ACTA ARITHMETICA
149.4 (2011)

Voronoĭ type criteria for lattice coverings with balls

by

Peter M. Gruber (Wien)

1. Introduction. Given a lattice L in $\mathbb{E}^d$, let $\sigma(L) > 0$ be minimum such that the balls $\{\sigma(L)B^d + l : l \in L\}$ cover $\mathbb{E}^d$. Denote the density of this lattice covering by $\vartheta(L)$. A major problem of the geometry of numbers is to determine the global and local minima of $\vartheta(L)$ as L ranges over the space of all lattices in $\mathbb{E}^d$. The global minima are known for $d = 2, 3, 4, 5$, the local minima for $d = 2, 3, 4$. See [16] for details and references. For general d, Barnes and Dickson [1] and Delone, Dolbilin, Ryshkov and Shtogrin [3] gave necessary and sufficient conditions for lattice coverings of (locally) minimum density, where the lattices satisfy certain restrictions. The restrictions were eliminated by Schürmann and Vallentin [17], and Schürmann [16] specified an algorithm to determine all lattice coverings with balls of (locally) minimum density. Up to similarities there are only finitely many such coverings for each d.

Using an alternative approach, we characterize lattice coverings of balls with different local minimum properties of the density. The characterizations are by means of Voronoĭ type properties of the lattices. For $d = 2$ it is precisely the regular hexagonal lattices which provide coverings with circles of semistationary, stationary, or minimum density. There are no lattice coverings with circles of ultraminimum density. For an alternative, more precise approach to the results in the planar case see [9]. All results of the present article may be stated in terms of non-homogeneous minima of positive definite quadratic forms.

The idea underlying the proofs is to identify lattices in $\mathbb{E}^d$ with points of $\mathbb{E}^{d(d+1)/2}$. The covering problems are then translated into transparent geometric problems in $\mathbb{E}^{d(d+1)/2}$. For a series of other applications of this idea see the report [8] and the references in the article [10]. Further tools are empty spheres and Delone simplices; see [16].

2010 *Mathematics Subject Classification*: 05B40, 11H31, 11H55, 52C15, 52C17.
Key words and phrases: lattice covering, ball, density, minimum property, Voronoĭ type result.

DOI: 10.4064/aa149-4-6

In all dimensions, lattice packings of balls of maximum density are necessarily of ultramaximum density (see [10]). Our results show that in the covering case an analogous result does not hold for all dimensions.

Lattice packing and covering problems for balls and convex bodies, typically, lead to investigations of the minimum points, resp. the points of $\mathbb{E}^d$ with maximum distance from the lattice, the so-called deep holes. The investigation of the deep holes, in turn, amounts to the study of the lattice points closest to them. Therefore, in general, covering problems are more difficult to deal with than corresponding packing problems and it is not surprising that our results on packings in [10] are farther reaching than the results on coverings in the present article. In spite of this we presume that, in analogy to the packing case, the results of this article can be extended to lattice coverings with smooth convex bodies (see [10]).

The books [2, 4, 7, 11, 13, 16, 21] provide general information on the geometry of numbers, the geometric theory of positive definite quadratic forms, and convex geometry.

Let lin, pos, conv, $\mathrm{relint}_{\mathcal{T}}$, diam, tr, V, ${}^{\mathcal{T}}$, ^{T} stand for linear, positive (better: non-negative) and convex hull, for the interior relative to the subspace $\mathcal{T}$ of $\mathbb{E}^{d(d+1)/2}$, and for diameter, trace, volume, orthogonal projection of $\mathbb{E}^{d(d+1)/2}$ onto $\mathcal{T}$, and transposition.

2. Local minimum properties of lattice coverings with balls. We begin with the introduction of needed notions.

Basic concepts. Let L be a *lattice* in Euclidean d-space $\mathbb{E}^d$, that is, the set of all integer linear combinations of d linearly independent vectors in $\mathbb{E}^d$. These vectors form a *basis* of L, and the volume of the parallelotope generated by the basis vectors is the *determinant* $d(L)$ of L. The *covering radius* $\sigma(L)$ of the Euclidean unit ball B^d with respect to L is given by

$$\sigma(L) = \min\{\sigma \geq 0 : \{\sigma B^d + l : l \in L\} \text{ is a covering of } \mathbb{E}^d\}.$$

In particular, $\{\sigma(L)B^d + l : l \in L\}$ is a covering of $\mathbb{E}^d$. Its *density* is

$$\vartheta(L) = \frac{\sigma(L)^d V(B^d)}{d(L)}.$$

The aim of this article is to study the local minimum properties of $\vartheta(\cdot)$ as L ranges over the space of lattices in $\mathbb{E}^d$.

