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On the topology of sums in powers of an algebraic number

by

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1. Introduction and auxiliary results. Let $q \in (1,2)$ and put

$$\Lambda_n(q) = \Big\{ \sum_{k=0}^{n} a_k q^k \;\Big|\; a_k \in \{-1,0,1\} \Big\},$$

and $\Lambda(q) = \bigcup_{n\ge 1} \Lambda_n(q)$. (It is obvious that the sets $\Lambda_n(q)$ are nested.) The question we want to address is the topological structure of $\Lambda(q)$. Is it dense? discrete? mixed?

The first important result has been obtained by A. Garsia [12]: if q is a Pisot number (an algebraic integer greater than 1 whose conjugates are less than 1 in modulus), then $\Lambda(q)$ is uniformly discrete. On the other hand, if q does not satisfy an algebraic equation with coefficients $0, \pm 1$, then it is a simple consequence of the pigeonhole principle that 0 is a limit point of $\Lambda(q)$, and thus it is dense—see below.

Surprisingly little is known about the case when q is a root of a polynomial with coefficients $0, \pm 1$. The most notable result is [11, Theorem I] in which the authors prove in particular that if $q < (1+\sqrt{5})/2$ and q is not Pisot, then $\Lambda(q)$ has a finite accumulation point.

In this paper we study this case and give two sufficient conditions for $\Lambda(q)$ to be dense. These conditions are rather general and cover a substantial subset of such q's—see Theorems 2.1 and 2.4.

Put

$$Y_n(q) = \Big\{ \sum_{k=0}^{n} a_k q^k \;\Big|\; a_k \in \{0,1\} \Big\}$$

and $Y(q) = \bigcup_{n\ge 1} Y_n(q)$. The set $Y(q)$ is discrete and we can write its elements in ascending order:

$$Y(q) = \{0 = y_0(q) < y_1(q) < y_2(q) < \cdots\}.$$

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Following [11], we define

$$l(q) = \varliminf_{n\to\infty} (y_{n+1}(q) - y_n(q)).$$

THEOREM 1.1 ([8]). *If* 0 *is a limit point of* $\Lambda(q)$, *then* $\Lambda(q)$ *is dense in* $\mathbb{R}$.

It is obvious that 0 is a limit point of $\Lambda(q)$ if and only if $l(q) = 0$. This yields

COROLLARY 1.2. *The set* $\Lambda(q)$ *is dense in* $\mathbb{R}$ *if and only if* $l(q) = 0$.

The purpose of this paper is to find some wide classes of algebraic q for which $l(q) = 0$.

Put for any $\beta \in \mathbb{C}$,

$$Y_n(\beta) = \Big\{ \sum_{k=0}^{n} a_k \beta^k \;\Big|\; a_k \in \{0,1\},\ 0 \le k \le n \Big\}$$

and $z_n(\beta) := \#Y_n(\beta)$. It is obvious that $z_n(\beta) \le 2^{n+1}$.

In order to estimate $z_n(\beta)$ for $|\beta| > 1$, it is useful to consider the set

$$A_\lambda := \Big\{ \sum_{k=0}^{\infty} a_k \lambda^k \;\Big|\; a_k \in \{0,1\},\ k \ge 0 \Big\}, \qquad \text{where } \lambda = \beta^{-1}.$$

We have $|\lambda| < 1$, so the series converges for any choice of the coefficients $a_k \in \{0,1\}$. It is easy to see that the set A_λ is compact, being the image of the infinite product space $\{0,1\}^\infty$ under a continuous mapping. It satisfies the set equation

$$A_\lambda = \lambda A_\lambda \cup (1 + \lambda A_\lambda),$$

and can be characterized as the unique compact set with this property [14]. It is thus the attractor of the iterated function system $\{z \mapsto \lambda z,\ z \mapsto \lambda z + 1\}$ in the complex plane; see [14] for details.

The sets A_λ with $|\lambda| < 1$ have been extensively studied in the "fractal" literature; see e.g. [2, 4, 15, 21] and the book [3, Chapter 8.2]. Note that some of these sources are concerned with the sets

$$\widetilde{A}_\lambda := \Big\{ \sum_{k=0}^{\infty} a_k \lambda^k \;\Big|\; a_k \in \{-1,1\},\ k \ge 0 \Big\},$$

however, it is clear that $A_\lambda = T(\widetilde{A}_\lambda)$, where $T(z) = \frac{1}{2}(z + (1-\lambda)^{-1})$, so all the results immediately transfer.

