p-adically analytic family of K-Banach modules that modulo ϖ is a constant family. These modules are parametrized by points k in a p-adic rigid analytic affinoid space Ω. We write $\mathbb{D}_k$ for the module with parameter k and $\mathbb{D}_k^0$ for the closed unit ball in $\mathbb{D}_k$. The families we have in mind are the modules of distributions constructed in [3], but we use a more general concept of analytic families of Banach modules, partly for clarity and partly so that variants of those in [3] may be used if necessary. We use homology in this paper because it is better suited to computer experiments, but it is straightforward to state and prove parallel results for cohomology.

Our main results, contained in Theorem 27 and Corollary 28, may be summarized as follows: Let $k_0 \in \Omega(K)$ and $\zeta_0 \in H_j(\Gamma, \mathbb{D}_{k_0}^0)$ be an ordinary Hecke eigenclass. We obtain a finite number of rigid analytic functions on Ω whose zero-locus is a Zariski-closed subspace $V \subset \Omega$ such that for each $k \in V$ there exists a nonzero homology class $\zeta_k \in H_j(\Gamma, \mathbb{D}_k^0)$ such that $\zeta_{k_0} = \zeta_0$, and for every $k \in V$, the reduction of ζ_k modulo ϖ is a Hecke eigenclass whose Hecke eigenvalues are congruent to those of ζ_0 modulo ϖ. Under the hypothesis of Theorem 30, the classes ζ_k will be non-ϖ-power-torsion.

A priori V might consist only of the single point k_0. We obtain a lower bound for the dimension of V computed as follows. Let α denote the system of Hecke eigenvalues on ζ_0 reduced modulo ϖ. Let d be the dimension over $\mathcal{O}/\varpi$ of the generalized α-eigenspace of $H_{j-1}(\Gamma, \mathbb{D}_{k_0}^0/\varpi\mathbb{D}_{k_0}^0)$. Then the codimension of V in Ω is at most d.

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Theorem 27 is actually more general. We input a chain z_0 which modulo ϖ becomes a cycle in $C_j(\Gamma, \mathbb{D}^0_{k_0}/\varpi\mathbb{D}^0_{k_0})$ but z_0 itself need not be a cycle. When z_0 is not itself a cycle, the subspace V may be empty. We hope in future work to find conditions which would imply that V is nonempty. This would provide a "bridge" from mod p homology to non-p-power-torsion homology in characteristic 0.

To prove the main results, we work directly with chains and use a variant of the Bockstein construction. We must assume that there is a Hecke operator u that acts completely continuously on the chain level and whose eigenvalue on ζ_0 is a p-adic unit. We use u to construct Hida-type idempotents on the chain level that project into a Hecke eigenspace.

We define and construct certain special kinds of orthonormal bases for the chains $C_*(\mathbb{D}_k)$ and use them to derive the main results of this paper in Section 7. In Section 8, we give an example for $\mathrm{GL}(3)/\mathbb{Q}$ in which the deformation space V is at least 2-dimensional. In principle we could compute the local equation of V near k_0 to any desired degree of accuracy.

In this example, the weight space Ω is 3-dimensional and we know from [2] that there cannot be a 3-dimensional space of deformations and that, modulo twisting, V contains at most finitely many classical points. The fact that there is a 2-dimensional space of deformations in this example can also be obtained (nonconstructively) by modifying an argument of Hida's in [10], as noted in the introduction to [2]. Similar results of Calegari and Mazur giving lower bounds on the dimension of the deformation space in the case of $\mathrm{GL}(2)/F$, F an imaginary quadratic field, may be found in [7].

The subspace V of Ω in Theorem 27, which parametrizes the deformations we construct, can be thought of as part of the projection to weight space of an "eigenvariety" E parametrizing the set of all p-adic deformations of Hecke eigenclasses of finite slope. A point in E is a pair (k, θ) where θ parametrizes a Hecke eigenclass in $H^*(\Gamma, \mathbb{D}_k)$. When Γ is an arithemtic group, the projection $E \to V$ is expected to be locally finite, so that E and V have locally the same dimension. There is presently no single agreed-upon definition of E. For two approaches, see [6] and [9].

Eric Urban [11] has made some conjectures on the dimension of E. We assume that the arithmetic group Γ is of an appropriate type, having p in its level. Let ℓ be the rank of the co-weight torus T. We denote by D_k an appropriate space of locally analytic distributions of weight k. Let θ be a system of Hecke eigenvalues occurring in the cohomology $H^*(\Gamma, D_k)$ with $k \in \mathrm{Hom}_{\mathrm{cont}}(T(\mathbb{Z}_p), \mathbb{C}_p^\times)$. A simple form of Urban's conjecture is the following:

Conjecture 1 (Urban). *Let $x = (k, \theta)$ be a point of the eigenvariety and assume that k is a classical weight. The irreducible components of* *the eigenvariety passing through x are all of dimension $\ell - \delta$ if and only if there exist two nonnegative integers a, b and a positive integer m such that*

(a) *the θ-generalized eigenspace of $H^r(\Gamma, D_k)$ is nonzero if and only if $a \le r \le b$ and its dimension is $m\binom{b-a}{r-a}$,*
(b) $\delta = b - a$.

There are cases where one expects several irreducible components of the eigenvariety of different dimensions. In those cases, Urban has an analogous conjecture.

Theorem 27 may be viewed as giving some evidence for Urban's conjecture in the case where $m = 1$, assuming that the generalized eigenspace of the cohomology mod ϖ with Hecke eigenvalues equal to those of θ (mod ϖ) has minimal possible dimension. For in this case, the integer d in our main result will coincide with the integer δ in Conjecture 1. (Cf. Theorem 5.1, p. 101 of [4].)

2. Arithmetic groups and chains. Let G be a reductive $\mathbb{Q}$-group, split at p. We fix Γ an arithmetic subgroup of $G(\mathbb{Q})$, S a subsemigroup of $G(\mathbb{Q})$, and $\mathcal{H} = \mathcal{H}(\Gamma, S)$ the Hecke algebra. We assume that $\mathcal{H}$ is commutative. Then $\mathcal{H}$ acts on the right of the homology of Γ with coefficients in any right $\mathbb{Z}_p[S]$-module M. Denote the homology of Γ with coefficients in M by $H_*(M)$.

By a theorem of Borel and Serre, any arithmetic group Γ is of type VFL (see, for example, [5, p. 218]). Therefore, by [5, Proposition 5.1, p. 197], Γ is of type FP_∞. Then by [5, Proposition 4.5, p. 195], the trivial Γ-module $\mathbb{Z}_p$ admits a resolution by free, finitely generated $\mathbb{Z}_p[\Gamma]$-modules.