Identify a (real) symmetric $d \times d$ matrix $A = (a_{ik})$ with the point

$$(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{2d}, \dots, a_{dd})^T \in \mathbb{E}^{d(d+1)/2}.$$

The symbols O and I denote the $d \times d$ zero and unit matrix. For symmetric $d \times d$ matrices $A = (a_{ik})$, $B = (b_{ik})$, the *inner product* and the *norm* are defined to be $A \cdot B = \sum a_{ik} b_{ik}$ and $\|A\| = (\sum a_{ik}^2)^{1/2}$. The dot $\cdot$ and the symbol $\|\cdot\|$ denote also the inner product and the norm on $\mathbb{E}^d$. For $u, l \in \mathbb{E}^d$ the *tensor product* $u \otimes l$ is the $d \times d$ matrix ul^T. Then $Al \cdot u = A \cdot u \otimes l$. The subspace $\mathcal{T} = \{A \in \mathbb{E}^{d(d+1)/2} : \operatorname{tr} A = A \cdot I = 0\}$ of $\mathbb{E}^{d(d+1)/2}$ has codimension 1 and I is a normal vector of it.

Minimum and Voronoĭ type properties. The density $\vartheta(\cdot)$ is said to be locally, lower *semistationary, stationary, minimum,* or *ultraminimum* at L if

$$\frac{\vartheta((I+A)L)}{\vartheta(L)} \begin{Bmatrix} \geq 1 + o(\|A\|) \\ = 1 + o(\|A\|) \\ \geq 1 \\ \geq 1 + \text{const}\,\|A\| \end{Bmatrix} \quad \text{as } A \to O,\ A \in \mathcal{T},$$

where an inequality or equality holds as $A \to O$ if it holds for all A with sufficiently small norm. Here the symbol $o(\cdot)$ may assume also negative values. The symbol const means a positive constant depending on L. Since the density does not change if L is replaced by a positive multiple of it, the restriction to matrices $A \in \mathcal{T}$ is not an essential one. We make it to avoid clumsy formulations.

A sphere S in $\mathbb{E}^d$ is *empty* with respect to L if there is no point of L in its interior. For simplicity, we assume that an empty sphere always contains points of L. Clearly,

(1) $\sigma(L)$ is the maximum radius of an empty sphere of L.

Let S be an empty sphere with radius $\sigma(L)$ and center c. After a translation by a suitable vector of L we may assume that $o \in S$. Later it will be shown that the other points of $L \cap S$ can be listed in the form $l_1, \dots, l_d$, $m_1, \dots, m_k$, $k \geq 0$, such that the following hold:

(2)
- $l_1, \dots, l_d$ are linearly independent,
- $c = \lambda_1 l_1 + \cdots + \lambda_d l_d$, where $0 \leq \lambda_i$, $\lambda_1 + \cdots + \lambda_d \leq 1$,
- $m_j = \mu_{j1} l_1 + \cdots + \mu_{jd} l_d$, where $\mu_j \in \mathbb{R}$.

Then $D = \operatorname{conv}\{o, l_1, \dots, l_d\}$ is a *Delone simplex* of L with (one vertex at o and) *circumsphere* S, *circumradius* $\sigma(L)$ and *circumcenter* c. (Note that in contrast to the usual definition of a Delone simplex, it is assumed here that the circumradius of a Delone simplex is equal to $\sigma(L)$.) Let