LEMMA 1.3.

(i) *If* $\lambda \in \mathbb{C}$ *with* $|\lambda| \in (1/2, 1)$, *then* $z_n(\lambda) = \#Y_n(\lambda) \ge |\lambda|^{-n-1}$ *for all* n.

(ii) *If* $\lambda \in \mathbb{C}$ *with* $2^{-1/2} \le |\lambda| < 1$, *and* $|\mathrm{Re}\,\lambda| \le |\lambda|^2 - 1/2$, *then* $z_n(\lambda) \ge |\lambda|^{-2(n+1)}$ *for all* n.

Proof. By the definition of the set A_λ, we have, for all $n \ge 0$,

$$A_\lambda = \bigcup_{z \in Y_n(\lambda)} (z + \lambda^{n+1} A_\lambda). \tag{1.1}$$

(i) Suppose that the set A_λ is connected, and let $u, v \in A_\lambda$ be such that $|u - v| = \mathrm{diam}(A_\lambda)$. Then there exists a "chain" of distinct subsets $A_j := z_j + \lambda^n A_\lambda \subset A_\lambda$, $j = 1, \ldots, m$, with $z_j \in Y_n(\lambda)$, such that $u \in A_1, v \in A_m$ and $A_j \cap A_{j+1} \ne \emptyset$ for all $j \le m - 1$. Therefore,

$$\begin{aligned} \mathrm{diam}(A_\lambda) &\le \sum_{j=1}^m \mathrm{diam}(A_j) = m\,\mathrm{diam}(\lambda^{n+1} A_\lambda) \\ &\le \#Y_n(\lambda)|\lambda|^{n+1}\,\mathrm{diam}(A_\lambda), \end{aligned}$$

and the claim follows. If, on the other hand, A_λ is disconnected, then $\lambda A_\lambda \cap (\lambda A_\lambda + 1) = \emptyset$. This is a general principle for attractors of iterated function systems with two contracting maps (see [13, 4] or [3, Chapter 8.2]). Therefore, in this case λ is not a zero of a power series with coefficients $\{-1, 0, 1\}$, much less of a polynomial, hence $z_n(\lambda) = 2^{n+1} > |\lambda|^{-n-1}$ for all n.

(ii) By [21, Prop. 2.6(i)], in view of the above remark concerning $\widetilde{A}_\lambda$, we know that A_λ has nonempty interior for all λ in the open unit disc such that $0 \le |\mathrm{Re}\,\lambda| \le |\lambda|^2 - 0.5$. Then from (1.1), for the Lebesgue measure $\mathcal{L}^2$, we have

$$\mathcal{L}^2(A_\lambda) \le \#Y_n(\lambda)\mathcal{L}^2(\lambda^{n+1} A_\lambda) = z_n(\lambda)|\lambda|^{2(n+1)}\mathcal{L}^2(A_\lambda),$$

as desired. ∎

Note that the proof of Lemma 1.3 does not use the fact that λ is nonreal. Hence we obtain the following result as a direct corollary:

LEMMA 1.4. *If* $q \in (1, 2)$, *then* $z_n(\pm q) \ge C q^n$ *for some* $C > 0$.

REMARKS 1.5.

(i) Lemma 1.4 for $+q$ was proved in [11], using the fact that $y_{n+1}(q) - y_n(q) \le 1$ for all n and any $q \in (1, 2)$.
(ii) With a bit more work one can show that in the setting of Lemma 1.3(i) we have $z_n(\lambda) \ge C_n|\lambda|^{-n}$ for some $C_n \uparrow \infty$, assuming that λ is nonreal. However, this is not needed in this paper.
(iii) It follows from the results of [7, 17] that for any $\varphi \ne 0, \pi$, the set A_λ has nonempty interior for $\lambda = re^{i\varphi}$ with r sufficiently close to 1, but it seems difficult to apply them in the absence of quantitative estimates.