Fix such a resolution. Denote the chains with coefficients in any right $\mathbb{Z}_p[\Gamma]$-module M with respect to this resolution by $C_*(M) = F_* \otimes_{\mathbb{Z}_p[\Gamma]} M$. Denote the boundary maps by ∂. If an element g of a ring or a group acts on an element m of a module on the right, we write the action as $m|g$. We have the following formula:

LEMMA 2. *For $i \ge 0$, let $\{f_{ij}\} \subset F_i$ be a free basis of F_i over $\mathbb{Z}_p[\Gamma]$ and $d_j \in M$. Write $\partial f_{ij} = \sum_b \lambda_b f_{i-1,b}|\gamma_b$ with $\lambda_b \in \mathbb{Z}_p$ and $\gamma_b \in \Gamma$. Then*

$$\partial\Big(\sum_j f_{ij} \otimes_\Gamma d_j\Big) = \sum_{j,b} \lambda_b f_{i-1,b} \otimes_\Gamma d_j|\gamma_b^{-1}$$

in $C_{i-1}(M)$.

Fix a free basis $\{f_{ij}\}$ of F_i over $\mathbb{Z}_p[\Gamma]$. Let r_i denote the rank of F_i over $\mathbb{Z}_p[\Gamma]$. For later use, define the following (noncanonical) isomorphisms of $\mathbb{Z}_p$-modules:

Definition 3. For any Γ-module M and any $i \geq 0$, let $\xi_i : C_i(M) \to M^{r_i}$ be defined by

$$\xi_i : \sum_j f_{ij} \otimes_\Gamma m_j \mapsto (m_j).$$

We lift the action of a single double coset in $\mathcal{H}$ from homology to chains as follows. Let $s \in S$. Set $\Delta = s^{-1}\Gamma s$, $\Delta^* = s\Gamma s^{-1}$. Write $\Gamma/\Gamma \cap \Delta^* = \coprod_t \gamma_t(\Gamma \cap \Delta^*)$. Define a new resolution of $\mathbb{Z}_p$ by $\mathbb{Z}_p[\Delta]$-modules and denote it by $F_*^\bullet$. The underlying $\mathbb{Z}_p$-modules are the same as in F_*, but the action of $\delta \in \Delta$ is given by $f \bullet \delta = f|s\delta s^{-1}$. Fix a homotopy equivalence $\tau : F_*^\bullet \to F_*$ of $\Gamma \cap \Delta$-modules, so that $\tau(f|s\delta s^{-1}) = \tau(f)|\delta$ for any $\delta \in \Gamma \cap \Delta$.

Define $\widetilde{s}$ on $C_*(M)$ by the formula

$$(f \otimes_\Gamma m)|\widetilde{s} = \sum_t \tau(f|\gamma_t) \otimes_\Gamma m|\gamma_t s, \tag{1}$$

extended by linearity. We have:

Lemma 4. *$\widetilde{s}$ is a well-defined $\mathbb{Z}_p$-linear chain map (i.e. commutes with ∂) and induces the action of the Hecke operator $\Gamma s \Gamma$ on $H_*(M)$.*

Remark 5. If M is an $A[S]$-module for some $\mathbb{Z}_p$-algebra A, we see that $C_*(M)$ is an A-module where A acts on the tensor product $F_* \otimes_{\mathbb{Z}_p[\Gamma]} M$ through the second factor. Then ∂ and $\widetilde{s}$ are A-module maps. Moreover, suppose that M_i is an $A_i[S]$-module for some $\mathbb{Z}_p$-algebra A_i, $i = 1, 2$, and that we have compatible homomorphisms of algebras $f : A_1 \to A_2$ and modules $\phi : M_1 \to M_2$. Then we obtain an induced chain map $\phi_* : C_*(M_1) \to C_*(M_2)$ which is linear with respect to f and $\widetilde{s}$-equivariant, by the formula

$$\sum_j f_{ij} \otimes_\Gamma m_i \mapsto \sum_j f_{ij} \otimes_\Gamma \phi(m_i).$$

3. ϖ-adic families of Banach modules. Let K_a denote a fixed algebraic closure of K. Use $\|\cdot\|$ to denote the norm on any complete K-algebra B. Let B^0 denote the closed unit ball in B and set $\overline{B} = B^0/\varpi B^0$. We use the definitions and conventions of [8] for Banach K-algebras and Banach modules over such algebras.

Definition 6. If D is a left A-Banach module over a Banach K-algebra A and also a right S-module, for a semigroup S that acts via A-module Banach morphisms of operator norm ≤ 1, such that the A- and S-actions commute, then we will say that D is an *A-S-module.*

Note that D^0 is then an A^0-S-module and $\overline{D}$ is an $\overline{A}$-S-module.

We say that D is *ON-able* if it has an ON (orthonormal) basis (see [8]). If $\{d_r\}$ is an ON basis for D over K, and B is any K-Banach module, then $\{1 \otimes d_r\}$ is an ON basis for $B \hat{\otimes}_K D$ over B. Since any K-Banach space is

ON-able over K, so is $B \hat{\otimes}_K D$ over B, and $(B \hat{\otimes}_K D)^0 = B^0 \hat{\otimes}_{\mathcal{O}} D^0$. The next lemma follows immediately from Lemma 1.1 of [8].

LEMMA 7. *Let A be a Banach K-algebra and D a Banach A-module such that $\|K\| = \|A\| = \|D\|$. Then a subset T of D is an ON basis for D over A if and only if $T \subset D^0$ and the image of T in $\overline{D}$ is a free basis of $\overline{D}$ over $\overline{A}$.*

Let Ω be a fixed K-affinoid rigid analytic space such that $\Omega(K) \neq \emptyset$. We denote its affinoid algebra by A_Ω. By $k \in \Omega$ we mean $k \in \Omega(K_a)$. Let $K(k)$ denote the field $\mathcal{O}_{\Omega,k}/\mathfrak{m}_{\Omega,k}$ (which is a finite extension of K), $\mathcal{O}_k$ its ring of integers, ϖ_k a uniformizer of it and $\mathbb{F}(k) = \mathcal{O}_k/(\varpi_k)$.