(3)
- $U(D) = \lambda_1 u_1 \otimes l_1 + \cdots + \lambda_d u_d \otimes l_d = \lambda_1 l_1 \otimes l_1 + \cdots + \lambda_d l_d \otimes l_d - c \otimes c$, where $u_i = l_i - c$,
- $V(D, m_j) = (\lambda_1 - \mu_{j1}) u_1 \otimes l_1 + \cdots + (\lambda_d - \mu_{jd}) u_d \otimes l_d + v_j \otimes m_j$
  $= (\lambda_1 - \mu_{j1}) l_1 \otimes l_1 + \cdots + (\lambda_d - \mu_{jd}) l_d \otimes l_d + m_j \otimes m_j - c \otimes c$,
  where $v_j = m_j - c$,

Two Delone simplices are *equivalent* if they differ by a lattice translation or a reflection in o. If two Delone simplices D, E are equivalent, then $U(D) = U(E)$ and similarly for V. Let $\mathcal{D}(L)$ be a maximal family of pairwise non-equivalent Delone simplices of L.

We propose to call the lattice L *paracomplete, semicomplete*, or *complete* if the following statements hold:

$$\text{(4)} \qquad \max_{D\in\mathcal{D}(L)} \min\{A\cdot U(D), A\cdot V(D,m_1),\dots,A\cdot V(D,m_k)\} \begin{Bmatrix} =0 \\ \geq 0 \\ >0 \end{Bmatrix}$$
$$\text{for } A\in\mathcal{T}\setminus\{O\}.$$

The meaning of these notions will be clear from the proof of Theorem 1.

Results. The main results of this note are the following criteria for minimum properties of the density of lattice coverings with balls.

THEOREM 1. *For $\vartheta(\cdot)$ and L the following statements hold:*

(i) *$\vartheta(\cdot)$ is stationary at L if and only if L is paracomplete.*
(ii) *$\vartheta(\cdot)$ is semistationary at L if and only if L is semicomplete.*
(iii) *$\vartheta(\cdot)$ is ultraminimum at L if and only if L is complete.*

In the case when the points of $\mathbb{E}^d$ with maximum distance from the lattice L, the *deep holes of* L, or the *points farthest from* L, and the points of L which are nearest to the deep holes are known or can be determined, the criteria apply effectively.

Completeness implies that $\operatorname{conv}\{U(D)^{\mathcal{T}} : D\in\mathcal{D}(L)\}$ is a proper convex polytope in the subspace $\mathcal{T}$ and thus has at least $\dim\mathcal{T}+1 = \frac{1}{2}d(d+1)$ vertices. This yields the following result.

COROLLARY 2. *If $\vartheta(\cdot)$ is ultraminimum at L then L has at least $\frac{1}{2}d(d+1)$ pairwise non-equivalent Delone simplices.*

This result is reminiscent of a theorem of Korkin and Zolotarev [12] which in geometric terms says that a lattice packing of balls which locally has maximum (and thus, by a result of the author [10], ultramaximum) density has kissing number at least $d(d+1)$.

A lattice L in $\mathbb{E}^2$ is *regular hexagonal* if it is similar to the lattice with basis

$$\{(1,0)^T, \left(\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3}\right)^T\}.$$

Regular hexagonal lattices are unique from many different viewpoints; see the book of Fejes Tóth [5] and the surveys of the author [6], Morgan and Bolton [14], and Saff and Kuijlaars [15]. The following result is a further example; for an alternative proof see [9].

COROLLARY 3. *Among the lattices in $\mathbb{E}^2$ it is precisely the regular hexagonal lattices at which the density $\vartheta(\cdot)$ is semistationary/stationary/minimum. The density is not ultraminimum at any lattice.*

Proofs of the theorem and its corollaries

Proof of Theorem 1. The proof is divided into four steps.

STEP 1.

(5) $\quad \mathcal{D}(L) \neq \emptyset$.

Let S be an empty sphere of L which has the maximum possible radius $\sigma(L)$. Then $L \cap S \neq \emptyset$ and after a translation by a suitable vector of L if necessary, we may assume that $o \in S$. Let c be the center of S. We show that

(6) $\quad c \in \operatorname{conv}(L \cap S)$.