Lemma 1.6. *If $\beta \in \mathbb{C} \setminus \{0\}$, then $z_n(\beta) = z_n(1/\beta)$.*

Proof. Define $\phi : Y_n(\beta) \to Y_n(1/\beta)$ as follows:

$$\phi\Big(\sum_{k=0}^{n} a_k \beta^k\Big) = \sum_{k=0}^{n} a_{n-k}(1/\beta)^k.$$

A relation $\sum_{k=0}^{n} a_k\beta^k = \sum_{k=0}^{n} b_k\beta^k$ is equivalent to $\sum_{k=0}^{n} a_k\beta^{k-n} = \sum_{k=0}^{n} b_k\beta^{k-n}$, which in turn is equivalent to $\phi(\sum_{k=0}^{n} a_k\beta^k) = \phi(\sum_{k=0}^{n} b_k\beta^k)$. Thus, ϕ is a bijection. ∎

Lemma 1.7. *Let $q \in (1,2)$. If $z_n(q) \gg q^n$ (i.e., $\overline{\lim}_{n\to\infty} q^{-n} z_n(q) = +\infty$), then $l(q) = 0$.*

Proof. Since $\sum_{k=0}^{n} a_k q^k < q^{n+1}/(q-1)$, the result follows immediately from the pigeonhole principle. ∎

Consequently, if q is not a root of a polynomial with coefficients $0, \pm 1$, then $z_n(q) = 2^{n+1}$, and $l(q) = 0$ (which is well known, of course—see, e.g., [8]). If q is such a root, it is obvious that $z_n(q) \ll 2^n$, and the problem becomes nontrivial. It is generally believed that $l(q) = 0$ unless q is Pisot, but this is probably a very tough conjecture.

2. Main results. We need some preliminaries. Put

$$L(q) = \overline{\lim_{n\to\infty}}\,(y_{n+1}(q) - y_n(q)).$$

Note that $L(q) = 0$ is equivalent to $y_{n+1}(q) - y_n(q) \to 0$ as $n \to \infty$. This condition was studied in the seminal paper [11]; in particular, it was shown that if $q < 2^{1/4} \approx 1.18921$ and q is not equal to the square root of the second Pisot number ≈ 1.17485, then $L(q) = 0$ (1). It was also shown in the same paper that $L(\sqrt{2}) = 0$.

It is worth noting that the two conditions $l(q) = 0$ and $L(q) = 0$ are, generally speaking, very different in nature; for instance, $l(q) = 0$ for all transcendental q, whereas $L(q) = 1$ for all $q \geq (1+\sqrt{5})/2$ (see, e.g., [10]) and no $q \in (\sqrt{2}, (1+\sqrt{5})/2)$ with $L(q) = 0$ is known.

Throughout this section we assume that $q \in (1,2)$ is a root of a polynomial with coefficients $0, \pm 1$. It is easy to show that in this case any conjugate of q is less than 2 in modulus.

Finally, recall that an algebraic integer $q > 1$ is called a *Perron number* if each of its conjugates is less than q in modulus.

Theorem 2.1. *If $q \in (1,2)$ is not a Perron number, then $l(q) = 0$. If, in addition, $q < \sqrt{2}$ and $-q$ is not a conjugate of q, then $L(q) = 0$.*

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(1) V. Komornik has recently shown [16] that the second condition can be removed, so $L(q) = 0$ if $q < 2^{1/4}$.

Proof. We first prove $l(q)=0$. We have three cases.

CASE 1: *q has a real conjugate p and $q<|p|$.* Since p is an algebraic conjugate of q, it follows from Galois theory that the map $\psi: Y_n(q)\to Y_n(p)$ given by $\psi(\sum_{i=0}^n a_i q^i)=\sum_{i=0}^n a_i p^i$ is a bijection. Hence $z_n(q)=z_n(p)\ge C|p|^n$ by Lemma 1.4 and $z_n(q)\gg q^n$. Now the claim follows from Lemma 1.7.

CASE 2: *q has a complex nonreal conjugate p and $q<|p|$.* This case is similar to Case 1: $z_n(q)=z_n(p)\ge C|p|^n$ by Lemma 1.3(i) and $z_n(q)\gg q^n$.

CASE 3: *q has a conjugate p and $q=|p|$.* Let f denote the minimal polynomial for q. Then $f(x)=g(x^m)$ for some $m\ge 2$ by [6]. Put $\beta=q^m$. We have

$$\begin{aligned}Y_{mk}(q)&=\{a_0+a_1\beta^{1/m}+a_2\beta^{2/m}+\cdots+a_{mk}\beta^n \mid a_i\in\{0,1\}\}\\&=\{A_1+\beta^{1/m}A_2+\beta^{2/m}A_3+\cdots+\beta^{(m-1)/m}A_m:\\&\qquad\qquad A_1\in Y_k(\beta),\ A_i\in Y_{k-1}(\beta),\ 2\le i\le m\}.\end{aligned}$$

Observe that any relation of the form

$$A_1+\beta^{1/m}A_2+\cdots+\beta^{(m-1)/m}A_m=A_1'+\beta^{1/m}A_2'+\cdots+\beta^{(m-1)/m}A_m'$$

implies $A_1=A_1',\ldots,A_m=A_m'$. Indeed, if q satisfies an equation $B_1+qB_2+\cdots+q^{m-1}B_m=0$ with $B_i\in\mathbb{Z}[q^m]$, then $qe^{2\pi ij/m}$ satisfies the same equation for $j=1,\ldots,m-1$, hence $B_i=0$ for all i. Thus, $z_{mk}(\beta^{1/m})=z_k(\beta)(z_{k-1}(\beta))^{m-1}$.

