

Connecting homomorphisms associated to Tate sequences

by

PAUL RICHARD BUCKINGHAM (Edmonton)

1. Introduction. In [19, 21], Tate constructs, for a Galois extension L/K of number fields with Galois group G , a *Tate sequence* $0 \rightarrow \mathcal{O}_{L,S}^\times \rightarrow A \rightarrow B \rightarrow X \rightarrow 0$ of $\mathbb{Z}[G]$ -modules whose extension class in $\text{Ext}_{\mathbb{Z}[G]}^2(X, \mathcal{O}_{L,S}^\times)$ is the so-called *Tate canonical class*, where $\mathcal{O}_{L,S}^\times$ is the group of S -units in L and X is a finitely generated, \mathbb{Z} -torsion-free module which will be defined in Section 2. This is done under the assumptions that S contains the ramified places and that the S -class-group of L is trivial.

Tate's construction shows that A and B may in fact be chosen finitely generated and cohomologically trivial. This has the consequence that the Galois cohomology of $\mathcal{O}_{L,S}^\times$ can be identified with that of X , after a dimension shift of 2. This is reminiscent of the Artin–Tate formulation of class field theory [1] in which the cohomology of the idele class-group is identified with the cohomology of \mathbb{Z} after a dimension shift of 2, with a similar statement in the local case. In fact, this is no coincidence, since it is via this interpretation of class field theory that the Tate sequence is constructed.

The sequence is a primary feature in numerous applications to multiplicative Galois module structure. Tate uses it in [21, Ch. II] to prove Stark's Conjecture for rational characters, and Chinburg employs it in the construction of his third Ω -invariant, which is central to his root number conjecture—see [4, 5]. In fact, the Tate sequence also appears in the definition of the lifted Ω -invariant in the Lifted Root Number Conjecture of Gruenberg–Ritter–Weiss [9]. Further, a variant of the sequence is used to construct the equivariant Tamagawa number in the Equivariant Tamagawa Number Conjecture (ETNC) for the motive $h^0(\text{Spec}(L))(0)$, as in Burns–Flach [2] for example.

2010 *Mathematics Subject Classification*: Primary 11R29; Secondary 11R34.

Key words and phrases: Tate sequence, class-group, Galois cohomology.

A significant step forward in the theory of Tate sequences was Ritter and Weiss’s Tate sequence “for small S ”, which allows the sequence to be constructed for an arbitrary set of places S . More precisely, they found in [15] a sequence

$$(1.1) \quad 0 \rightarrow \mathcal{O}_{L,S}^\times \rightarrow A \rightarrow B \rightarrow \nabla \rightarrow 0$$

for a Galois extension L/K of number fields with Galois group G , with no restriction on the set S other than that it contains the infinite places. Whereas in the preceding theory of Tate sequences, the module in place of ∇ was the \mathbb{Z} -torsion-free $\mathbb{Z}[G]$ -module X whose Galois structure is described in terms of very basic arithmetic information, ∇ itself fits into an exact sequence

$$(1.2) \quad 0 \rightarrow \text{Cl}_S(L) \rightarrow \nabla \rightarrow X \rightarrow 0.$$

An early use of this refined Tate sequence “for small S ” was in the proof by Ritter and Weiss [16] of the Strong Stark Conjecture for abelian extensions of \mathbb{Q} . More recent applications have been the consideration of the minus part of the ETNC for tame extensions, as in Nickel [13], and the study of Fitting ideals of (duals of) minus parts of class-groups, as in Greither [8].

Despite the sequence’s growing use in multiplicative Galois module structure, one aspect of it that has not yet been determined is the precise effect of the connecting homomorphisms that are naturally associated to the sequence. Our aim is to make certain such maps explicit, under some assumptions on the set S , namely that S will contain the infinite and ramified places and at least one place with full decomposition group. We expect to be able to remove this last assumption in the future. However, we emphasize that the S -class-group will not be assumed trivial. The maps we make explicit are the connecting homomorphism $H^{-2}(G, X) \rightarrow H^{-1}(G, \text{Cl}_S(L))$ associated to (1.2), and the map $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times)$ that results from $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^{-1}(G, \nabla)$ together with the appropriate connecting homomorphisms (i.e. starting in dimension -1 then 0) obtained by splitting (1.1) into two short exact sequences.

After building up preliminaries in Sections 2–5, we compute the maps in Sections 6 and 7, and give some corollaries in Section 8.

2. Notation, assumptions and conventions. We will assume throughout that L/K is Galois with Galois group G , and S will denote a finite set of places of K containing the infinite and ramified places. By a minor abuse of notation, $\mathcal{O}_{L,S}$ will denote the S_L -integers in L , where S_L consists of the places of L above those in S . Thus $\mathcal{O}_{L,S}^\times$ is the group of S_L -units in L . The S -class-group of K will be written $\text{Cl}_S(K)$, and the

S_L -class-group of L will be just $\text{Cl}_S(L)$, the redundant subscript L again being dropped.

By L_S , we mean the Hilbert S_L -class field of L , that is, the maximal unramified abelian extension of L in which all places in S_L split completely. $\text{Gal}(L_S/L)$ is isomorphic, via the Artin map, to $\text{Cl}_S(L)$.

We will denote the element $\sum_{\sigma \in G} \sigma \in \mathbb{Z}[G]$ by N . Multiplication by N induces an endomorphism of $\text{Cl}_S(L)$, but we stress that this is different from the map $\text{Nm} : \text{Cl}_S(L) \rightarrow \text{Cl}_S(K)$, induced by the norm of ideals, that is featured in Section 8.

All number fields will lie in a fixed algebraic closure $\bar{\mathbb{Q}}$ of \mathbb{Q} . If \mathfrak{p} is a place of K , we will fix once and for all a place of $\bar{\mathbb{Q}}$ above \mathfrak{p} . Then given any number field F containing K , $\mathfrak{p}(F)$ will denote the place of F below the chosen place of $\bar{\mathbb{Q}}$ above \mathfrak{p} . For shorthand, we will denote the completion $F_{\mathfrak{p}(F)}$ by $F_{\mathfrak{p}}$, and if F/K is Galois, the decomposition group $\text{Gal}(F/K)_{\mathfrak{p}(F)}$ will be denoted simply $\text{Gal}(F/K)_{\mathfrak{p}}$. Note that if $\mathfrak{p} \in S$, then since $\mathfrak{p}(L)$ splits completely in L_S , restriction $\text{Gal}(L_S/K)_{\mathfrak{p}} \rightarrow G_{\mathfrak{p}}$ defines an isomorphism, and we denote by $\iota_{\mathfrak{p}}$ the composition

$$G_{\mathfrak{p}} \xrightarrow{\cong} \text{Gal}(L_S/K)_{\mathfrak{p}} \rightarrow \text{Gal}(L_S/K).$$

For a finite place \mathfrak{p} of K , let $\tilde{\varphi}_{\mathfrak{p}}$ denote a lift in $\text{Gal}(L_{\mathfrak{p}}^{\text{ur}}/K_{\mathfrak{p}})$ of the Frobenius of $K_{\mathfrak{p}}^{\text{ur}}/K_{\mathfrak{p}}$, and $\bar{\varphi}_{\mathfrak{p}}$ its image in $\text{Gal}(L_S/K)$. Further, if \mathfrak{p} is unramified in L/K , then $\varphi_{\mathfrak{p}}$ will be the associated Frobenius element in $G_{\mathfrak{p}}$.

An important object appearing throughout the article is the Galois module X defined below:

DEFINITION 2.1. Let Y be the free abelian group on S_L , and X the kernel of the augmentation map $Y \rightarrow \mathbb{Z}$ sending every place $\mathfrak{P} \in S_L$ to 1.

To give some context, we remark that X appears in the Dirichlet regulator map, i.e. the isomorphism

$$\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_{L,S}^{\times} \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} X, \quad 1 \otimes u \mapsto \sum_{w \in S_L} \log \|u\|_w w,$$

where the absolute values $\|\cdot\|_w$ are normalized in the particular canonical way which makes the product formula hold, as in [12, Ch. III, Section 1].

For any group G , ΔG will denote its augmentation ideal, that is, the kernel of the augmentation map $\mathbb{Z}[G] \rightarrow \mathbb{Z}$ which sends each group element to 1.

2.1. Key assumptions. In Sections 4 and 5.1, we assume that L/K is Galois and S contains the ramified places. In Sections 5.2 and 6–8, we further assume that there is a place $\mathfrak{p}_0 \in S$ such that $G_{\mathfrak{p}_0} = G$. We observe that this last assumption forces G to be solvable, since then G is the Galois group of a Galois extension of local fields.