Otherwise c and the convex polytope $\operatorname{conv}(L \cap S)$ can be separated by a hyperplane. Let u be a normal vector of this hyperplane which points into the halfspace containing c. Then, moving S in the direction u through a sufficiently small distance yields a sphere of radius $\sigma(L)$ which contains no point of L and thus gives rise to an empty sphere of radius greater than $\sigma(L)$, a contradiction, concluding the proof of (6). A similar argument shows that

(7) $\quad \operatorname{conv}(L \cap S)$ is proper, that is, $\operatorname{int}\operatorname{conv}(L \cap S) \neq \emptyset$.

Simple induction shows that each proper convex polytope with one vertex marked is the union of proper simplices, all vertices of which are among the vertices of the polytope, one being the marked vertex. This together with (7) and (6) implies that $L \cap S$ can be listed in the form $o, l_1, \dots, l_d, m_1, \dots, m_k$, where $l_1, \dots, l_d$ are linearly independent (i.e. (2) holds) and such that $D = \operatorname{conv}\{o, l_1, \dots, l_d\} \in \mathcal{D}(L)$ (thus (5) holds).

STEP 2.

(8) $\quad$ Let $D = \operatorname{conv}\{o, l_1, \dots, l_d\}$ be a proper simplex with circumcenter c. For each $A \in \mathcal{T}$ with sufficiently small norm the following hold: If $c + h$ is the circumcenter of $(I + A)D$, then

$$h = M^{-T}a + \tfrac{1}{2}M^{-T}b - AM^{-T}a + O(\|A\|^3) = M^{-T}a + O(\|A\|^2)$$

as $A \to O$, $A \in \mathcal{T}$, where

$$M = (l_1, \dots, l_d), \qquad a = \begin{pmatrix} A \cdot u_1 \otimes l_1 \\ \vdots \\ A \cdot u_d \otimes l_d \end{pmatrix}, \qquad b = \begin{pmatrix} A^2 \cdot l_1 \otimes l_1 \\ \vdots \\ A^2 \cdot l_d \otimes l_d \end{pmatrix},$$

and $u_i = l_i - c$.

The center $c+h$ of $(I+A)D$ satisfies the relations

$$\begin{aligned}(c+h)^2 &= (l_i + Al_i - c - h)^2 = (l_i - c + Al_i - h)^2\\ &= (l_i - c)^2 + 2Al_i \cdot (l_i - c) - 2h \cdot (l_i - c) + (Al_i)^2 - 2Al_i \cdot h + h^2.\end{aligned}$$

Since $(l_i - c)^2 = c^2$, $l_i - c = u_i$, $l_i \cdot h = l_i^T h$, it follows that

$$c \cdot h = l_i \cdot h - u_i \cdot h = A \cdot u_i \otimes l_i - h \cdot u_i + \tfrac{1}{2} A^2 \cdot l_i \otimes l_i - Al_i \cdot h,$$

i.e.,

$$l_i^T h + (Al_i)^T h = A \cdot u_i \otimes l_i + \tfrac{1}{2} A^2 \cdot l_i \otimes l_i$$

and thus, since $A = A^T$,

$$(M^T + M^T A)h = a + \tfrac{1}{2} b, \quad \text{i.e.,} \quad (I + A)h = M^{-T} a + \tfrac{1}{2} M^{-T} b.$$

Taking into account that $(I+A)^{-1} = I - A + \cdots$, we deduce that

$$h = M^{-T} a + \tfrac{1}{2} M^{-T} b - AM^{-T} a + O(\|A\|^3) = M^{-T} a + O(\|A\|^2),$$

concluding the proof of (8).

As a consequence of (8) it will be show that

$$\begin{aligned}&(c+h)^2 = (l_i + Al_i - c - h)^2 = c^2 + 2A \cdot U(D) + O(\|A\|^2),\\ &(m_j + Am_j - c - h)^2 = c^2 + 2A \cdot V(D, m_j) + O(\|A\|^2)\\ &\text{as } A \to O,\ A \in \mathcal{T}.\end{aligned} \tag{9}$$

Since $c = \lambda_1 l_1 + \cdots + \lambda_d l_d$, $m_j = \mu_{j1} l_1 + \cdots + \mu_{jd} l_d$, we have

$$M^{-1} c = (\lambda_1, \ldots, \lambda_d)^T, \qquad M^{-1} m_j = (\mu_{j1}, \ldots, \mu_{jd})^T.$$