DEFINITION 8. Let $\mathbb{D}$ be a K-Banach space and S a semigroup. Set $\mathbb{D}_\Omega = A_\Omega \hat{\otimes}_K \mathbb{D}$ with the obvious A_Ω-Banach module structure, with A_Ω acting on the left. A *ϖ-adic family* of S-modules over Ω of type $\mathbb{D}$ is an A_Ω-S-module structure on the A_Ω-Banach module $\mathbb{D}_\Omega$.

Let $\mathbb{D}_\Omega$ be a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$. Given $k \in \Omega$, let $\mathrm{ev}_k : A_\Omega \to K(k)$ denote evaluation at k. We obtain the $K(k)$-S-module $\mathbb{D}_k := K(k) \hat{\otimes}_{A_\Omega,\mathrm{ev}_k} \mathbb{D}_\Omega$. We denote the resulting S-action on $\mathbb{D}_k$ by $d|^k s$.

For each $k \in \Omega$, transitivity of tensor product gives a natural identification $\mathbb{D}_k = K(k) \otimes_K \mathbb{D}$ as $K(k)$-Banach spaces. For $s \in S$, $\mathbb{D}_k^0|^k s \subset \mathbb{D}_k^0$.

If $y \in \mathbb{D}_\Omega = A_\Omega \hat{\otimes} \mathbb{D}$, let $y(k)$ denote the specialization of y at $k \in \Omega$. We extend specialization to chains as follows: Apply Remark 5 to the case $A_1 = A_\Omega$, $A_2 = K(k)$, $M_1 = \mathbb{D}_\Omega$ and $M_2 = \mathbb{D}_k$, with f being evaluation at k and ϕ specialization at k. Denote ϕ_* by $\sigma_k : C_i(\mathbb{D}_\Omega) \to C_i(\mathbb{D}_k)$. If $s \in S$, this is an $\widetilde{s}$-equivariant chain map. We can describe σ_k alternatively in terms of the ξ_i's of Definition 3 as the map $\xi_{i,k}^{-1} \circ \phi^{r_i} \circ \xi_{i,\Omega}$.

Suppose $\mathbb{D}_\Omega$ is a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$. The chains $C_j(\mathbb{D}_\Omega)$ are given the structure of A_Ω-Banach module via the isomorphism $\xi_j : C_j(\mathbb{D}_\Omega) \to \mathbb{D}_\Omega^{r_i} = A_\Omega \hat{\otimes} \mathbb{D}^{r_j}$ (Definition 3). Then $C_j(\mathbb{D}_\Omega^0) = C_j(\mathbb{D}_\Omega)^0$.

Denote the boundary maps by ∂_Ω in $C_*(\mathbb{D}_\Omega)$ and ∂_k in $C_*(\mathbb{D}_k)$ for any $k \in \Omega$. From Lemmas 2 and 4 we obtain:

LEMMA 9. *Let $\mathbb{D}_\Omega$ be a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$ and $j \geq 0$. Then $\widetilde{s}$ (given by formula (1)) acting on the right of $C_j(\mathbb{D}_\Omega)$ is a morphism of A_Ω-Banach modules of operator norm ≤ 1. Similarly, $\partial_\Omega : C_j(\mathbb{D}_\Omega) \to C_{j-1}(\mathbb{D}_\Omega)$ is a morphism of A_Ω-Banach modules of operator norm ≤ 1, and it commutes with $\widetilde{s}$. These statements remain true if Ω is replaced with k and A_Ω with $K(k)$, for any $k \in \Omega$.*

4. Families that are constant modulo ϖ

DEFINITION 10. The ϖ-adic family $\mathbb{D}_\Omega$ of S-modules over Ω of type $\mathbb{D}$ is constant modulo ϖ if there exists an S-module structure on $\overline{\mathbb{D}}$ such that

$\overline{\mathbb{D}}_\Omega$ is isomorphic as an $\overline{A}_\Omega$-S module to $\overline{A}_\Omega \otimes_{\mathbb{F}} \overline{\mathbb{D}}$, where S acts on the latter via the right factor and $\overline{A}_\Omega$ via the left factor.

Example 11. The following construction may be found in detail in [3]. Recall that G is a reductive $\mathbb{Q}$-group split at p. Let $K = \mathbb{Q}_p$.

Let T denote a maximal K-split torus of G_p. Let $\mathcal{X}$ denote the K-rigid analytic space that parametrizes $\mathbb{C}_p$-valued characters of $T(\mathbb{Z}_p)$. If $k \in \mathcal{X}(K_a)$, we use the notation t^k to denote the evaluation of k on t. We fix an open K-affinoid neighborhood Ω of k_0 with the properties

(1) $A_\Omega = \mathcal{O}_{\mathcal{X}}(\Omega)$ is a Tate algebra;
(2) for any $t \in T(\mathbb{Z}_p)$, the function on Ω sending t to t^k is in A_Ω;
(3) $t^k \equiv t^{k_0} \pmod p$ for all $k \in \Omega$ and $t \in T(\mathbb{Z}_p)$.

Let I be an Iwahori subgroup of $G(\mathbb{Z}_p)$ containing $T(\mathbb{Z}_p)$. Let X denote the corresponding "big cell". It is a p-adic manifold. There is a cone T^+ in $T(\mathbb{Q}_p)$ such that the semigroup T^+I acts naturally on X on the right. We construct $\mathbb{D} := \mathbb{D}_{k_0}$ as a family of distributions on X, making T act through the character k_0. Then $\mathbb{D}_\Omega$ is a family of T^+I-modules over Ω of type $\mathbb{D}$ which is constant modulo ϖ. For each k, T^+I acts on $\mathbb{D}_k$ through the character k. Let $\Gamma' \subset G(\mathbb{Z})$ be a congruence subgroup and $S' \subset G(\mathbb{Q})$ a subsemigroup containing Γ'. Set $\Gamma = \Gamma' \cap I$ and $S = S' \cap T^+I$. We assume that (Γ, S) is a Hecke pair with commutative Hecke algebra.

We choose an element $\pi \in T^+$ such that all the negative roots evaluated on π have positive p-adic ord. Then Hypothesis 17 below will hold if Hypothesis 16 holds for this π. In this case, Theorem 6.4.1 of [3] implies that any ordinary package of Hecke eigenvalues occurring on $H_*(\mathbb{D}_k)$ also occurs in $H_*(V_k)$ if k is integral dominant and V_k is the finite-dimensional irreducible module with weight k.