3. Group cohomology. Cohomology will be Tate cohomology throughout. We will use the following model for Tate cohomology groups in negative degrees, which we shall think of as homology groups, that is, $H^i(G, A) = H_{-i-1}(G, A)$ for $i < -1$: For $n \geq 0$, let P_n be the free *right* $\mathbb{Z}[G]$ -module on G^n , and for $n \geq 1$ define a boundary map $d_n : P_n \rightarrow P_{n-1}$ by sending $[\sigma_1, \dots, \sigma_n]$ to

$$[\sigma_1, \dots, \sigma_{n-1}] \cdot \sigma_n + \sum_{i=1}^{n-1} (-1)^{n-i} [\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_n] + (-1)^n [\sigma_2, \dots, \sigma_n].$$

Then $P_\bullet \xrightarrow{\text{aug}} \mathbb{Z} \rightarrow 0$ is a free resolution of \mathbb{Z} as a *right* $\mathbb{Z}[G]$ -module, and for a *left* $\mathbb{Z}[G]$ -module A , we view $H_i(G, A)$ as the i th homology group of the chain complex $P_\bullet \otimes_{\mathbb{Z}[G]} A$ when $i > 0$. The Tate cohomology group $H^{-1}(G, A)$ will be taken to mean

$$(3.1) \quad \frac{\{a \in A \mid \sum_{\sigma \in G} \sigma a = 0\}}{\mathbb{Z}\{(\sigma - 1)a \mid \sigma \in G, a \in A\}}.$$

Note that the denominator in (3.1) is indeed a $\mathbb{Z}[G]$ -submodule, i.e. is closed under the action of G .

LEMMA 3.1. *Let G be a finite group and H a subgroup. There is a well-defined group isomorphism*

$$H^{\text{ab}} \rightarrow H^{-2}(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z})$$

sending $\sigma[H, H]$ to $\overline{[\sigma] \otimes 1 \otimes 1}$.

Proof. That H^{ab} is isomorphic to $H^{-2}(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} \mathbb{Z})$ is simply Shapiro’s Lemma—as in [22, Lemma 6.3.2] for example—together with the fact that $H^{-2}(H, \mathbb{Z}) \simeq H^{\text{ab}}$. The explicit description of the isomorphism is left to the reader. ■

3.1. Extension classes. Suppose that A and C are $\mathbb{Z}[G]$ -modules and that C is \mathbb{Z} -free.

LEMMA 3.2. *There is a canonical isomorphism*

$$\text{Ext}_{\mathbb{Z}[G]}^1(C, A) \simeq H^1(G, \text{Hom}_{\mathbb{Z}}(C, A)).$$

Proof. Since this is well known, we only sketch the proof. Supposing we have an exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ of $\mathbb{Z}[G]$ -modules, choose a section $s : C \rightarrow B$ of the \mathbb{Z} -module homomorphism $B \rightarrow C$, so that $s \in \text{Hom}_{\mathbb{Z}}(C, B)$ maps to 1_C in $\text{Hom}_{\mathbb{Z}}(C, C)$. This is possible by the assumption on C . Then for each $\sigma \in G$, $\sigma s - s$ is the image of a unique $f_\sigma \in \text{Hom}_{\mathbb{Z}}(C, A)$. The map $\sigma \mapsto f_\sigma$ is a 1-cocycle $G \rightarrow \text{Hom}_{\mathbb{Z}}(C, A)$.

Conversely, given a 1-cocycle $f : G \rightarrow \text{Hom}_{\mathbb{Z}}(C, A)$, we can endow the direct sum $B = A \oplus C$ of \mathbb{Z} -modules with a G -action by

$$(3.2) \quad \sigma(a, c) = (\sigma a + f(\sigma)(\sigma c), \sigma c).$$

One checks that these constructions pass to mutually inverse maps between $\text{Ext}_{\mathbb{Z}[G]}^1(C, A)$ and $H^1(G, \text{Hom}_{\mathbb{Z}}(C, A))$. ■

LEMMA 3.3. *Suppose $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of $\mathbb{Z}[G]$ -modules such that C is \mathbb{Z} -free. Viewing its extension class ξ as an element of $H^1(G, \text{Hom}_{\mathbb{Z}}(C, A))$, the connecting homomorphism $H^i(G, C) \rightarrow H^{i+1}(G, A)$ is given by following cup-product with ξ by the evaluation map*

$$H^{i+1}(G, C \otimes_{\mathbb{Z}} \text{Hom}_{\mathbb{Z}}(C, A)) \rightarrow H^{i+1}(G, A).$$

Proof. See [3, Ch. XII, Prop. 6.1]. ■

By Lemma 3.2, $\text{Ext}_{\mathbb{Z}[G]}^1(\Delta G, A) \simeq H^1(G, \text{Hom}_{\mathbb{Z}}(\Delta G, A))$ for each $\mathbb{Z}[G]$ -module A . On the other hand, the exact sequence

$$0 \rightarrow A \rightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A) \rightarrow \text{Hom}_{\mathbb{Z}}(\Delta G, A) \rightarrow 0$$

together with the cohomological triviality of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ gives an isomorphism $H^1(G, \text{Hom}_{\mathbb{Z}}(\Delta G, A)) \rightarrow H^2(G, A)$. Thus we have:

LEMMA 3.4. *For any $\mathbb{Z}[G]$ -module A , $\text{Ext}_{\mathbb{Z}[G]}^1(\Delta G, A) \simeq H^2(G, A)$.*

4. The modules $W_{S'}$, $R_{S'}$ and $B_{S'}$

4.1. The modules $W_{S'}$ and $R_{S'}$. We let S' denote a finite set of places of K containing S , and further satisfying:

- (i) $\bigcup_{\mathfrak{p} \in S'} G_{\mathfrak{p}} = G$,
- (ii) $\text{Cl}_{S'}(L) = 0$.

Such an S' can always be chosen, by the Chebotarev Density Theorem. Following [15, Section 1], we define

$$W_{S'} = \bigoplus_{\mathfrak{p} \in S} \Delta_{\mathfrak{p}} G \oplus \bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z}[G],$$

where $\Delta_{\mathfrak{p}} G$ is the left ideal in $\mathbb{Z}[G]$ generated by $\Delta G_{\mathfrak{p}}$. We observe that this description of $W_{S'}$ relies on S containing the ramified primes. For a treatment of the general case, see [15] itself.

As in [15, Section 4], we now define $R_{S'}$ to be the kernel of the map $W_{S'} \rightarrow \Delta G$ that is inclusion on $\Delta_{\mathfrak{p}} G$ and left multiplication by $\varphi_{\mathfrak{q}} - 1$ on the copy of $\mathbb{Z}[G]$ corresponding to $\mathfrak{q} \in S' \setminus S$.

4.2. The module $B_{S'}$. After [15, Section 4], we let $B_{S'}$ be the kernel of the map

$$\bigoplus_{\mathfrak{p} \in S'} \mathbb{Z}[G] \rightarrow \mathbb{Z}[G]$$

that is the identity on the copy of $\mathbb{Z}[G]$ corresponding to $\mathfrak{p} \in S$, and multiplication by $\varphi_{\mathfrak{q}} - 1$ on the copy of $\mathbb{Z}[G]$ corresponding to $\mathfrak{q} \in S' \setminus S$. We note that $B_{S'}$ is projective, and therefore cohomologically trivial. To provide more context, we note that $B_{S'}$ is the module B appearing in (1.1). The map $R_{S'} \rightarrow B_{S'}$ in (4.5) below is induced by the inclusion map

$$W_{S'} \rightarrow \bigoplus_{\mathfrak{p} \in S'} \mathbb{Z}[G].$$

The map $B_{S'} \rightarrow X$ is induced by the map

$$\bigoplus_{\mathfrak{p} \in S'} \mathbb{Z}[G] \rightarrow Y$$

that is zero on $\mathbb{Z}[G]$ corresponding to $\mathfrak{q} \in S' \setminus S$ and sends $\alpha \in \mathbb{Z}[G]$ corresponding to $\mathfrak{p} \in S$ to $\alpha\mathfrak{p}(L)$.

4.3. Class field theory and diagrams. Denote the idele class-group of L by C_L . We let

$$0 \rightarrow C_L \rightarrow \mathcal{V} \rightarrow \Delta G \rightarrow 0$$

be the extension corresponding under the isomorphism of Lemma 3.4 to the global fundamental class in $H^2(G, C_L)$. In fact, for concreteness, we will take the following description of \mathcal{V} : Suppose the element of $H^1(G, \text{Hom}_{\mathbb{Z}}(\Delta G, C_L))$ corresponding to the global fundamental class is represented by the 1-cocycle f . Then we view \mathcal{V} as $C_L \oplus \Delta G$ (direct sum as \mathbb{Z} -modules) with G -action given as in (3.2).