Thus

$$\begin{aligned}(c+h)^2 &= c^2 + 2c \cdot h + h^2 = c^2 + 2c^T M^{-T} a + O(\|A\|^2)\\ &= c^2 + 2(M^{-1}c)^T a + O(\|A\|^2)\\ &= c^2 + 2A \cdot \{\lambda_1 u_1 \otimes l_1 + \cdots + \lambda_d u_d \otimes l_d\} + O(\|A\|^2)\\ &= c^2 + 2A \cdot U(D) + O(\|A\|^2) \quad \text{as } A \to O,\ A \in \mathcal{T},\end{aligned}$$

and

$$\begin{aligned}(m_j &+ Am_j - c - h)^2 = (m_j - c + Am_j - h)^2\\ &= (m_j - c)^2 + 2Am_j \cdot (m_j - c) - 2h \cdot m_j + 2h \cdot c + O(\|A\|^2)\\ &= c^2 + 2A \cdot v_j \otimes m_j - 2m_j^T h + 2c^T h + O(\|A\|^2)\\ &= c^2 + 2A \cdot v_j \otimes m_j - 2m_j^T M^{-T} a + 2c^T M^{-T} a + O(\|A\|^2)\\ &= c^2 + 2A \cdot \{v_j \otimes m_j - (\mu_{j1} u_1 \otimes l_1 + \cdots + \mu_{jd} u_d \otimes l_d)\\ &\qquad + (\lambda_1 u_1 \otimes l_1 + \cdots + \lambda_d u_d \otimes l_d)\} + O(\|A\|^2)\\ &= c^2 + 2A \cdot V(D, m_j) + O(\|A\|^2) \quad \text{as } A \to O,\ A \in \mathcal{T}.\end{aligned}$$

Step 3. Let $\mathcal{S}(L)$ be the finite family of proper simplices with vertices in L, one being the origin o, and diameter at most $3\sigma(L)$. We will show that

(10) there is $\varepsilon > 0$ such that for any $D \in \mathcal{S}(L)$ and $A \in \mathcal{T}$ with $\|A\| < \varepsilon$ the following holds: if D is not a Delone simplex of L, then $(I+A)D$ is not a Delone simplex of $(I+A)L$.

Clearly, the following statement holds:

Let $D \in \mathcal{S}(L)$. Then $D \notin \mathcal{D}(L)$ if and only if either

- the circumradius of D is $\neq \sigma(L)$, or
- the circumsphere of D is not empty, or
- the circumcenter is not contained in D.

Since $\mathcal{S}(L)$ is a finite family of simplices, this implies the following:

(11) There is $\delta > 0$ such that for any $D \in \mathcal{S}(L)$ we have the following, where c is the circumcenter of D: $D \notin \mathcal{D}(L)$ if and only if either

- the circumradius σ of D satisfies $|\sigma - \sigma(L)| \geq \delta$, or
- there is $m \in L$ such that $\|m - c\| \leq \sigma - \delta$, or
- c has distance at least δ from D.

Since $\mathcal{S}(L)$ is finite and for $D \in \mathcal{S}(L)$ the circumradius and the circumcenter of $(I+A)D$ and the points of $(I+A)L$ vary continuously with $A \in \mathcal{T}$, we obtain the following:

There is $\varepsilon > 0$ such that for any $D \in \mathcal{S}(L)$ and $A \in \mathcal{T}$ with $\|A\| < \varepsilon$ the following hold: If $D \notin \mathcal{D}(L)$, then [$(I+A)D$ satisfies at least one of the three conditions in (11) with D, L, δ replaced by $(I+A)D$, $(I+A)L, \delta/2$ and thus] $(I+A)D \notin \mathcal{D}((I+A)L)$.

The proof of (10) is complete.