Now, if $q\ge 2^{1/m}$, then $\beta\ge 2$, so $z_k(\beta)=2^{k+1}$, and we see from the above argument that for $n=mk$ we have $z_n(q)\ge C2^n\gg q^n$. Otherwise $z_n(q)\ge z_n(\beta)\ge C\beta^n\gg q^n$. Hence by Lemma 1.7, $l(q)=0$.

Let us now prove the second part of the theorem. Suppose $q<\sqrt{2}$ is not Perron and $-q$ is not its conjugate; then q has a conjugate $\alpha\ne -q$ with $|\alpha|\ge q$. Thus, q^2 has a conjugate α^2, and $|\alpha|^2\ge q^2$ with $\alpha^2\ne q^2$. If $|\alpha|>\sqrt{2}$, then α^2 (and hence q^2) is not a root of a $-1,0,1$ polynomial. Otherwise, we can apply the first part of this theorem to q^2. In either case, $l(q^2)=0$, whence by [10, Theorem 5], $L(q)=0$. ∎

REMARK 2.2. Stankov [22] has proved a similar result for the set

$$\mathcal{A}(q)=\Big\{\sum_{k=0}^n a_k q^k \;\Big|\; a_k\in\{-1,1\},\ n\ge 1\Big\}. \tag{2.1}$$

More precisely, he has shown that if $\mathcal{A}(q)$ is discrete, then all *real* conjugates of q are of modulus strictly less than q.

COROLLARY 2.3. *If $q\in(1,2)$ is the square root of a Pisot number and not itself Pisot, then $l(q)=0$.*

Proof. If $q = \sqrt{\beta}$ and β is Pisot, then either $-q$ is a conjugate of q, or q is Pisot. ■

THEOREM 2.4.

(i) *Suppose* $q \in (1,2)$ *has a conjugate* α *such that* $|\alpha|q < 1$. *Then* $l(q) = 0$, *and consequently* $\Lambda(q)$ *is dense in* $\mathbb{R}$.
(ii) *Suppose* $q \in (1,2)$ *has a nonreal conjugate* α *such that* $|\alpha|q = 1$. *Then* $l(q) = 0$.

If, in addition, $q < \sqrt{2}$ *in either case, then* $L(q) = 0$.

Proof. (i) As above, we have $z_n(q) = z_n(\alpha)$. On the other hand, by Lemma 1.6, $z_n(\alpha) = z_n(1/\alpha)$, and by Lemmas 1.4 and 1.3, $z_n(1/\alpha) \geq C(|1/\alpha|)^n$. Hence $z_n(q) \geq C(|1/\alpha|)^n \gg q^n$, in view of $|\alpha q| < 1$. Hence by Lemma 1.7, $l(q) = 0$.

If $q < \sqrt{2}$, then q^2 has a conjugate α^2, and $q^2|\alpha|^2 < 1$. Hence $l(q^2) = 0$, whence $L(q) = 0$.

(ii) Denote $\alpha_1 = q$, $\alpha_2 = \alpha$, and $\alpha_3 = \overline{\alpha}$. Since $|\alpha|q = 1$ and α is nonreal, we have three conjugates satisfying $\alpha_1^2\alpha_2\alpha_3 = 1$. Smyth [20, Lemma 1] characterizes such situations, but it is easier for us to proceed directly. The Galois group of the minimal polynomial for q is transitive, so there is an automorphism of the Galois group mapping α_1 to α_2. We deduce that $\alpha_2^2\alpha_i\alpha_j = 1$ for some distinct conjugates α_i and α_j of α_1. But this implies $\max\{|\alpha_i|, |\alpha_j|\} \geq \alpha_1 = q$, hence q is not a Perron number, and $l(q) = 0$ by Theorem 2.1.