Ritter and Weiss also define local versions $V_{\mathfrak{p}}$ of \mathcal{V} for each $\mathfrak{p} \in S'$. For $\mathfrak{p} \in S$, $V_{\mathfrak{p}}$ is simply the analogous construction to \mathcal{V} with C_L replaced by $L_{\mathfrak{p}}^{\times}$ and the global fundamental class replaced by the local one. For $\mathfrak{p} \in S' \setminus S$, the definition is more subtle, but still uses local class-field-theoretic data. We will only need the definitions in a very simple situation, which we will turn to in Section 7.2. The reader wishing to see the complete definition may refer to [15, Sections 1, 3]. Following Ritter and Weiss, we set

$$(4.1) \quad V_{S'} = \left(\bigoplus_{\mathfrak{p} \in S'} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_{\mathfrak{p}}]} V_{\mathfrak{p}} \right) \oplus \left(\bigoplus_{\mathfrak{P} \notin S'_L} U_{\mathfrak{P}} \right),$$

where $U_{\mathfrak{P}}$ is the group of units in $L_{\mathfrak{P}}$. Note that the first direct sum runs through primes \mathfrak{p} of K (in S'), whereas the second direct sum runs through primes \mathfrak{P} of L (not above S').

The important diagrams involving the modules $W_{S'}$, $R_{S'}$ and $B_{S'}$ are

$$(4.2) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{L,S}^\times & \longrightarrow & A & \longrightarrow & R_{S'} \xrightarrow{s} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J_S & \longrightarrow & V_{S'} & \longrightarrow & W_{S'} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_L & \longrightarrow & \mathcal{V} & \longrightarrow & \Delta G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \text{Cl}_S(L) & \longrightarrow & 0 & & 0 \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

and

$$(4.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & R_{S'} & \longrightarrow & B_{S'} & \longrightarrow & X \longrightarrow 0 \\ & & \downarrow s & & \downarrow t & & \downarrow 1 \\ 0 & \longrightarrow & \text{Cl}_S(L) & \longrightarrow & \nabla & \longrightarrow & X \longrightarrow 0 \end{array}$$

In (4.2), the module A is defined to be the kernel of $V_{S'} \rightarrow \mathcal{V}$, and (4.2) is simply the snake diagram arising from the middle two rows. The surjectivity of $V_{S'} \rightarrow \mathcal{V}$ in (4.2) and the origin of (4.3) are treated in [15, Section 4].

$R_{S'}$ fits into an exact sequence

$$(4.4) \quad 0 \rightarrow \mathcal{O}_{L,S}^\times \rightarrow A \rightarrow R_{S'} \xrightarrow{s} \text{Cl}_S(L) \rightarrow 0,$$

where A is the (cohomologically trivial) $\mathbb{Z}[G]$ -module appearing in the Tate sequence (1.1). Ritter and Weiss term the map s the “snake map” because it is the snake map of diagram (4.2)—see the discussion following [15, Theorem 1]. The significance of the snake map is as follows: Applying $\text{Hom}_{\mathbb{Z}}(-, \text{Cl}_S(L))$ to the short exact sequence

$$(4.5) \quad 0 \rightarrow R_{S'} \rightarrow B_{S'} \rightarrow X \rightarrow 0$$

occurring in (4.3), and remembering that X is \mathbb{Z} -free, we obtain the exact sequence

$$0 \rightarrow \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L)) \rightarrow \text{Hom}_{\mathbb{Z}}(B_{S'}, \text{Cl}_S(L)) \rightarrow \text{Hom}_{\mathbb{Z}}(R_{S'}, \text{Cl}_S(L)) \rightarrow 0.$$

Upon identifying $\text{Ext}_{\mathbb{Z}[G]}^1(X, \text{Cl}_S(L))$ with $H^1(G, \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L)))$, the ex-

tension class of (1.2) is the image of the class of $-s$ under the connecting homomorphism

$$H^0(G, \text{Hom}_{\mathbb{Z}}(R_{S'}, \text{Cl}_S(L))) \rightarrow H^1(G, \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L))).$$

Since (1.2) is central to our purpose, getting a strong hold on s will be one of our main goals, and will be carried out in Section 5.

5. The snake map

5.1. Explicit description of s . In this section, L/K is a Galois extension of number fields and S a finite set of places of K containing the infinite and ramified ones. We let $G^S = \text{Gal}(L_S/K)$.

We now come to the description of the snake map $s : R_{S'} \rightarrow \text{Cl}_S(L)$ given in [15, Section 5]. It is defined first of all as a prime-by-prime map $W_{S'} \rightarrow \mathcal{H}$ where

$$\mathcal{H} = \frac{\Delta G^S}{\text{Ker}(\Delta G^S \rightarrow \Delta G)\Delta G^S}.$$

(We note that an element σ of G acts on \mathcal{H} by left multiplication by any preimage of σ in G^S .)

For $\mathfrak{p} \in S$, the map $\Delta_{\mathfrak{p}}G \rightarrow \mathcal{H}$ sends $\beta\alpha$ to $\beta(\overline{\iota_{\mathfrak{p}}(\alpha)})$ for $\alpha \in \Delta G_{\mathfrak{p}}$ and $\beta \in \mathbb{Z}[G]$, where the bar denotes class in \mathcal{H} .

Recall the element $\bar{\varphi}_{\mathfrak{q}} \in G^S = \text{Gal}(L_S/K)$ defined in Section 2. For $\mathfrak{q} \in S' \setminus S$, the image of an element β inside the copy of $\mathbb{Z}[G]$ in $W_{S'}$ corresponding to \mathfrak{q} is mapped to $\beta(\bar{\varphi}_{\mathfrak{q}})$ in \mathcal{H} .

There is an embedding $\text{Gal}(L_S/L) \rightarrow \mathcal{H}$ which sends σ to the class of $\sigma - 1$, and the restriction of the map $W_{S'} \rightarrow \mathcal{H}$ to $R_{S'}$ has its image in $\text{Im}(\text{Gal}(L_S/L) \rightarrow \mathcal{H})$. Identifying $\text{Gal}(L_S/L)$ with $\text{Cl}_S(L)$, we have thus described the snake map $s : R_{S'} \rightarrow \text{Cl}_S(L)$.

5.2. The extension class of ∇ . We now assume that there exists a finite place $\mathfrak{p}_0 \in S$ such that $G_{\mathfrak{p}_0} = G$. For each $\mathfrak{p} \in S$, fix a set $D_{\mathfrak{p}}$ of representatives for $(G/G_{\mathfrak{p}})_{\text{left}}$ containing 1, and given $\sigma \in G$ let $\rho_{\mathfrak{p}}(\sigma)$ be the chosen representative of the coset $\sigma G_{\mathfrak{p}}$. It is possible to choose S' as in Section 4 with the further condition that every place $\mathfrak{q} \in S' \setminus S$ splits completely in L/K . For this, we are already using that $\bigcup_{\mathfrak{p} \in S} G_{\mathfrak{p}} = G$. We are assuming more, of course: $G_{\mathfrak{p}_0} = G$. With S' chosen in this way, we have

$$(5.1) \quad R_{S'} = \text{Ker}\left(\bigoplus_{\mathfrak{p} \in S} \Delta_{\mathfrak{p}}G \rightarrow \Delta G\right) \oplus \bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z}[G].$$

DEFINITION 5.1. Given $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$, $\sigma \in G$ and $\tau \in D_{\mathfrak{p}}$, let $r_{\sigma, \tau}^{(\mathfrak{p})}$ be the element of $R_{S'} = \text{Ker}(\bigoplus_{\mathfrak{p} \in S} \Delta_{\mathfrak{p}}G \rightarrow \Delta G) \oplus \bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z}[G]$ which has $\sigma\rho_{\mathfrak{p}}(\sigma^{-1}\tau) - \tau$ in the \mathfrak{p} -component, $\tau - \sigma\rho_{\mathfrak{p}}(\sigma^{-1}\tau)$ in the \mathfrak{p}_0 -component, and zero elsewhere.

LEMMA 5.2. Define $g : G \rightarrow \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L))$ by sending $\sigma \in G$ to the map

$$\tau \mathfrak{p}(L) - \mathfrak{p}_0(L) \mapsto s(r_{\sigma, \tau}^{(\mathfrak{p})})$$

for $\tau \in D_{\mathfrak{p}}$, $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$. Then the image of $-\bar{s} \in H^0(G, \text{Hom}_{\mathbb{Z}}(R_{S'}, \text{Cl}_S(L)))$ in $H^1(G, \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L)))$ is \bar{g} .