STEP 4.

$$\begin{aligned}(12)\qquad &\frac{\vartheta((I+A)L)}{\vartheta(L)}\\ &= 1 + \frac{d}{\sigma(L)^2} \max_{D \in \mathcal{D}(L)} \min\{A \cdot U(D), A \cdot V(D, m_1), \dots, A \cdot V(D, m_k)\}\\ &\quad + O(\|A\|^2) \quad \text{ as } A \to O,\ A \in \mathcal{T}.\end{aligned}$$

To prove (12), the following consequences of (9) are needed:

- If D is a Delone simplex of L and $(I+A)D$ a Delone simplex of $(I+A)L$, then
$$\begin{aligned}\sigma((I+A)L)^2 &= \sigma(L)^2 + 2A \cdot U(D) + O(\|A\|^2)\\ &\leq \sigma(L)^2 + 2A \cdot V(D, m_j) + O(\|A\|^2)\end{aligned}$$
for $j = 1, \dots, k$ as $A \to O, A \in \mathcal{T}$.

- If D is a Delone simplex of L with center c but $(I+A)L$ is not a Delone simplex of $(I+A)L$, then

$$\begin{aligned}
&\sigma((I+A)L)^2\\
&\geq (\text{radius of the largest empty sphere of } (I+A)L \text{ with center } c+h)^2\\
&= \min\{(c+h)^2, (m_1+Am_1-c-h)^2, \dots, (m_k+Am_k-c-h)^2\}\\
&= \sigma(L)^2 + 2\min\{A\cdot U(D), A\cdot V(D,m_1), \dots, A\cdot V(D,m_k)\}\\
&\qquad + O(\|A\|^2) \quad \text{as } A\to O,\ A\in\mathcal{T}.
\end{aligned}$$

Assertion (10) says that for all $A\in\mathcal{T}$ with sufficiently small norm the following holds: If a simplex $(I+A)D$ is a Delone simplex of $(I+A)L$, where $D\in\mathcal{S}(L)$, then D is a Delone simplex of L. By (5) there exist Delone simplices of $(I+A)L$ of this form. Therefore the above consequences of (9), together with the formula (see [10])

$$\det(I+A) = 1-\tfrac{1}{2}\|A\|^2 + O(\|A\|^3) \quad \text{as } A\to O,\ A\in\mathcal{T},$$

yield (12):

$$\begin{aligned}
&\frac{\vartheta((I+A)L)}{\vartheta(L)} = \frac{\sigma((I+A)L)^d}{\sigma(L)^d\det(I+A)}\\
&= \Bigg(1+\frac{2}{\sigma(L)^2}\max_{D\in\mathcal{D}(L)}\min\{A\cdot U(D), A\cdot V(D,m_1), \dots, A\cdot V(D,m_k)\}\\
&\qquad + O(\|A\|^2)\Bigg)^{d/2}\big(1-\tfrac{1}{2}\|A\|^2+O(\|A\|^3)\big)^{-1}\\
&= 1+\frac{d}{\sigma(L)^2}\max_{D\in\mathcal{D}(L)}\min\{A\cdot U(D), A\cdot V(D,m_1), \dots, A\cdot V(D,m_k)\}\\
&\qquad + O(\|A\|^2) \quad \text{as } A\to O,\ A\in\mathcal{T}.
\end{aligned}$$

The equivalences in the theorem are consequences of the identity (12). ∎

Proof of Corollary 2. This is an immediate consequence of the theorem. ∎

Proof of Corollary 3. Let L be a lattice in $\mathbb{E}^2$. By considering the lattice points which provide the first and second successive minima of L with respect to B^2 and choosing a suitable Cartesian coordinate system, we may assume that L has a basis of the form

$$\{l_1=(1,0)^T,\ l_2=(r,s)^T\} \quad \text{where } 0\leq r\leq \tfrac{1}{2},\ s>0,\ r^2+s^2\geq 1.$$

Furthermore, L has one equivalence class of Delone triangles, from which we choose the representative

$$D = \operatorname{conv}\{o, l_1, l_2\}.$$

First, the following will be shown:

(13) Let $0 < r \le 1/2$. Then ϑ is semistationary/stationary/minimum at L if and only if L is regular hexagonal. For no lattice is the density ultraminimum.