If $q < \sqrt{2}$, then $q^2|\alpha^2| = 1$, and (ii) applies to q^2, unless $\alpha^2 \in \mathbb{R}$. In the latter case $\alpha = \pm i/q$, whence the minimal polynomial for q contains only powers divisible by 4. Hence the minimal polynomial for q^2 contains only even powers, which implies that $-q^2$ is conjugate to q^2, whence q^2 is not Perron, and $l(q^2) = 0$. ■

REMARK 2.5. If $|\alpha|q = 1$ and α is real, we do not know if $l(q) = 0$. In fact, this includes the interesting (and probably, difficult) case of Salem numbers (2).

DEFINITION 2.6. We say that an algebraic integer $q > 1$ is *anti-Pisot* if it has only one conjugate less than 1 in modulus and at least one conjugate greater than 1 in modulus other than q itself.

COROLLARY 2.7. *If* $q \in (1,2)$ *is anti-Pisot and also a root of a polynomial with coefficients in* $\{-1,0,1\}$, *then* $l(q) = 0$.

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(2) Recall that an algebraic number $q > 1$ is called a *Salem number* if all its conjugates have absolute value no greater than 1, and at least one has absolute value exactly 1.

Proof. Let $\alpha = \alpha_1, \alpha_2, \dots, \alpha_{k-1}, q$ be all the conjugates of q. We have $|\prod_{j=1}^{k-1} \alpha_j| q = 1$, because q satisfies an algebraic equation with coefficients $0, \pm 1$, whence its minimal polynomial must have a constant term ± 1.

Suppose $|\alpha| < 1$; then it is clear that $\alpha \in \mathbb{R}$ (since it is unique). If $|\alpha_2| > 1$ and $|\alpha_j| \geq 1$ for $j = 3, \dots, k-1$, then it is obvious that $|\alpha| q \leq |\alpha_2|^{-1} < 1$, i.e., the condition of Theorem 2.4(i) is satisfied. ∎

3. Examples

EXAMPLE 3.1. Let $q \approx 1.22074$ be the positive root of $x^4 = x + 1$. Then q has a single conjugate $\alpha \approx -0.72449$ inside the open unit disc and no conjugates of modulus 1, whence q is anti-Pisot, and by Corollary 2.7, $l(q) = 0$. Furthermore, $q < \sqrt{2}$, whence $L(q) = 0$ as well.

Note that $q > 2^{1/4}$, so we cannot derive the latter claim immediately from [11, Theorem IV].

EXAMPLE 3.2. An example of q with a real conjugate α which is not anti-Pisot but still satisfies the condition of Theorem 2.4(i), is the appropriate root of $x^5 = x^4 + x^2 + x - 1$. Here $q \approx 1.52626$ and $\alpha \approx 0.59509$.

EXAMPLE 3.3. For the equation $x^5 = x^4 - x^2 + x + 1$ we have $q \approx 1.26278$ and $|\alpha| \approx 0.74090$ so $|\alpha| q \approx 0.93559$ (and $\alpha \notin \mathbb{R}$). By Theorem 2.4(i), $L(q) = 0$.

EXAMPLE 3.4. For the equation $x^8 = x^7 + x^6 + x^5 - x^4 - x^3 - x^2 + x - 1$ we have $q \approx 1.52501$. Among its conjugates is $\alpha \approx 0.3741 + 0.52404i$ with $|\alpha| \approx 0.64387 < 1/q = 0.65574$, so again $l(q) = 0$ by Theorem 2.4(i). Note that $q > \sqrt{2}$ so we cannot claim $L(q) = 0$.

EXAMPLE 3.5. The following example illustrates Theorem 2.4(ii). Let $q \approx 1.19863$ be the largest root of $x^{12} = x^9 + x^6 + x^3 - 1$; then $\alpha = \zeta q^{-1}$ is a root of this equation as well, where ζ is any complex nonreal cubic root of unity. Hence $q|\alpha| = 1$, and Theorem 2.4(ii) applies, i.e., $L(q) = 0$. Note that $q = \sqrt[3]{\beta}$, where β is a quartic Salem number.

EXAMPLE 3.6. For the equation $x^{11} = x^{10} + x^9 - x^6 + x^4 - x^2 - 1$ we have $q \approx 1.5006$. Among its conjugates is $\lambda \approx 0.02625 + 0.7414i$. Theorem 2.4 does not apply, but we can use Lemma 1.3(ii) to obtain

$$z_n(q) = z_n(\lambda) \geq |\lambda|^{-2(n+1)} \approx 1.81696^{n+1} \gg q^n,$$

which implies that $l(q) = 0$. Note that Lemma 1.3(ii) indeed applies, because $0.02625 \approx \operatorname{Re} \lambda < |\lambda|^2 - 1/2 \approx 0.05037$.