Proof. We follow the diagram

$$(5.2) \quad \begin{array}{ccc} & & \text{Map}(G, \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L))) \\ & & \downarrow \\ \text{Hom}_{\mathbb{Z}}(B_{S'}, \text{Cl}_S(L)) & \longrightarrow & \text{Map}(G, \text{Hom}_{\mathbb{Z}}(B_{S'}, \text{Cl}_S(L))) \\ \downarrow & & \\ \text{Hom}_{\mathbb{Z}}(R_{S'}, \text{Cl}_S(L)) & & \end{array}$$

from bottom-left to top-right. Note that

$$B_{S'} = \text{Ker} \left(\bigoplus_{\mathfrak{p} \in S} \mathbb{Z}[G] \rightarrow \mathbb{Z}[G] \right) \oplus \bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z}[G].$$

We first look for a \mathbb{Z} -splitting of $0 \rightarrow R_{S'} \rightarrow B_{S'} \rightarrow X \rightarrow 0$. We use the \mathbb{Z} -basis

$$\bigcup_{\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}} \{ \tau \mathfrak{p}(L) - \mathfrak{p}_0(L) \mid \tau \in D_{\mathfrak{p}} \}$$

of X . For $\mathfrak{p} \in S$, let $e_{\mathfrak{p}}$ be the element of $\bigoplus_{\mathfrak{p} \in S} \mathbb{Z}[G]$ which has 1 in the \mathfrak{p} -component and zero everywhere else. Then a lift of $\tau \mathfrak{p}(L) - \mathfrak{p}_0(L)$ to $B_{S'}$ is $\tau e_{\mathfrak{p}} - \tau e_{\mathfrak{p}_0}$. Therefore if Z is the \mathbb{Z} -span of these lifts,

$$B_{S'} = R_{S'} \oplus Z.$$

Define $\mu : B_{S'} \rightarrow \text{Cl}_S(L)$ by $\mu|_{R_{S'}} = -s$ and $\mu|_Z = 0$. Let f be the image of μ under the horizontal map in (5.2), i.e. if $\sigma \in G$, then $f(\sigma) = \sigma \mu - \mu$. Now take $\tau \in D_{\mathfrak{p}}$ for some $\mathfrak{p} \in S$. Then

$$(\sigma \mu)(\tau e_{\mathfrak{p}} - \tau e_{\mathfrak{p}_0}) = \sigma \mu(\sigma^{-1} \tau e_{\mathfrak{p}} - \sigma^{-1} \tau e_{\mathfrak{p}_0}),$$

and one sees that $\sigma^{-1} \tau e_{\mathfrak{p}} - \sigma^{-1} \tau e_{\mathfrak{p}_0}$ decomposes as

$$(5.3) \quad \begin{aligned} & [(\sigma^{-1} \tau - \rho_{\mathfrak{p}}(\sigma^{-1} \tau))e_{\mathfrak{p}} + (\rho_{\mathfrak{p}}(\sigma^{-1} \tau) - \sigma^{-1} \tau)e_{\mathfrak{p}_0}] \\ & \quad + [\rho_{\mathfrak{p}}(\sigma^{-1} \tau)e_{\mathfrak{p}} - \rho_{\mathfrak{p}}(\sigma^{-1} \tau)e_{\mathfrak{p}_0}] \end{aligned}$$

in $R_{S'} \oplus Z$. Also, we recognize the first bracketed element in (5.3) as $-\sigma^{-1} r_{\sigma, \tau}^{(\mathfrak{p})}$. Thus

$$(\sigma \mu)(\tau e_{\mathfrak{p}} - \tau e_{\mathfrak{p}_0}) = -\sigma \mu(\sigma^{-1} r_{\sigma, \tau}^{(\mathfrak{p})}) = \sigma s(\sigma^{-1} r_{\sigma, \tau}^{(\mathfrak{p})}) = s(r_{\sigma, \tau}^{(\mathfrak{p})}).$$

Since $\mu(\tau e_{\mathfrak{p}} - \tau e_{\mathfrak{p}_0}) = 0$, we therefore obtain

$$f(\sigma)(\tau e_{\mathfrak{p}} - \tau e_{\mathfrak{p}_0}) = s(r_{\sigma, \tau}^{(\mathfrak{p})}).$$

Further, the map $g : G \rightarrow \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L))$ appearing in the statement of the lemma has image f in $\text{Map}(G, \text{Hom}_{\mathbb{Z}}(B_{S'}, \text{Cl}_S(L)))$. The lemma is proven. ■

DEFINITION 5.3. If $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$, then $\iota_{\mathfrak{p}_0}(\tau)^{-1} \iota_{\mathfrak{p}}(\tau) \in \text{Gal}(L_S/L)$, and we let its image in $\text{Cl}_S(L)$ under the Artin map be $\mathfrak{c}_{\mathfrak{p}}(\tau)$.

LEMMA 5.4. If $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$, $\tau \in D_{\mathfrak{p}}$ and $\sigma \in G$, then

$$s(r_{\sigma, \tau}^{(\mathfrak{p})}) = \tau \mathfrak{c}_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau)).$$

Proof. Since s is a $\mathbb{Z}[G]$ -module homomorphism, the statement of the lemma is equivalent to $s(\tau^{-1} r_{\sigma, \tau}^{(\mathfrak{p})}) = \mathfrak{c}_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau))$. Now, recalling the notation $e_{\mathfrak{p}}$ introduced in the proof of Lemma 5.2, we see that

$$\tau^{-1} r_{\sigma, \tau}^{(\mathfrak{p})} = (\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau) - 1)e_{\mathfrak{p}} - (\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau) - 1)e_{\mathfrak{p}_0},$$

whose image in \mathcal{H} is

$$\overline{\iota_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau)) - 1} - \overline{\iota_{\mathfrak{p}_0}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau)) - 1}$$

by the discussion of Section 5.1. However, this is simply

$$(5.4) \quad \overline{\iota_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau)) - \iota_{\mathfrak{p}_0}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau))}.$$

Now, one checks that if $\sigma_1, \sigma_2 \in G^S$ and $\sigma_2^{-1} \sigma_1 \in \text{Gal}(L_S/L)$, then $\sigma_2^{-1} \sigma_1$ has image $\overline{\sigma_1 - \sigma_2}$ in \mathcal{H} . Therefore the element of (5.4) is the image of

$$\iota_{\mathfrak{p}_0}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau))^{-1} \iota_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau))$$

under $\text{Gal}(L_S/L) \rightarrow \mathcal{H}$. Thus $s(\tau^{-1} r_{\sigma, \tau}^{(\mathfrak{p})}) = \mathfrak{c}_{\mathfrak{p}}(\tau^{-1} \sigma \rho_{\mathfrak{p}}(\sigma^{-1} \tau))$ as desired. ■

For each $\mathfrak{q} \in S' \setminus S$, let $s_{\mathfrak{q}} : \mathbb{Z}[G] \rightarrow \text{Cl}_S(L)$ be the restriction of the snake map to the copy of $\mathbb{Z}[G]$ in $R_{S'}$ corresponding to \mathfrak{q} .

LEMMA 5.5. $s_{\mathfrak{q}}(1) = \overline{\mathfrak{q}(L)\mathcal{O}_{L,S}}$, where $\mathfrak{q}(L)$ is the distinguished prime of L above \mathfrak{q} .

Proof. The map $\mathbb{Z}[G] \rightarrow R_{S'} \rightarrow \mathcal{H}$ corresponding to \mathfrak{q} sends 1 to $\overline{\bar{\varphi}_{\mathfrak{q}} - 1}$ (recall the definition of $\bar{\varphi}_{\mathfrak{p}}$ in Section 2), which is the image of $\bar{\varphi}_{\mathfrak{q}}$ under $\text{Gal}(L_S/L) \rightarrow \mathcal{H}$. Note that $\bar{\varphi}_{\mathfrak{q}}$ is indeed in $\text{Gal}(L_S/L)$ because $K_{\mathfrak{q}} = L_{\mathfrak{q}}$. We also observe that $\bar{\varphi}_{\mathfrak{q}}$ is, in this case, just the Frobenius in L_S/L associated to $\mathfrak{q}(L)$, and therefore the image of $\overline{\mathfrak{q}(L)\mathcal{O}_{L,S}}$ under the Artin map $\text{Cl}_S(L) \rightarrow \text{Gal}(L_S/L)$ is $\bar{\varphi}_{\mathfrak{q}}$, completing the proof. ■

6. The map $H^{-2}(G, X) \rightarrow H^{-1}(G, \text{Cl}_S(L))$. As in Section 5.2, we assume that S contains the infinite and ramified places and that there exists a finite place $\mathfrak{p}_0 \in S$ such that $G_{\mathfrak{p}_0} = G$. We will assume this for the remainder of the article.