We return to the general situation. If $\mathbb{D}_\Omega$ is constant modulo ϖ, then $d|^{k_1}s \equiv d|^{k_2}s \pmod \varpi$ for any $k_1, k_2 \in \Omega$, $d \in \mathbb{D}^0$ and $s \in S$. Thus, the action of S on $\overline{x} \in \overline{\mathbb{D}}_k$ is independent of k and will be denoted simply by $\overline{x}|s$. Also the homology groups $H_*(\overline{\mathbb{D}}_k)$ as $\mathcal{H}$-modules are independent of k.

Fix $k_0 \in \Omega(K)$. The following lemma follows directly from the definitions:

Lemma 12. *Let $\mathbb{D}_\Omega$ be a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$ that is constant modulo ϖ. Let $i \geq 0$ and $s \in S$.*

(1) *Define $\eta_k : C_i(\overline{\mathbb{D}}_k) \to \mathcal{O}_k/\varpi \otimes_{\mathbb{F}} C_i(\overline{\mathbb{D}}_{k_0})$ by*

$$\eta_k\Big(\sum_\beta f_\beta \otimes_\Gamma \overline{d}_\beta\Big) = \sum_\beta f_\beta \otimes_\Gamma \overline{d}_\beta$$

for any $f_\beta \in F_i$ and $\overline{d}_\beta \in \overline{\mathbb{D}}$. Then η_k is an isomorphism of $\mathcal{O}_k/\varpi$-Banach modules which is equivariant for the action of $\widetilde{s}$.

(2) *Define $\eta_\Omega : C_i(\overline{\mathbb{D}}_\Omega) \to \overline{A}_\Omega \otimes_{\mathbb{F}} C_i(\overline{\mathbb{D}}_{k_0})$ by*

$$\eta_\Omega\Big(\sum_\beta f_\beta \otimes_\Gamma (\overline{a}_\beta \otimes_{\mathbb{F}} \overline{d}_\beta)\Big) = \sum_\beta \overline{a}_\beta \otimes_{\mathbb{F}} (f_\beta \otimes_\Gamma \overline{d}_\beta)$$

for any $f_\beta \in F_i$, $\overline{d}_\beta \in \overline{\mathbb{D}}$, and $\overline{a}_\beta \in \overline{A}_\Omega$. Then η_Ω is an isomorphism of $\overline{A}_\Omega$-Banach modules which is equivariant for the action of $\widetilde{s}$.

(3) *$\eta_k \circ \sigma_k = \sigma_k \circ \eta_\Omega$, where as above σ_k denotes specialization at k.*

Definition 13. Let $\hat{x} \in C_i(\mathbb{D}^0_{k_0})$. A *proper lift* of $\hat{x}$ is an element $x^\circ \in C_i(\mathbb{D}^0_\Omega)$ such that

(1) $\sigma_{k_0}(x^\circ) = \hat{x}$;
(2) $\eta_k(\rho(\sigma_k(x^\circ))) = \rho(\hat{x})$ for all $k \in \Omega$, where ρ denotes reduction modulo ϖ.

Lemma 14. *Let $\mathbb{D}_\Omega$ be a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$ that is constant modulo ϖ. Then proper lifts always exist.*

Proof. Let $\hat{x} \in C_i(\mathbb{D}^0_{k_0})$. Let $\xi_\Omega : C_i(\mathbb{D}_\Omega) \to \mathbb{D}^{r_i}_\Omega = A_\Omega \hat{\otimes} \mathbb{D}^{r_i}_{k_0}$ and $\xi_{k_0} : C_i(\mathbb{D}_{k_0}) \to \mathbb{D}^{r_i}_{k_0}$ be the maps defined in Definition 3. Let 1_Ω denote the constant function 1 on Ω. Then $\xi^{-1}_{i,\Omega}(1_\Omega \otimes \xi_{k_0}(\hat{x}))$ is a proper lift of $\hat{x}$. ∎

The following lemma gives us a general way to find ON bases for chain spaces.

Lemma 15. *Let $\mathbb{D}_\Omega$ be a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$ that is constant modulo ϖ. Let $\{\hat{z}_m\}$ be a set of elements in $C_i(\mathbb{D}^0_{k_0})$. Let $\overline{z}_m$ be the reduction modulo ϖ of $\hat{z}_m$ in $C_i(\overline{\mathbb{D}}_{k_0})$. For each m, choose a proper lift z°_m of $\hat{z}_m$. Assume that $\{\overline{z}_m\}$ is an $\mathbb{F}$-basis of $C_i(\overline{\mathbb{D}}_{k_0})$. Then $\{z^\circ_m\}$ is an ON basis of $C_i(\mathbb{D}_\Omega)$.*

Proof. By Lemma 7 it is enough to show that $\{\rho(z^\circ_m)\}$ is a free $\overline{A}_\Omega$-basis of $C_i(\overline{\mathbb{D}}_\Omega)$. By Lemma 12 it suffices to see this after applying η_Ω. But $\eta_\Omega(\rho(z^\circ_m)) = \overline{1}_\Omega \otimes \overline{z}_m \in \overline{A}_\Omega \otimes_{\mathbb{F}} C_i(\overline{\mathbb{D}}_{k_0})$, as may be seen by specializing at each k, using Lemma 12(3), and the definition of proper lift. ∎

5. Hypotheses on the "U_p" operator. For the rest of this paper we assume that $\mathbb{D}_\Omega$ is a ϖ-adic family of S-modules over Ω of type $\mathbb{D}$ and constant modulo ϖ. Let $\overline{\mathbb{F}}$ be the residue field of K_a, which is an algebraic closure of $\mathbb{F}$.

Fix $\alpha : \mathcal{H} \to \overline{\mathbb{F}}$ a ring homomorphism. For any $\mathcal{H} \otimes \overline{\mathbb{F}}$-module W, denote by W_α the generalized α-eigenspace of W.

Fix $\pi \in S$ and a p-adic unit $\lambda \in \mathcal{O}$ and set $u = \lambda^{-1}\Gamma\pi\Gamma \in \mathcal{H}$. We shall assume the following Hypotheses 16 and 17.

HYPOTHESIS 16. $\alpha(u) = 1$.

Since $\alpha(u) = 1$, we have $H_j(\widetilde{\mathbb{D}}_k)_\alpha \subset H_j(\widetilde{\mathbb{D}}_k)|u$.

Set $\upsilon = \lambda^{-1}\widetilde{\pi}$, where $\widetilde{\pi}$ on $C_*(\mathbb{D}_\Omega)$ is given by formula (1) in Section 2 (for $s = \pi$). Thus, for any j and any $k \in \Omega$, we obtain $|\upsilon$ acting on $C_j(\mathbb{D}_\Omega)$, stabilizing $C_j(\mathbb{D}^0_\Omega)$, and $|^k\upsilon$ acting on $C_j(\mathbb{D}_k)$, stabilizing $C_j(\mathbb{D}^0_k)$.