Since $0 < r \le 1/2$, the points o, l_1, l_2 are the only points of L on the circumcircle of D. Then there is no point m_j and we have to calculate only $U(D)$. The circumcenter of D is

$$c = \left(\frac{1}{2}, t\right), \quad \text{where} \quad \frac{1}{4} + t^2 = \left(r - \frac{1}{2}\right)^2 + (s-t)^2.$$

Since

$$c = \lambda_1 l_1 + \lambda_2 l_2, \quad \text{where} \quad \lambda_1 = \frac{1}{2} - \frac{rt}{s},\ \lambda_2 = \frac{t}{s},$$

we have

$$\begin{aligned} U(D) &= \left(\frac{1}{2} - \frac{rt}{s}\right)\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} + \frac{t}{s}\begin{pmatrix}r\\s\end{pmatrix} \otimes \begin{pmatrix}r\\s\end{pmatrix} - \begin{pmatrix}1/2\\t\end{pmatrix} \otimes \begin{pmatrix}1/2\\t\end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4} - \frac{rt}{s} + \frac{r^2 t}{s} & rt - \frac{t}{2} \\ rt - \frac{t}{2} & st - t^2 \end{pmatrix}. \end{aligned}$$

Thus

ϑ is semistationary at L

$\Leftrightarrow$ $\max\{A \cdot U(D)\} = A \cdot U(D) \ge 0$ for $A \in \mathcal{T}$

$\Leftrightarrow$ $A \cdot U(D) = 0$ for $A \in \mathcal{T}$

$\Leftrightarrow$ $U(D) = \lambda I$ for suitable $\lambda \in \mathbb{R}$

(note that I is orthogonal to the subspace $\mathcal{T}$ of codimension 1)

$\Leftrightarrow rt - \dfrac{t}{2} = 0,\ \dfrac{1}{4} - \dfrac{rt}{s} + \dfrac{r^2 t}{s} = st - t^2$

$\Leftrightarrow r = \dfrac{1}{2},\ \dfrac{1}{4} - \dfrac{t}{2s} + \dfrac{t}{4s} = st - t^2$

$\Leftrightarrow r = \dfrac{1}{2},\ s - t = 4s^2 t - 4st^2 = 4st(s-t)$

$\Leftrightarrow r = \dfrac{1}{2},\ st = \dfrac{1}{4}$

(note that $r^2 + s^2 \ge 1$, $\frac{1}{4} + t^2 = \left(r - \frac{1}{2}\right)^2 + (s-t)^2 = (s-t)^2$ and thus $s \ge \frac{\sqrt{3}}{2}$, $t \le \frac{1}{2\sqrt{3}}$,

$s = t + \sqrt{\frac{1}{4} + t^2} \le \frac{1}{2\sqrt{3}} + \sqrt{\frac{1}{12} + \frac{1}{4}} = \frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{2}$)

$\Leftrightarrow r = \dfrac{1}{2},\ s = \dfrac{\sqrt{3}}{2},\ t = \dfrac{1}{2\sqrt{3}}$

$\Leftrightarrow$ L is regular hexagonal.

If the density is stationary/minimum/ultraminimum at L, then it is semistationary at L. A simple calculation shows that in the regular hexagonal case we have $U(D) = (1/6)I$. Hence L is paracomplete and semicomplete, but not complete. By the theorem the density then is semistationary, stationary, but not ultraminimum at L. It is well-known that it is minimum. The proof of (13) is finished.

Second, the following statement holds:

(14) Let $r = 0$, $s \geq 1$. Then ϑ is not semistationary/stationary/minimum /ultraminimum at L.

In this case

$$\{l_1 = (1,0)^T, l_2 = (0,s)^T\}, \qquad c = \left(\frac{1}{2}, \frac{s}{2}\right)^T, \qquad m = (1,s)^T,$$
$$c = \frac{1}{2}l_1 + \frac{1}{2}l_2, \qquad m = l_1 + l_2$$

and thus

$$U(D) = \frac{1}{4}\begin{pmatrix} 1 & -s \\ -s & s^2 \end{pmatrix}, \qquad V(D,m) = \frac{1}{4}\begin{pmatrix} 1 & 3s \\ 3s & s^2 \end{pmatrix}.$$

Thus the theorem shows that

$$\begin{aligned} &\vartheta \text{ is semistationary} \\ &\quad \Leftrightarrow \min\{A \cdot U(D), A \cdot V(D,m)\} \geq 0 \text{ for } A \in \mathcal{T} \\ &\quad \Leftrightarrow \min\{a_{11} - 2sa_{12} + s^2 a_{22}, a_{11} + 6sa_{12} + s^2 a_{22}\} \geq 0 \text{ for } A \in \mathcal{T} \\ &\quad \Leftrightarrow \min\{-2sa_{12}, 6sa_{12}\} \geq (1 - s^2)a_{22} \text{ for } A \in \mathcal{T}. \end{aligned}$$

Since the latter inequality is not satisfied by all $A \in \mathcal{T}$, the density ϑ is not semistationary at L. Note that if the density is stationary/minimum/ultraminimum, it is semistationary too. The proof of (14) is finished.