LEMMA 6.1. $H^{-2}(G, X)$ is generated over \mathbb{Z} by

$$\overline{[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))} \mid \mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}, \tau \in G_{\mathfrak{p}}\}.$$

Proof. Note that the exact sequence $0 \rightarrow X \rightarrow Y \rightarrow \mathbb{Z} \rightarrow 0$ gives an exact sequence

$$(6.1) \quad 0 \rightarrow H^{-2}(G, X) \rightarrow H^{-2}(G, Y) \rightarrow H^{-2}(G, \mathbb{Z}) \rightarrow 0.$$

Indeed, the assumption $G_{\mathfrak{p}_0} = G$ implies that $H^{-1}(G, X)$ is zero and that $H^{-3}(G, Y) \rightarrow H^{-3}(G, \mathbb{Z})$ is surjective. Now, Y is isomorphic to the module $\bigoplus_{\mathfrak{p} \in S} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_{\mathfrak{p}}]} \mathbb{Z}$, so by Lemma 3.1, there is an isomorphism

$$\bigoplus_{\mathfrak{p} \in S} G_{\mathfrak{p}}^{\text{ab}} \rightarrow H^{-2}(G, Y), \quad (\tau_{\mathfrak{p}}[G_{\mathfrak{p}}, G_{\mathfrak{p}}])_{\mathfrak{p}} \mapsto \sum_{\mathfrak{p} \in S} \overline{[\tau_{\mathfrak{p}}] \otimes \mathfrak{p}(L)}.$$

Given $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$ let $x_{\mathfrak{p}}(\tau)$ be the element of $\bigoplus_{\mathfrak{p} \in S} G_{\mathfrak{p}}^{\text{ab}}$ with $\tau[G_{\mathfrak{p}}, G_{\mathfrak{p}}]$ in the \mathfrak{p} -component, $\tau^{-1}[G_{\mathfrak{p}_0}, G_{\mathfrak{p}_0}]$ in the \mathfrak{p}_0 -component and the commutator group $[G_{\mathfrak{q}}, G_{\mathfrak{q}}]$ in the \mathfrak{q} -component for $\mathfrak{q} \in S \setminus \{\mathfrak{p}_0, \mathfrak{p}\}$. Observe then that $\text{Ker}(\bigoplus_{\mathfrak{p} \in S} G_{\mathfrak{p}}^{\text{ab}} \rightarrow G^{\text{ab}})$ is generated by

$$\{x_{\mathfrak{p}}(\tau) \mid \mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}, \tau \in G_{\mathfrak{p}}\}.$$

This makes use of the fact that $G_{\mathfrak{p}_0} = G$.

Now, $x_{\mathfrak{p}}(\tau)$ maps to

$$\overline{[\tau] \otimes \mathfrak{p}(L) - [\tau] \otimes \mathfrak{p}_0(L)} = \overline{[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))},$$

which lies in the image of the injective map $H^{-2}(G, X) \rightarrow H^{-2}(G, Y)$. Since $\text{Ker}(\bigoplus_{\mathfrak{p} \in S} G_{\mathfrak{p}}^{\text{ab}} \rightarrow G^{\text{ab}})$ is generated by the elements $x_{\mathfrak{p}}(\tau)$ as above, we obtain an injective map

$$\text{Ker}\left(\bigoplus_{\mathfrak{p} \in S} G_{\mathfrak{p}}^{\text{ab}} \rightarrow G^{\text{ab}}\right) \rightarrow H^{-2}(G, X)$$

which sends $x_{\mathfrak{p}}(\tau)$ to $\overline{[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))}$. Since both sides have the same order (they can be shown to be isomorphic by using the sequence in (6.1) directly, together with Shapiro’s Lemma), this map is an isomorphism. This completes our proof. ■

REMARK. It is clear from the definition of the elements $x_{\mathfrak{p}}(\tau)$ in the above proof that, in the statement of Lemma 6.1, the elements τ need only be taken from a set of representatives for the left cosets of $[G_{\mathfrak{p}}, G_{\mathfrak{p}}]$ in $G_{\mathfrak{p}}$.

LEMMA 6.2. Let A be a $\mathbb{Z}[G]$ -module and $\xi \in H^1(G, A)$ be represented by the 1-cocycle $g : G \rightarrow A$. Then the map

$$H^{-2}(G, X) \rightarrow H^{-1}(G, X \otimes_{\mathbb{Z}} A)$$

obtained by taking cup-product with ξ sends the element $\overline{[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))}$ to $\overline{(\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\tau)}$, where $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$.

Proof. We dimension shift using the commutativity of the diagram

$$\begin{array}{ccc}
 H^{-2}(G, X) & \xrightarrow{c_1} & H^{-1}(G, X \otimes_{\mathbb{Z}} A) \\
 \downarrow \delta_1 & & \downarrow \delta_2 \\
 H^{-1}(G, \Delta G \otimes_{\mathbb{Z}} X) & \xrightarrow{c_2} & H^0(G, \Delta G \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} A)
 \end{array}$$

where c_1 and c_2 are obtained by cupping with ξ . Given $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$, we have $\delta_1([\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))) = \overline{(\tau - 1) \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))}$. By [18, Appendix to Chapter XI, Lemma 2], the image of this under c_2 is represented in $H^0(G, \Delta G \otimes_{\mathbb{Z}} X \otimes_{\mathbb{Z}} A)$ by

$$\begin{aligned}
 & - \sum_{\sigma \in G} \sigma((\tau - 1) \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))) \otimes g(\sigma) \\
 &= - \sum_{\sigma \in G} (\sigma\tau - 1) \otimes (\sigma\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\sigma) \\
 & \quad + \sum_{\sigma \in G} (\sigma - 1) \otimes (\sigma\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\sigma) \\
 &= - \sum_{\sigma \in G} (\sigma - 1) \otimes (\sigma\tau^{-1}\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\sigma\tau^{-1}) \\
 & \quad + \sum_{\sigma \in G} (\sigma - 1) \otimes (\sigma\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\sigma) \\
 &= - \sum_{\sigma \in G} (\sigma - 1) \otimes (\sigma\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes (g(\sigma\tau^{-1}) - g(\sigma)) \\
 &= - \sum_{\sigma \in G} (\sigma - 1) \otimes (\sigma\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes \sigma g(\tau^{-1}).
 \end{aligned}$$

The image under δ_2^{-1} of the class of this is

$$\begin{aligned}
 -\overline{(\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\tau^{-1})} &= \overline{(\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes \tau^{-1}g(\tau)} \\
 &= \overline{(\mathfrak{p}(L) - \mathfrak{p}_0(L)) \otimes g(\tau)}. \blacksquare
 \end{aligned}$$

PROPOSITION 6.3. *Under the above assumptions, the connecting homomorphism $H^{-2}(G, X) \rightarrow H^{-1}(G, \text{Cl}_S(L))$ sends $[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))$, with $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$, to $\mathfrak{c}_{\mathfrak{p}}(\tau)$. (See Definition 5.3 for $\mathfrak{c}_{\mathfrak{p}}(\tau)$.)*

Proof. We apply Lemma 6.2 in the case $A = \text{Hom}_{\mathbb{Z}}(X, \text{Cl}_S(L))$ and take g as in Lemma 5.2, and combine this with Lemma 3.3. Given $\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}$ and $\tau \in G_{\mathfrak{p}}$, the image of $[\tau] \otimes (\mathfrak{p}(L) - \mathfrak{p}_0(L))$ in $H^{-1}(G, \text{Cl}_S(L))$ is represented by $g(\tau)(\mathfrak{p}(L) - \mathfrak{p}_0(L)) = s(r_{\tau,1}^{(\mathfrak{p})})$. By Lemma 5.4, $s(r_{\tau,1}^{(\mathfrak{p})}) = \mathfrak{c}_{\mathfrak{p}}(\tau)$. \blacksquare

We can now describe $H^1(G, \mathcal{O}_{L,S}^{\times})$ as a quotient of $H^{-1}(G, \text{Cl}_S(L))$.

DEFINITION 6.4. Let $\overline{\mathcal{C}}_S(L/K)$ denote the subgroup of $H^{-1}(G, \text{Cl}_S(L))$ generated by

$$\overline{\{c_p(\tau) \mid p \in S \setminus \{p_0\}, \tau \in G_p \setminus \{1\}\}}.$$

We remark that the bar in $\overline{\mathcal{C}}_S(L/K)$ is intended to reflect that we are working in cohomology. A similar object $\mathcal{C}_S(L/K)$ defined for the class-group in its entirety will appear in Corollary 8.8.

PROPOSITION 6.5. *There is a canonical isomorphism*

$$H^{-1}(G, \text{Cl}_S(L))/\overline{\mathcal{C}}_S(L/K) \simeq H^1(G, \mathcal{O}_{L,S}^\times).$$

Proof. In light of Proposition 6.3, and the existence of the canonical isomorphism $H^{-1}(G, \nabla) \simeq H^1(G, \mathcal{O}_{L,S}^\times)$ arising from the Tate sequence (1.1), it suffices to show that $H^{-1}(G, X) = 0$. To demonstrate the vanishing of this cohomology group, consider the exact sequence

$$H^{-2}(G, Y) \rightarrow H^{-2}(G, \mathbb{Z}) \rightarrow H^{-1}(G, X) \rightarrow 0.$$

Viewing $H^{-2}(G, Y)$ as $\bigoplus_{p \in S} G_p^{\text{ab}}$ and $H^{-2}(G, \mathbb{Z})$ as G^{ab} , we see that the map $H^{-2}(G, Y) \rightarrow H^{-2}(G, \mathbb{Z})$ is surjective, showing that indeed $H^{-1}(G, X) = 0$. We have used in particular that $G_{p_0} = G$. ■