HYPOTHESIS 17. For each $j \geq 0$, υ acts completely continuously on $C_j(\mathbb{D}_\Omega)$.

If π acts completely continuously on $\mathbb{D}_\Omega$, then Hypothesis 17 holds automatically.

Set $\widetilde{\mathbb{D}}_k = \mathbb{D}^0_k/\varpi_k\mathbb{D}^0_k$. The next lemma follows immediately from complete continuity and the fact that $\alpha(u) = 1$.

LEMMA 18. *Let $j \geq 0$ and $k \in \Omega$. The following modules are finitely generated:* (1) $C_j(\overline{\mathbb{D}}_\Omega)|\upsilon$ *over* $\overline{A}_\Omega$*;* (2) $C_i(\widetilde{\mathbb{D}}_k)|\upsilon$ *over* $\mathbb{F}(k)$*;* (3) $H_j(\widetilde{\mathbb{D}}_k)|u$ *over* $\mathbb{F}(k)$*;* (4) $H_j(\widetilde{\mathbb{D}}_k)_\alpha$ *over* $\mathbb{F}(k)$.

DEFINITION 19. Set $d_i(k) = \dim_{\mathbb{F}(k)} H_{i-1}(\widetilde{\mathbb{D}}_k)_\alpha$.

LEMMA 20. *Let $j \geq 0$, $\{t_a \mid a = 1, \ldots, n\} \subset \mathcal{H}$, $\{c_a \mid a = 1, \ldots, n\} \subset \mathcal{O}_{K_a}$. Let $\tau = (\widetilde{t}_1 - c_1)\cdots(\widetilde{t}_n - c_n)$ where each $\widetilde{t}_a$ is the lift of t_a to the chain level given by formula* (1). *Let*

$$e(\upsilon\tau) = \lim_{m\to\infty}(\upsilon\tau)^{m!}.$$

Then the limit exists and $e(\upsilon\tau)$ acts on $C_j(\mathbb{D}_\Omega)$ as an A_Ω-linear idempotent (*hence of norm* 1, *unless* $e(\upsilon\tau) = 0$).

Proof. This follows from Lemma 31 in Section 9 applied to $\xi_i^{-1}\circ(\upsilon\tau)\circ\xi_i$ (where ξ_i is the isometry defined in Definition 3). ∎

We are going to create this idempotent when τ is chosen to project onto the α-eigenspace, as follows:

LEMMA 21. *Let J be a finite set of nonnegative integers and $k \in \Omega$. There exists a finite extension K'/K and $T \in \mathcal{H}\otimes K'$ such that for all $j \in J$,*

(1) *T induces an idempotent in* $\mathrm{End}_{\mathbb{F}(k)}(H_j(\widetilde{\mathbb{D}}_k)|u)$, *and*

(2) *T projects $H_j(\widetilde{\mathbb{D}}_k)|u$ onto $H_j(\widetilde{\mathbb{D}}_k)_\alpha$.*

Proof. The finite-dimensional $\mathbb{F}(k)$-vector space $\bigoplus_J H_j(\widetilde{\mathbb{D}}_k)|u$ can be decomposed over the algebraic closure $\overline{\mathbb{F}}$ into generalized $\mathcal{H}$-eigenspaces. For each homomorphism $\beta : \mathcal{H} \to \overline{\mathbb{F}}$ such that $\beta \neq \alpha$ and with the property that $[\bigoplus_J H_j(\widetilde{\mathbb{D}}_k)|u]_\beta \neq 0$, choose $T_\beta \in \mathcal{H}$ such that $\alpha(T_\beta) \neq \beta(T_\beta)$. Let e_β be the cardinality of $\mathbb{F}(k)[\beta(T_\beta)]^\times$. Fix a lifting of β to $\hat{\beta} : \mathcal{H} \to \mathcal{O}_{K_a}$. Then we may take

$$T = \prod_{\beta \neq \alpha} (T_\beta - \hat{\beta}(T_\beta))^{e_\beta \delta}$$

where δ is a sufficiently large power of p. ■

Fix $k \in \Omega$ and choose T_β's as in the proof of Lemma 21. We lift each T_β to a chain map $\widetilde{T}_\beta$ as in equation (1) of Section 2. Set

$$\tau = \prod_{\beta \neq \alpha} (\widetilde{T}_\beta - \hat{\beta}(T_\beta))^{e_\beta \delta}$$

where the product is taken in any (fixed) order. We also enlarge K if necessary, so that we may assume that $k \in \Omega(K)$ and all $\hat{\beta}(T_\beta) \in K$.

DEFINITION 22. With the choices above, set $e = e(\upsilon\tau)$.

LEMMA 23. *e is an A_Ω-linear idempotent on $\bigoplus_J H_j(\mathbb{D}_\Omega)$ and for each $k \in \Omega$, $\bigoplus_J H_j(\widetilde{\mathbb{D}}_k)|e = \bigoplus_J H_j(\widetilde{\mathbb{D}}_k)_\alpha$.*

Proof. This follows from the definition of e, the fact that u acts invertibly on $\bigoplus_J H_j(\widetilde{\mathbb{D}}_k)_\alpha$, and Lemma 21. ■

6. ON bases of type (k, e, j). Recall that we have denoted the reduction map modulo ϖ by ρ. We now define a special kind of ON basis for $C_j(\mathbb{D}_\Omega)$.

Denote groups of boundaries by B_* and groups of cycles by Z_*. By Hypothesis 17, $B_j(\overline{\mathbb{D}}_k)|e$ is a finite-dimensional $\mathbb{F}$-vector space. The following lemma is obvious because ρ, e and ∂ all commute with each other.

LEMMA 24. *$\{\rho(\partial_k \hat{v}) \mid \hat{v} \in C_{j+1}(\mathbb{D}_k^0)|e\} = B_j(\overline{\mathbb{D}}_k)|e \subset Z_j(\overline{\mathbb{D}}_k)|e$.*

DEFINITION 25. Let $j \in J$ and assume that $j+1 \in J$. We fix $k \in \Omega(K)$, chain maps υ and τ as above, and denote by e the idempotent $e(\upsilon\tau)$.