EXAMPLE 3.7. Consider the equation $x^{18} = -x^{16} + x^{14} + x^{11} + x^{10} + \cdots + x + 1$ (no powers missing between x^{10} and 1). It has a root $q \approx 1.22289$, and the conjugates largest in modulus are $u, \overline{u}$ approximately equal to $-.03958 \pm 1.3109i$. Then Theorem 2.1 implies $L(q) = 0$.

It is worth mentioning that there is another way to obtain this result. Consider q^2 and its conjugates $u^2, \overline{u}^2$. We claim that although $|u^2| < 2$, u^2, and hence q^2, is not a zero of a $-1,0,1$ polynomial (whence $l(q^2) = 0$, which implies $L(q) = 0$).

Indeed, if it were, then $q^{-2}, u^{-2}, (\overline{u})^{-2}$ would also be zeros of such a polynomial. However, the product of these three numbers is ≈ 0.226024, so this is impossible, in view of the following

CLAIM. *Suppose z_1, z_2, z_3 are three different roots of a $-1,0,1$ polynomial. Then $|z_1 z_2 z_3| \geq 1/2 \cdot (4/3)^{-3/2} = 0.32476\ldots$.*

This claim is a slight generalization of [5, Theorem 2]; see [19, Theorem 2.4].

EXAMPLE 3.8. Finally, an example of q for which none of our criteria works is the real root of $x^5 = x^4 + x^3 - x + 1$. Here $q \approx 1.54991$, and the other four conjugates are nonreal, with the moduli ≈ 1.04492 and ≈ 0.76871 respectively.

Another example is any Salem number $q \in (1,2)$, for instance $q \approx 1.72208$ which is a root of $x^4 = x^3 + x^2 + x - 1$ (which is of course none other than β from Example 3.5).

4. Final remarks and open problems

4.1. Our first remark concerns the case $q \in (m, m+1)$ with $m \geq 2$. Here the natural definition for $\Lambda(q)$ is

$$\Lambda(q) = \Big\{ \sum_{k=0}^{n} a_k q^k \;\Big|\; a_k \in \{-m, -m+1, \ldots, m-1, m\},\ n \geq 1 \Big\}.$$

Theorem 2.4 holds for this case, provided $\alpha \in \mathbb{R}$ (and so does Case 1 of Theorem 2.1)—the proof is essentially the same. The case of nonreal α is less straightforward, since there is no ready-to-apply complex machinery for $m \geq 2$. (Basically, we need that if α is a zero of a polynomial with coefficients in $\{-m, \ldots, m\}$, then the attractor of the iterated function system $\{\alpha z + j\}_{j=0}^{m}$ in the complex plane is connected. This can be verified for $m = 2, 3$ but we do not know if this is true in general.) Note also that an analogue of Theorem 1.1 for $m \geq 2$ has been proved in [9].

4.2. We do not know whether the extra condition that $-q$ is not a conjugate of q is really necessary in the second claim of Theorem 2.1. In particular, is it true that $L(\sqrt{\varphi}) = 0$ if φ is the golden ratio?

4.3. In [18, Proposition 1.2] it is shown that if $q < \sqrt{2}$ and q^2 is not a root of a polynomial with coefficients $0, \pm 1$, then the set $\mathcal{A}(q)$ given by (2.1) is dense in $\mathbb{R}$. In fact, what the authors use in their proof is the condition $l(q^2) = 0$. Consequently, Theorems 2.1 and 2.4 provide sufficient conditions for $\mathcal{A}(q)$ to be dense in the case when q^2 satisfies an algebraic equation with coefficients $0, \pm 1$.

4.4. Is $l(q) = 0$ for q in Example 3.8 and suchlike?

4.5. All our criteria suggest that $l(q) = 0$ implies $L(q) = 0$ for $q < \sqrt{2}$. Is this really the case?

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Added in proof (June 2011). In the recent paper by Sh. Akiyama and V. Komornik [1] several results mentioned in the introductory part of the present paper have been significantly improved, namely:

- If $q \in (1, \sqrt{2}]$ is non-Pisot, then $l(q) = 0$ and $\mathcal{A}(q)$ is dense in $\mathbb{R}$.
- If $q \in (\sqrt{2}, 2)$ is non-Pisot, then $\Lambda(q)$ has a finite accumulation point.
- For $q \in (1, 2^{1/3}]$ we have $L(q) = 0$.

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