7. The map $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times)$. We now wish to make the canonical isomorphism appearing in Proposition 6.5 explicit. The Tate sequence (1.1) is obtained by combining the short exact sequence

$$(7.1) \quad 0 \rightarrow \mathcal{O}_{L,S}^\times \rightarrow A \rightarrow \text{Ker}(s) \rightarrow 0$$

arising from (4.4) with the short exact sequence

$$(7.2) \quad 0 \rightarrow \text{Ker}(t) \rightarrow B_{S'} \rightarrow \nabla \rightarrow 0$$

arising from the middle vertical map in (4.3), observing that $\text{Ker}(s) = \text{Ker}(t)$. Therefore the map $H^i(G, \text{Cl}_S(L)) \rightarrow H^{i+2}(G, \mathcal{O}_{L,S}^\times)$ factors as

$$\begin{aligned} H^i(G, \text{Cl}_S(L)) &\rightarrow H^i(G, \nabla) \rightarrow H^{i+1}(G, \text{Ker}(t)) \\ &\rightarrow H^{i+1}(G, \text{Ker}(s)) \xrightarrow{\delta_2} H^{i+2}(G, \mathcal{O}_{L,S}^\times), \end{aligned}$$

where the second map is the connecting homomorphism arising from (7.2), and the last map, δ_2 , is the connecting homomorphism arising from (7.1). However, the composition of the first three maps is just the connecting homomorphism $\delta_1 : H^i(G, \text{Cl}_S(L)) \rightarrow H^{i+1}(G, \text{Ker}(s))$ arising from the exact sequence

$$0 \rightarrow \text{Ker}(s) \rightarrow R_{S'} \rightarrow \text{Cl}_S(L) \rightarrow 0.$$

In particular, the map $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times)$ is the composition

$$H^{-1}(G, \text{Cl}_S(L)) \xrightarrow{\delta_1} H^0(G, \text{Ker}(s)) \xrightarrow{\delta_2} H^1(G, \mathcal{O}_{L,S}^\times).$$

7.1. The first connecting homomorphism, δ_1 . By our choice of the set S' , $\text{Cl}_S(L)$ can be generated by the classes of the ideals of L above those in $S' \setminus S$. Further, since for $\mathfrak{c} \in \text{Cl}_S(L)$ and $\sigma \in G$ we have $\overline{\sigma\mathfrak{c}} = \bar{\mathfrak{c}}$ in $H^{-1}(G, \text{Cl}_S(L))$, an element of this cohomology group can be represented by an ideal class of the form

$$(7.3) \quad \mathfrak{c} = \sum_{\mathfrak{q} \in S' \setminus S} a_{\mathfrak{q}} \overline{\mathfrak{q}(L)\mathcal{O}_{L,S}}$$

with the $a_{\mathfrak{q}}$ in \mathbb{Z} , writing $\text{Cl}_S(L)$ additively. By assumption, the fractional ideal $\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}}N}$ of $\mathcal{O}_{L,S}$ is principal and therefore is generated over $\mathcal{O}_{L,S}$ by a for some $a \in L^\times$.

By Lemma 5.5, a lift of \mathfrak{c} to $R_{S'}$ is the element $(a_{\mathfrak{q}})_{\mathfrak{q}}$ of $\bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z}[G]$. Applying N , we see that $\delta_1(\bar{\mathfrak{c}})$ is represented by the element $(a_{\mathfrak{q}}N)_{\mathfrak{q}}$ of $\text{Ker}(s)$, that is, $a_{\mathfrak{q}}N$ in the \mathfrak{q} -component for $\mathfrak{q} \in S' \setminus S$ and 0 elsewhere.

7.2. The second connecting homomorphism, δ_2 . We describe the map $V_{S'} \rightarrow \mathcal{V}$. Having done this, it will be easy to see that the natural choice of a lift of $(a_{\mathfrak{q}}N)_{\mathfrak{q}}$ to $V_{S'}$ under $V_{S'} \rightarrow W_{S'}$ does indeed lie in $A = \text{Ker}(V_{S'} \rightarrow \mathcal{V})$ —see Lemma 7.1 below. The corresponding 1-cocycle $G \rightarrow \mathcal{O}_{L,S}^\times$ will then drop right out, as in Theorem 7.2 below.

So, recall from (4.1) that $V_{S'} = (\bigoplus_{\mathfrak{p} \in S'} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_{\mathfrak{p}}]} V_{\mathfrak{p}}) \oplus (\bigoplus_{\mathfrak{P} \notin S'} U_{\mathfrak{P}})$, where $V_{\mathfrak{p}}$ is described for $\mathfrak{p} \in S$ in [15, Section 1]. By [20, pp. 195–196], there is $\lambda_{\mathfrak{p}} \in \text{Hom}_{\mathbb{Z}}(\Delta G_{\mathfrak{p}}, C_L)$ such that we can choose the map $V_{\mathfrak{p}} \rightarrow \mathcal{V}$ to send (b, x) to $(\varsigma_{\mathfrak{p}}(b)\lambda_{\mathfrak{p}}(x), x)$ for $b \in L_{\mathfrak{p}}^\times$ and $x \in \Delta G_{\mathfrak{p}}$, where $\varsigma_{\mathfrak{p}} : L_{\mathfrak{p}}^\times \rightarrow C_L$ is the canonical inclusion.

As for the description of $V_{\mathfrak{q}}$ for $\mathfrak{q} \in S' \setminus S$, we see from [15, Prop. 2] that under our assumptions, $V_{\mathfrak{q}} = U_{\mathfrak{q}} \oplus \mathbb{Z}$ as \mathbb{Z} -modules, noting that $G_{\mathfrak{q}}$ is trivial since \mathfrak{q} splits completely in L/K . (We are also using our notation convention that for a finite place \mathfrak{q} of K , $U_{\mathfrak{q}}$ denotes $U_{\mathfrak{q}(L)}$.) Thus the \mathfrak{q} -component of $V_{S'}$ is $(\mathbb{Z}[G] \otimes_{\mathbb{Z}} U_{\mathfrak{q}}) \oplus \mathbb{Z}[G]$. The map $(\mathbb{Z}[G] \otimes_{\mathbb{Z}} U_{\mathfrak{q}}) \oplus \mathbb{Z}[G] \rightarrow \mathcal{V}$ sends $(\alpha \otimes u, \beta)$ to $(\varsigma_{\mathfrak{q}}(u)^\alpha \varsigma_{\mathfrak{q}}(\pi_{\mathfrak{q}})^\beta, 0)$ where $\pi_{\mathfrak{q}}$ is a fixed uniformizer of $L_{\mathfrak{q}}$.

Finally, all unit groups $U_{\mathfrak{P}}$ appearing in $V_{S'}$ are mapped to 0 in \mathcal{V} .

By definition of a (see the sentence following (7.3)), for all $\sigma \in G$ we have $a\sigma(\pi_{\mathfrak{q}})^{-a_{\mathfrak{q}}}$ equal to a unit $\tilde{u}_{\mathfrak{q},\sigma}$ in $L_{\sigma\mathfrak{q}(L)}$, which in turn can be expressed as $\sigma(u_{\mathfrak{q},\sigma})$ for a unique unit $u_{\mathfrak{q},\sigma}$ in $L_{\mathfrak{q}}$. Further, a is a unit in $L_{\mathfrak{P}}$ for every \mathfrak{P} not above S' .

Let v be the element of $V_{S'}$ which has $\sum_{\sigma \in G} \sigma \otimes (\sigma^{-1}(a), 0)$ in the \mathfrak{p} -component for $\mathfrak{p} \in S$, $(\sum_{\sigma \in G} \sigma \otimes u_{\mathfrak{q},\sigma}, a_{\mathfrak{q}}N)$ in the \mathfrak{q} -component for $\mathfrak{q} \in S' \setminus S$, and $a \in U_{\mathfrak{P}} \subseteq L_{\mathfrak{P}}$ in the \mathfrak{P} -component for every prime \mathfrak{P} of L not above S' .

LEMMA 7.1. *With notation as above, the element v of $V_{S'}$ is a lift of $(a_{\mathfrak{q}}N)_{\mathfrak{q}}$ to $A = \text{Ker}(V_{S'} \rightarrow \mathcal{V})$.*

Proof. We first show that $v \in \text{Ker}(V_{S'} \rightarrow \mathcal{V})$. For $\mathfrak{p} \in S$, the image of $\sum_{\sigma \in G} \sigma \otimes (\sigma^{-1}(a), 0)$ is

$$\sum_{\sigma \in G} \sigma(\varsigma_{\mathfrak{p}}(\sigma^{-1}(a)), 0) = \left(\prod_{\sigma \in G} \varsigma_{\mathfrak{p}}(\sigma^{-1}(a))^{\sigma}, 0 \right),$$

and the element of C_L appearing in the left-hand entry is represented by the idele having a in all the components above \mathfrak{p} and 1 everywhere else.