An *ON basis of type* (k, e, j) is a triple $(\mathcal{B}, X, Y)$ where X and Y are finite sets, $X \in C_j(\mathbb{D}_k^0)|e$, $Y \in C_{j+1}(\mathbb{D}_k^0)|e$ and $\mathcal{B}$ is an ON basis of $C_j(\mathbb{D}_\Omega)$ such that

$$\mathcal{B} = \{b_p, w_q, c_r\} \subset C_j(\mathbb{D}_\Omega^0),$$

where

(1) for each q, there exists $\hat{x}_q \in X$ such that w_q is a proper lift of $\hat{x}_q$ in $C_j(\mathbb{D}_\Omega^0)|e$;
(2) for each p, there exists $\hat{y}_p \in Y$ such that $b_p = \partial_\Omega(y_p^\circ)$ for some proper lift y_p° of $\hat{y}_p$ in $C_{j+1}(\mathbb{D}_\Omega^0)|e$;
(3) the set $\{\rho(\partial_k \hat{y}) \mid \hat{y} \in Y\}$ is an $\mathbb{F}$-basis of $B_j(\overline{\mathbb{D}}_k)|e$, and $\{\rho(\hat{x}), \rho(\partial_k \hat{y}) \mid \hat{x} \in X, \hat{y} \in Y\}$ is an $\mathbb{F}$-basis of $Z_j(\overline{\mathbb{D}}_k)|e$.

Lemma 26. *An ON basis* $\{b_p, w_q, c_r\}$ *of type* (k, e, j) *exists for* $C_j(\mathbb{D}_\Omega)$. *The index* q *takes on* $d = d_{j+1}(k)$ *values* (*Definition* 19).

Proof. We know that $C_j(\overline{\mathbb{D}}_k)|e$ is finite and that

$$Z_j(\overline{\mathbb{D}}_k)|e/B_j(\overline{\mathbb{D}}_k)|e = H_j(\overline{\mathbb{D}}_k)|e = H_j(\overline{\mathbb{D}}_k)_\alpha$$

has $\mathbb{F}$-dimension d (see Lemma 23). Therefore, in view of Lemma 24, we can choose a finite set $Y = \{\hat{y}_p\} \subset C_{j+1}(\mathbb{D}_k^0)|e$ such that

$$\{\rho(\partial_k \hat{y}) \mid \hat{y} \in Y\}$$

is an $\mathbb{F}$-basis of $B_j(\overline{\mathbb{D}}_k)|e$, and a finite set $X = \{\hat{x}_q\} \subset C_j(\mathbb{D}_k^0)|e$ of cardinality d such that

$$\{\rho(\hat{x}), \rho(\partial_k \hat{y}) \mid \hat{x} \in X,\, \hat{y} \in Y\}$$

is an $\mathbb{F}$-basis of $Z_j(\overline{\mathbb{D}}_k)|e$. Let w_q be a proper lift of $\hat{x}_q$ in $C_j(\mathbb{D}_\Omega^0)|e$ and let $b_p = \partial_\Omega(y_p^\circ)$ where y_p° is a proper lift of $\hat{y}_p$ in $C_{j+1}(\mathbb{D}_\Omega^0)|e$. Next choose $\hat{c}_r \in C_j(\mathbb{D}_k^0)$ such that $\{\rho(\hat{y}_p), \rho(\hat{x}_q), \rho(\hat{c}_r)\}$ is an $\mathbb{F}$-basis of $C_j(\overline{\mathbb{D}}_k)$. Let c_r be a proper lift of $\hat{c}_r$. Using Lemma 15 we see that $\{b_p, w_q, c_r\}$ is an ON basis of type (k, e, j) of $C_j(\mathbb{D}_\Omega^0)$. ∎

7. Main theorem

Theorem 27. *Fix* i *and set* $J = \{i-1, i, i+1\}$. *Let* υ, τ, *and* e *be as in Definition* 22 *for* $k = k_0$. *Let* $(\mathcal{B}, X, Y) = \{b_p, w_q, c_r\}$ *be an ON basis of* $C_{i-1}(\mathbb{D}_\Omega)$ *of type* $(k_0, e, i-1)$. *For each* $\hat{y}_p \in Y$, *set* $\beta_p = y_p^\circ$, *so that* $b_p = \partial_\Omega(\beta_p)$.

Consider a chain $z \in C_i(\mathbb{D}_{k_0}^0)$. *Let* $\overline{z} \in C_i(\overline{\mathbb{D}}_{k_0})$ *be the reduction modulo* ϖ *of* z. *Assume that* $\overline{z}$ *is a cycle and let* ζ *denote its homology class. Assume further that* $\zeta \in H_i(\overline{\mathbb{D}}_{k_0})_\alpha - \{0\}$. *Choose a proper lift* z° *of* z.

Write

$$\partial_\Omega(z^\circ)|e = \sum_p f_p b_p + \sum_{q=1}^{d_i(k_0)} g_q w_q + \sum_r h_r c_r \tag{2}$$

for some functions $f_p, g_q, h_r \in A_\Omega$.

Let

$$Z = (z^\circ)|e - \sum_p f_p \beta_p$$

and V *be the zero locus of the ideal generated by the* g_q *in* A_Ω. *Then*

(1) *If* $k \in V$, *then* $Z(k)$ *is a cycle in* $Z_i(\mathbb{D}_k^0)$ *and the homology class of the reduction modulo* ϖ *of* $Z(k)$ *is congruent to* ζ *modulo* ϖ *and in particular is nonzero.*

(2) *If* $k \in V$, *then for any* r, $h_r(k) = 0$.

Proof. Recall that for any $k \in \Omega$, ϖ_k is a uniformizer in $\mathcal{O}_k$. Write $\overline{a} \in \mathcal{O}_k/\varpi$ for the reduction of a modulo ϖ (not modulo ϖ_k) for $a \in \mathcal{O}_k$.

(1) First note that in fact $f_p, g_q, h_r \in A^0_\Omega$, because the left hand side of equation (2) is integral and $\mathcal{B}$ is an ON basis. Since the family $\mathbb{D}_\Omega$ is constant modulo ϖ and $\overline{z}$ is a cycle, we obtain, upon reducing formula (2) modulo ϖ and specializing at any $k \in \Omega$,

$$0 = \sum_p \overline{f}_p(k)\overline{b}_p(k) + \sum_q \overline{g}_q(k)\overline{w}_q(k) + \sum_r \overline{h}_r(k)\overline{c}_r(k).$$

By the freeness of $\{\overline{b}_p(k), \overline{w}_q(k), \overline{c}_r(k)\}$ over $\mathcal{O}_k/\varpi$, we conclude that $0 \equiv f_p(k) \equiv g_q(k) \equiv h_r(k) \pmod{\varpi}$ for all k, p, q, r. In particular, for all k, $Z(k) \equiv z|^k e \pmod{\varpi}$. Therefore, if we reduce $Z(k)$ modulo ϖ, the resulting cycle is homologous to $\zeta|e = \zeta$.