Having proved (13) and (14), we have completed the proof of Corollary 3. ∎

Acknowledgements. For valuable hints I am obliged to Achill Schürmann and Tony Thompson.

References

[1] E. S. Barnes and T. J. Dickson, *Extreme coverings of n-space by spheres*, J. Austral. Math. Soc. 7 (1967), 115–127.

[2] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Grundlehren Math. Wiss. 290, Springer, New York, 1999.

[3] B. N. Delone, N. P. Dolbilin, S. S. Ryshkov and M. I. Shtogrin, *A new construction of the theory of lattice coverings of an n-dimensional space by congruent balls*, Izv. Akad. Nauk SSSR Ser. Mat. 34 (1970), 289–298 (in Russian).

[4] P. Erdős, P. M. Gruber and J. Hammer, *Lattice Points*, Longman Sci. & Tech., Harlow, 1989.
[5] L. Fejes Tóth, *Lagerungen in der Ebene, auf der Kugel und im Raum*, 2nd ed., Grundlehren Math. Wiss. 65, Springer, Berlin, 1972.
[6] P. M. Gruber, *In many cases optimal configurations are almost regular hexagonal*, Rend. Circ. Mat. Palermo (2) Suppl. 65, Part II (2000), 121–145.
[7] —, *Convex and Discrete Geometry*, Grundlehren Math. Wiss. 336, Springer, Berlin, 2007.
[8] —, *Application of an idea of Voronoĭ*, a report, in preparation.
[9] —, *Extremum properties of lattice packing and covering with circles*, in preparation.
[10] —, *Application of an idea of Voronoĭ to lattice packing*, in preparation.
[11] P. M. Gruber and C. G. Lekkerkerker, *Geometry of Numbers*, 2nd ed., North-Holland, Amsterdam, 1987.
[12] A. N. Korkin et E. I. Zolotarev, *Sur les formes quadratiques positives*, Math. Ann. 11 (1877), 242–292.
[13] J. Martinet, *Perfect Lattices in Euclidean Spaces*, Grundlehren Math. Wiss. 327, Springer, Berlin, 2003.
[14] F. Morgan and R. Bolton, *Hexagonal economic regions solve the location problem*, Amer. Math. Monthly 109 (2002), 165–172.
[15] E. B. Saff and A. B. J. Kuijlaars, *Distributing many points on a sphere*, Math. Intelligencer 19 (1997), 5–11.
[16] A. Schürmann, *Computational Geometry of Positive Definite Quadratic Forms*, Amer. Math. Soc., Providence, RI, 2009.
[17] A. Schürmann and F. Vallentin, *Computational approaches to lattice packing and covering problems*, Discrete Comput. Geom. 35 (2006), 73–116.
[18] G. F. Voronoĭ, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Première mémoire: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math. 133 (1908), 97–178; [20], Vol. II, 171–238.
[19] —, *Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Deuxième mémoire. Recherches sur les parallélloèdres primitifs I, II*, J. Reine Angew. Math. 134 (1908), 198–287 and 136 (1909), 67–181; [20], Vol. II, 239–368.
[20] —, *Collected Works, Vols. I–III*, Izdat. Akad. Nauk Ukrain. SSSR, Kiev, 1952–1953.
[21] C. Zong, *Sphere Packings*, Springer, New York, 1999.

Peter M. Gruber
Department of Discrete Mathematics
and Geometry
Vienna University of Technology
Wiedner Hauptstraße 8–10/1046
A-1040 Wien, Austria
E-mail: peter.gruber@tuwien.ac.at

Received on 16.9.2010
and in revised form on 25.1.2011 (6492)