Now consider $\mathfrak{q} \in S' \setminus S$. The image of $(\sum_{\sigma \in G} \sigma \otimes u_{\mathfrak{q},\sigma}, a_{\mathfrak{q}}N)$ in \mathcal{V} is $\sum_{\sigma \in G} (\varsigma_{\mathfrak{q}}(u_{\mathfrak{q},\sigma} \pi_{\mathfrak{q}}^{a_{\mathfrak{q}}})^{\sigma}, 0)$. Using $a = \sigma(u_{\mathfrak{q},\sigma} \pi_{\mathfrak{q}}^{a_{\mathfrak{q}}})$, the element $\prod_{\sigma \in G} \varsigma_{\mathfrak{q}}(u_{\mathfrak{q},\sigma} \pi_{\mathfrak{q}}^{a_{\mathfrak{q}}})^{\sigma}$ of C_L is represented by the idele having a in all the components above \mathfrak{q} and 1 everywhere else. Remembering the contribution of $a \in U_{\mathfrak{P}}$ for each \mathfrak{P} not above S' , which gets mapped to $(x, 0)$ where x is the idele class represented by the idele with a in the \mathfrak{P} -component and 1 everywhere else, we find that v is mapped to $(\bar{a}, 0) = (1, 0) \in \mathcal{V}$, and so lies in A .

That v is a lift of $(a_{\mathfrak{q}}N)_{\mathfrak{q}}$ is clear. ■

Recall the ideal class c defined in (7.3) and the element a of L^{\times} appearing just after.

THEOREM 7.2. *The image of \bar{c} under $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^{\times})$ is represented by the 1-cocycle $\tau \mapsto \tau(a)/a$.*

Proof. Take $\tau \in G$. By the construction of the relevant connecting homomorphism associated to $0 \rightarrow \mathcal{O}_{L,S}^{\times} \rightarrow A \rightarrow \text{Ker}(s) \rightarrow 0$, we deduce that $\tau v - v$ must be a principal idele with S -unit entries, so it suffices to determine what that S -unit is in just one component. By considering \mathfrak{P} not above S' , it is clear that it must be $\tau(a)/a$ as claimed, but we can verify this by calculating the idele in all components.

First consider $\mathfrak{q} \in S' \setminus S$. Then

$$(\tau - 1) \left(\sum_{\sigma \in G} \sigma \otimes u_{\mathfrak{q},\sigma}, a_{\mathfrak{q}}N \right) = \left(\sum_{\sigma \in G} (\tau - 1)\sigma \otimes u_{\mathfrak{q},\sigma}, 0 \right),$$

and the element of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} U_{\mathfrak{q}}$ in the left-hand entry is

$$\begin{aligned} \sum_{\sigma \in G} (\tau\sigma) \otimes u_{\mathfrak{q},\sigma} - \sum_{\sigma \in G} \sigma \otimes u_{\mathfrak{q},\sigma} &= \sum_{\sigma \in G} \sigma \otimes (u_{\mathfrak{q},\tau^{-1}\sigma} u_{\mathfrak{q},\sigma}^{-1}) \\ &= \sum_{\sigma \in G} \sigma \otimes \sigma^{-1}(\sigma(u_{\mathfrak{q},\tau^{-1}\sigma})\sigma(u_{\mathfrak{q},\sigma})^{-1}). \end{aligned}$$

Under the identification of $\mathbb{Z}[G] \otimes_{\mathbb{Z}} U_{\mathfrak{q}}$ with $\bigoplus_{\Omega|\mathfrak{q}} U_{\Omega}$, this corresponds to the element of $\bigoplus_{\Omega|\mathfrak{q}} U_{\Omega}$ having $\sigma(u_{\mathfrak{q},\tau^{-1}\sigma})\sigma(u_{\mathfrak{q},\sigma})^{-1}$ in the $\sigma\mathfrak{q}(L)$ -component. However, by definition $\sigma(u_{\mathfrak{q},\tau^{-1}\sigma})\sigma(u_{\mathfrak{q},\sigma})^{-1} = \tau(a)/a$.

Now consider $\mathfrak{p} \in S$. Then

$$\begin{aligned} (\tau - 1) \sum_{\sigma \in G} \sigma \otimes (\sigma^{-1}(a), 0) &= \sum_{\sigma \in G} (\tau\sigma - \sigma) \otimes (\sigma^{-1}(a), 0) \\ &= \sum_{\sigma \in G} \sigma \otimes (\sigma^{-1}(\tau(a)/a), 0), \end{aligned}$$

which is the image under $\bigoplus_{\mathfrak{p}|\mathfrak{p}} L_{\mathfrak{p}}^{\times} \rightarrow V_{S'}$ of the element having $\tau(a)/a$ in every component. This completes the verification. ■

Incidentally, as an extra check, we can verify independently that for each $\tau \in G$, $\tau(a)/a \in \mathcal{O}_{L,S}^{\times}$. Namely, since

$$a\mathcal{O}_{L,S} = \prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}}N}$$

and $\tau N = N$, we see that $\tau(a)\mathcal{O}_{L,S} = a\mathcal{O}_{L,S}$, i.e. $\tau(a)/a \in \mathcal{O}_{L,S}^{\times}$.

8. Corollaries. We give a sequence of corollaries of Theorem 7.2 which lead to Corollary 8.8.

Recall once again the ideal class \mathfrak{c} defined in (7.3) and the element a of L^{\times} defined just after.

COROLLARY 8.1. *With the above notation,*

$$\bar{\mathfrak{c}} \in \text{Ker}(H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^{-1}(G, \mathcal{O}_{L,S}^{\times})) \quad \text{if and only if} \quad a \in \mathcal{O}_{L,S}^{\times} \cdot K^{\times}.$$

Proof. By Theorem 7.2, the cohomology class $\bar{\mathfrak{c}}$ lies in the specified kernel if and only if the 1-cocycle

$$G \rightarrow \mathcal{O}_{L,S}^{\times}, \quad \sigma \mapsto \sigma(a)/a,$$

is a 1-coboundary. But $\sigma \mapsto \sigma(a)/a$ is a 1-coboundary if and only if there is $u \in \mathcal{O}_{L,S}^{\times}$ such that $\sigma(a)/a = \sigma(u)/u$ for all $\sigma \in G$, which is the same as saying that a/u is fixed by G , i.e. $a/u \in K^{\times}$. ■

Let \mathcal{F} be the kernel of the composition

$$\bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z} \rightarrow \text{Cl}_S(L) \xrightarrow{N} \text{Cl}_S(L),$$

where the first map is given by sending an element $(a_{\mathfrak{q}})_{\mathfrak{q}}$ to the class of the ideal $\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}}}$. (There is no relationship with the \mathcal{F} used in the proof of Lemma 3.1.) Thus \mathcal{F} surjects onto $H^{-1}(G, \text{Cl}_S(L))$. Note that we also have a map $\mathcal{F} \rightarrow \text{Cl}_S(K)$ given by restricting

$$\bigoplus_{\mathfrak{q} \in S' \setminus S} \mathbb{Z} \rightarrow \text{Cl}_S(K), \quad (a_{\mathfrak{q}})_{\mathfrak{q}} \mapsto \overline{\prod_{\mathfrak{q} \in S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}}},$$

to \mathcal{F} .

COROLLARY 8.2. *The maps*

$$\mathcal{F} \rightarrow H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times) \quad \text{and} \quad \mathcal{F} \rightarrow \text{Cl}_S(K)$$

have the same kernel.

Proof. Take $(a_q)_q \in \mathcal{F}$ in the kernel of the first map, that is to say,

$$\prod_{q \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_q N} = a\mathcal{O}_{L,S}$$

for some $a \in \mathcal{O}_{L,S}^\times \cdot K^\times$ by Corollary 8.1. Of course, we may assume $a \in K^\times$. Since \mathfrak{q} splits completely in L/K for $\mathfrak{q} \in S' \setminus S$,

$$\nu_{\mathfrak{q}}(a) = \nu_{\mathfrak{q}(L)}(a) = a_{\mathfrak{q}}.$$

Since $a \in \mathcal{O}_{K,S'}^\times$, we have

$$a\mathcal{O}_{K,S} = \prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}},$$

and so the class of $\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}}$ is zero in $\text{Cl}_S(K)$. Thus $(a_q)_q$ is in the kernel of the second map.

Conversely, take $(a_q)_q \in \mathcal{F}$ such that $\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}}$ is principal as an ideal of $\mathcal{O}_{K,S}$, i.e.

$$\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}} = b\mathcal{O}_{K,S}$$

for some $b \in K^\times$. Choose $a \in L^\times$ such that $\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}} N} = a\mathcal{O}_{L,S}$, as in the construction of the map $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times)$. Then

$$a\mathcal{O}_{L,S} = \prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}} N} = \left(\prod_{\mathfrak{q} \in S' \setminus S} \mathfrak{q}^{a_{\mathfrak{q}}} \right) \mathcal{O}_{L,S} = b\mathcal{O}_{L,S},$$

so $a \in \mathcal{O}_{L,S}^\times \cdot K^\times$. By Corollary 8.1, this says that $(a_q)_q$ is in the kernel of the first map. ■

COROLLARY 8.3. *There is an embedding $H^1(G, \mathcal{O}_{L,S}^\times) \rightarrow \text{Cl}_S(K)$.*

Proof. By Corollary 8.2, we have

$$H^1(G, \mathcal{O}_{L,S}^\times) \xrightarrow{\cong} \mathcal{F} / \text{Ker}(\mathcal{F} \rightarrow \text{Cl}_S(K)) \rightarrow \text{Cl}_S(K),$$

the second map being injective. ■

We observe that the above corollary gives an alternative justification that $H^1(G, \mathcal{O}_{L,S}^\times)$ embeds into $\text{Cl}_S(K)$ to that found in [14, Cor. 2].