Set $d = d_i(k_0)$. By equation (2),

$$\partial_\Omega Z = \sum_{q=1}^{d} g_q w_q + \sum_r h_r c_r.$$

Therefore for any $k \in \Omega$,

$$\partial_k(Z(k)) = \sum_{q=1}^{d} g_q(k) w_q(k) + \sum_r h_r(k) c_r(k).$$

Fix a k such that $g_q(k) = 0$ for all q. Then

$$\partial_k(Z(k)) = \sum_r h_r(k) c_r(k).$$

Now suppose that $\sum_r h_r(k)c_r(k) \neq 0$. Let m be the largest integer such that $h_r(k)$ is divisible by ϖ_k^m for all r. Since ∂_k and $|^k e$ both commute with multiplication by constants in $K(k)$, and because $Z|e = Z$, we see that

$$\partial_k\left(\frac{Z(k)}{\varpi_k^m}\right) = \sum_r \frac{h_r(k)}{\varpi_k^m} c_r(k)$$

is fixed under $|^k e$. Also, for some r, ϖ_k^{m+1} does not divide $h_r(k)$.

Although $Z(k)/\varpi_k^m$ is not necessarily integral, the right hand side of the preceding formula shows that $\partial_k(Z(k)/\varpi_k^m)$ is integral, i.e. it is in $C_{i-1}(\mathbb{D}_k^0)$, because of how we chose m. Now reduce both sides modulo ϖ_k. Because $\partial_k^2 = 0$, the left hand side modulo ϖ_k reduces to a cycle in $Z_{i-1}(\mathbb{D}_k^0/\varpi_k\mathbb{D}_k^0)|e \approx \mathcal{O}_k/\varpi_k \otimes_{\mathcal{O}} Z_{i-1}(\overline{\mathbb{D}}_{k_0})|e$. Denoting reduction modulo ϖ_k by a tilde, we see that the left hand side modulo ϖ_k is in the $\mathcal{O}_k/\varpi_k$-span of $\{\tilde{b}_p(k), \tilde{w}_q(k)\}$.

But the right hand side reduces modulo ϖ_k to something nonzero in the $\mathcal{O}_k/\varpi_k$-span of $\{\tilde{c}_r(k)\}$. This contradicts the freeness of $\{\tilde{b}_p(k), \tilde{w}_q(k), \tilde{c}_r(k)\}$ over $\mathcal{O}_k/\varpi_k$. Hence $\partial_k(Z(k)) = \sum_r h_r(k)c_r(k) = 0$.

(2) What we have just seen implies that if $k \in V$ then $h_r(k) = 0$ for all r, since the c_r are part of an ON basis. ∎

We do not know how V depends on the various choices made. In general, V may be empty. We can get a lower bound on the dimension of V as follows.

Corollary 28. *Assume that A_Ω is a Tate algebra and let $d = d_i(k_0)$. Assume $k_0 \in \Omega(K)$ and $\eta_{k_0} \in H_i(\mathbb{D}^0_{k_0})$ are given such that the reduction modulo ϖ of η_{k_0} is an α-eigenclass. Then there exists a Zariski-closed subspace V of Ω of dimension at least $\dim \Omega - d$ such that for each $k \in V$, there exists $\eta_k \in H_i(\mathbb{D}^0_k)$ such that $\tilde{\eta}_k \in H_i(\mathbb{D}^0_k/\varpi_k \mathbb{D}^0_k)_\alpha - \{0\}$. The family $\{\eta_k\}$ is analytic in the sense that there exists $Z \in A_V \otimes C_i(\mathbb{D}_{k_0})$ such that for each $k \in V$, $Z(k)$ is a cycle and η_k is its homology class. Moreover, if $\eta_{k_0}|^{k_0}e = \eta_{k_0}$, then the homology class of $Z(k_0)$ is η_{k_0}.*

Proof. Let z in Theorem 27 be a cycle that represents η_{k_0}. Then $\partial_{k_0} z = 0$. From equation (2) specialized at k_0, we obtain

$$0 = \partial_{k_0} z|^{k_0} e = \sum_p f_p(k_0) b_p(k_0) + \sum_{q=1}^{d} g_q(k_0) w_q(k_0) + \sum_r h_r(k_0) c_r(k_0).$$

Since $\{b_p(k_0), w_q(k_0), c_r(k_0)\}$ form an orthonormal basis, the coefficients in the sums must all vanish. In particular, $g_q(k_0) = 0$ for all q, so that $k_0 \in V$.

The Tate algebra A_Ω is a catenary ring. Since $k_0 \in V$, V is not empty. Hence the dimension of V is at least $\dim \Omega - d$ by the Hauptidealsatz and Lemma 26. The analytic nature of $Z(k)$ as a function of k is clear from its definition in Theorem 27. The last assertion of the corollary is clear. ∎

Remark 29. Assume that $\eta_{k_0}|e = \eta_{k_0}$ and that η_{k_0} is a Hecke eigenclass. The homology class of the cycle $Z(k)$ may not be a Hecke eigenclass. However, by Lemma 32 below, we can find a dense open affinoid subset U of a finite integral cover of V and an analytic family $Z'(k')$ over U consisting of cycles whose homology classes are Hecke eigenclasses whose eigenvalues modulo $\varpi_{k'}$ are given by α. If in addition a certain multiplicity 1 result holds for the homology over V, as specified in (3) of Lemma 32, we can find U such that there is a point k_* in U over k_0 and the Hecke eigenvalues of the homology class of $Z'(k_*)$ are the same as those of η_{k_0}.

Some or all of the homology classes η_k of Corollary 28 may be annihilated by a power of ϖ_k. The following theorem gives us a way to rule out this possibility under a certain condition.

Theorem 30. *Suppose $H_{i+1}(\mathbb{D}^0_{k_0}/\varpi \mathbb{D}^0_{k_0})_\alpha = 0$. Then for each $k \in V$, the homology class of $Z(k)$ in Corollary 28 is non-ϖ_k-power-torsion.*

Proof. This follows from a standard Bockstein argument. ∎

8. Remarks on computations. In this section, we use the families $\mathbb{D}_\Omega$ outlined in Example 11 in Section 4. The deformation space V in Theorem 27 is effectively computable up to any desired degree of accuracy. We can construct an ON basis of type $(k_0, e, i-1)$ in such a way that there is a partition of the r-indices into R_1 and R_2, with R_1 finite with the following properties:

(1) b_p, w_q and the c_r, $r \in R_1$, form a finite ON basis for $C_{i-1}(\mathbb{D}^0_\Omega)|e$;
(2) the c_r, $r \in R_2$, form an ON basis for $C_{i-1}(\mathbb{D}^0_\Omega)|(1-e)$.

For any n, the computation modulo ϖ^n of f_p, g_q and the $h_r, r \in R_1$, in equation (2) up to any desired degree of accuracy in the k-variable is a finite computation. We do not need to compute the h_r, $r \in R_2$. In particular we can compute the g_q modulo ϖ^n and thus their common zero set V modulo ϖ^n up to any degree of accuracy.

To get a resolution F_*, we can use an explicit finite cell complex for a classifying space of a normal torsion-free subgroup of Γ.

Consider the case of $\mathrm{GL}(3,\mathbb{Q})$. Fix a positive integer N. Let $S_0(N)$ denote the subgroup of $\mathrm{GL}(3,\mathbb{Z})$ consisting of matrices whose first row is congruent to $(*,0,0)$ modulo N, and with positive determinant. Set $\Gamma_0(N) = S_0(N) \cap \mathrm{SL}(3,\mathbb{Z})$.

Fix a prime p not dividing N. Let I denote the Iwahori subgroup of $\mathrm{GL}(3,\mathbb{Z}_p)$ consisting of matrices that become upper triangular modulo p. Set $S = S_0(N) \cap I$ and $\Gamma = \Gamma_0(N) \cap I$. We work with the Hecke pair (Γ, S). If ℓ is a prime, let $D_{\ell,i}$ denote the diagonal matrix with $3-i$ 1's and then i ℓ's down the diagonal, for $1 \le i \le 3$. Let $T_{\ell,i}$ denote the Hecke operator corresponding to the double coset $\Gamma D_{\ell,i} \Gamma$.

We choose $N = 61$, $p = 5$, and $k_0 =$ the trivial character. In [1] we computed the homology of $\Gamma_0(61)$ with trivial coefficients, together with the Hecke eigenvalues for $\ell \le 29$. Choose a square root w of -3 in $(\mathbb{Q}_5)_a$ and set $K = \mathbb{Q}_5[w]$. Up to scalar multiples, there is a unique quasicuspidal Hecke eigenclass in $H_3(\Gamma_0(61), K)$ with $T_{2,1}$ acting by w. (A "quasicuspidal" homology class is one that is not in the image of the Borel–Serre boundary homology.) This class satisfies all of our hypotheses. Using Theorem 30, we can show that the eigenclasses resulting from the application of Theorem 27 are non-p-power-torsion.

Theorem 6.4.1 in [3] plus unpublished computations of David Pollack show that $d = 1$ for this example. By the main results of [2], if we look at the subset V_0 of V consisting of $k \in V$ such that k is trivial on all $\mathrm{diag}(1,1,x) \in T(\mathbb{Z}_p)$, then at most finitely many of the weights in V_0 can be dominant integral. Therefore, in this example, the dimension of V is exactly 2. I hope to report in the future on computations in collaboration with David Pollack to determine approximations to the equation of V.

9. Lemmas. In this section we state two lemmas whose proofs use standard techniques and will be omitted.

LEMMA 31. *Let A be a reduced K-affinoid Banach algebra, and D an ON-able Banach K-module. Let $D_A = A \hat{\otimes} D$. Let $f_0 : D \to D$ and $f_1 : D_A \to D_A$ be completely continuous maps of norm ≤ 1, linear over K and A respectively. Let $f = f_0 + \varpi f_1 : D_A \to D_A$. Then $\lim_{m\to\infty} f^{m!}$ exists and is an A-linear idempotent. The rate of convergence is uniform over* $\mathrm{Sp}(A)$.

LEMMA 32. *Let $Z(k)$, $k \in V$, be the cycles constructed in Corollary* 28. *Let V' be an irreducible component of V with K-affinoid algebra R'. Assume that for every $k \in V'$, the homology class of $Z(k)$ (which we know is nonzero) is non-ϖ_k-power-torsion.*

(1) *There exists a finite, integral extension R'' of R', a special open subset $U \subset \mathrm{Sp}(R'')$, and $w \in \mathcal{O}_U \hat{\otimes}_{R'} H_i(R' \hat{\otimes}_{A_\Omega} \mathbb{D}_\Omega)|e$ such that w is a generalized λ-eigenvector for the action of the Hecke algebra $\mathcal{H}$ for some character $\lambda : \mathcal{H} \to \mathcal{O}_U$, w does not vanish at any point of U and w can be represented by a cocycle which is analytic on U.*
(2) *Let $\lambda_0 : \mathcal{H} \to K$ be the character giving the action of $\mathcal{H}$ on $Z(k_0)$. Let F'' be the field of fractions of R''. Take λ as in* (1). *Suppose there exists a point $k_* \in \mathrm{Sp}(R'')$ above k_0 such that, for every character $\mu : \mathcal{H} \to R''$ (other than λ) for which the generalized μ-eigenspace of $F'' \hat{\otimes}_{R'} H_i(R' \hat{\otimes}_{A_\Omega} \mathbb{D}_\Omega)|e$ is nonzero, there exist T_μ such that $\lambda(T_\mu)(k_*) \ne \mu(T_\mu)(k_*)$. Then we may choose U to contain k_*, and $\lambda(T)(k_*) = \lambda_0(T)$ for all $T \in \mathcal{H}$.*
(3) *Suppose there exists a point $k_* \in \mathrm{Sp}(R'')$ above k_0 such that, for at most one character $\nu : \mathcal{H} \to R''$, $F'' \hat{\otimes}_{R'} H_i(R' \hat{\otimes}_{A_\Omega} \mathbb{D}_\Omega)|e$ has nonzero generalized ν-eigenspace and $\nu(T)(k_*) = \lambda_0(T)$ for every $T \in \mathcal{H}$. Then such a ν does exist, and in* (1) *we may take $\lambda = \nu$ and U containing k_*.*

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Avner Ash
Department of Mathematics
Boston College
Chestnut Hill, MA 02467, U.S.A.
E-mail: ashav@bc.edu

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