DEFINITION 8.4. Let $\text{Nm} : \text{Cl}_S(L) \rightarrow \text{Cl}_S(K)$ be the homomorphism on class-groups induced by the norm of ideals. Denote the resulting map $H^{-1}(G, \text{Cl}_S(L)) \rightarrow \text{Cl}_S(K)$ by $\overline{\text{Nm}}$.

COROLLARY 8.5. *There is an exact sequence*

$$0 \rightarrow \overline{\mathcal{C}}_S(L/K) \rightarrow H^{-1}(G, \text{Cl}_S(L)) \xrightarrow{\overline{\text{Nm}}} \text{Cl}_S(K).$$

(Recall the definition of $\overline{\mathcal{C}}_S(L/K)$ in 6.4.)

Proof. By Proposition 6.5, $\overline{\mathcal{C}}_S(L/K)$ is the kernel of $H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times)$, which is the kernel of

$$(8.1) \quad H^{-1}(G, \text{Cl}_S(L)) \rightarrow H^1(G, \mathcal{O}_{L,S}^\times) \rightarrow \text{Cl}_S(K)$$

by Corollary 8.3. It remains to show that the map in (8.1) is $\overline{\text{Nm}}$. An element x of $H^{-1}(G, \text{Cl}_S(L))$ is represented by an ideal class containing an ideal of the form

$$\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}(L)\mathcal{O}_{L,S})^{a_{\mathfrak{q}}}$$

with $(a_{\mathfrak{q}})_{\mathfrak{q}} \in \mathcal{F}$. If y is the image of x in $H^1(G, \mathcal{O}_{L,S}^\times)$, then the image of y in $\text{Cl}_S(K)$ is obtained by first lifting y to \mathcal{F} via $H^{-1}(G, \text{Cl}_S(L))$, and then applying the map $\mathcal{F} \rightarrow \text{Cl}_S(K)$. However, such a lift of y is $(a_{\mathfrak{q}})_{\mathfrak{q}}$, whose image in $\text{Cl}_S(K)$ is the ideal class containing

$$\prod_{\mathfrak{q} \in S' \setminus S} (\mathfrak{q}\mathcal{O}_{K,S})^{a_{\mathfrak{q}}}.$$

This ideal class is clearly $\overline{\text{Nm}}(x)$. ■

Corollary 8.5 establishes that $\overline{\mathcal{C}}_S(L/K) = \text{Ker}(\overline{\text{Nm}})$. We remark in passing, although we will not use this fact, that this kernel is in fact the “divisor knot” as defined by Jehne in [11, Ch. I, Section 1]—see the isomorphism before Theorem 1 of [11] and the definition in [11, (1.5)]. (Jehne’s divisor knot contains the earlier divisor knot of Scholz [17].)

Now, consider the following definition:

DEFINITION 8.6. Let $D(\text{Cl}_S(L))$ be the subgroup of $\text{Cl}_S(L)$ generated by

$$\{(\sigma - 1)\mathfrak{c} \mid \mathfrak{c} \in \text{Cl}_S(L), \sigma \in G\}.$$

Observe that $D(\text{Cl}_S(L))$ is closed under the action of G and is therefore a $\mathbb{Z}[G]$ -submodule of $\text{Cl}_S(L)$.

Noting that

$$(8.2) \quad D(\text{Cl}_S(L)) \subseteq \text{Ker}(\text{Nm}) \subseteq \text{Ker}(N : \text{Cl}_S(L) \rightarrow \text{Cl}_S(L)),$$

we obtain an exact sequence

$$0 \rightarrow D(\text{Cl}_S(L)) \rightarrow \text{Ker}(\text{Nm}) \rightarrow \text{Ker}(\overline{\text{Nm}}) \rightarrow 0.$$

Thus we arrive at an exact sequence

$$(8.3) \quad 0 \rightarrow D(\text{Cl}_S(L)) \rightarrow \text{Ker}(\text{Nm}) \rightarrow \overline{\mathcal{C}}_S(L/K) \rightarrow 0.$$

DEFINITION 8.7. Let $\mathcal{C}_S(L/K)$ be the subgroup of $\text{Cl}_S(L)$ generated by

$$\{\mathfrak{c}_{\mathfrak{p}}(\tau) \mid \mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}, \tau \in G_{\mathfrak{p}} \setminus \{1\}\}.$$

The following gives a description of the kernel of the norm map $\text{Cl}_S(L) \rightarrow \text{Cl}_S(K)$.

COROLLARY 8.8. *There is an exact sequence*

$$0 \rightarrow D(\text{Cl}_S(L)) + \mathcal{C}_S(L/K) \rightarrow \text{Cl}_S(L) \xrightarrow{\text{Nm}} \text{Cl}_S(K) \rightarrow 0.$$

(But see the Remark, after the proof, concerning existing literature.)

Proof. We see from (8.2) and Corollary 8.5 that $D(\text{Cl}_S(L))$ and $\mathcal{C}_S(L/K)$ are in the kernel of Nm. Conversely, take $\mathfrak{c} \in \text{Ker}(\text{Nm})$, so that $\bar{\mathfrak{c}} \in \text{Ker}(\overline{\text{Nm}})$, and write $\bar{\mathfrak{c}} \in H^{-1}(G, \text{Cl}_S(L))$ as

$$\bar{\mathfrak{c}} = \sum_{\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}} \sum_{\tau \in G_{\mathfrak{p}} \setminus \{1\}} a_{\mathfrak{p},\tau} \overline{\mathfrak{c}_{\mathfrak{p}}(\tau)}$$

with $a_{\mathfrak{p},\tau} \in \mathbb{Z}$. Then $\mathfrak{c} - \sum_{\mathfrak{p} \in S \setminus \{\mathfrak{p}_0\}} \sum_{\tau \in G_{\mathfrak{p}} \setminus \{1\}} a_{\mathfrak{p},\tau} \mathfrak{c}_{\mathfrak{p}}(\tau)$ is in the kernel of $\text{Ker}(\text{Nm}) \rightarrow \overline{\mathcal{C}}_S(L/K)$ and therefore is in $D(\text{Cl}_S(L))$ by (8.3), thus showing that $\mathfrak{c} \in D(\text{Cl}_S(L)) + \mathcal{C}_S(L/K)$.

As for the surjectivity of $\text{Cl}_S(L) \rightarrow \text{Cl}_S(K)$, this follows from the fact that \mathfrak{p}_0 remains non-split in L/K . ■

REMARK. The author is aware that a description of the kernel of the norm map as found in Corollary 8.8 should follow from, for example, work of Jaulent [10], Jehne [11], Fröhlich [6] and Furuta [7] on knots, central classes and genus fields. We include our derivation of this description as an illustration of a different approach, via Ritter and Weiss’s version of the Tate sequence, a derivation that, after our having obtained Theorem 7.2, is reasonably short.

Acknowledgments. The author would like to thank Al Weiss for a number of interesting discussions, and the referee for providing helpful remarks.

References

- [1] E. Artin and J. Tate, *Class Field Theory*, W. A. Benjamin, New York, 1968.
- [2] D. Burns and M. Flach, *Tamagawa numbers for motives with (noncommutative) coefficients, II*, Amer. J. Math. 125 (2003), 475–512.
- [3] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, NJ, 1956.
- [4] T. Chinburg, *On the Galois structure of algebraic integers and S-units*, Invent. Math. 74 (1983), 321–349.
- [5] —, *Exact sequences and Galois module structure*, Ann. of Math. (2) 121 (1985), 351–376.

- [6] A. Fröhlich, *Central Extensions, Galois Groups, and Ideal Class Groups of Number Fields*, Contemp. Math. 24, Amer. Math. Soc., Providence, RI, 1983.
- [7] Y. Furuta, *The genus field and genus number in algebraic number fields*, Nagoya Math. J. 29 (1967), 281–285.
- [8] C. Greither, *Determining Fitting ideals of minus class groups via the equivariant Tamagawa number conjecture*, Compos. Math. 143 (2007), 1399–1426.
- [9] K. W. Gruenberg, J. Ritter, and A. Weiss, *A local approach to Chinburg’s root number conjecture*, Proc. London Math. Soc. (3) 79 (1999), 47–80.
- [10] J.-F. Jaulent, *L’arithmétique des l -extensions*, Publ. Math. Fac. Sci. Besançon, Faculté des Sciences, Univ. de Franche-Comté, Besançon, 1986.
- [11] W. Jehne, *On knots in algebraic number theory*, J. Reine Angew. Math. 311/312 (1979), 215–254.
- [12] J. Neukirch, *Algebraic Number Theory*, Grundlehren Math. Wiss. 322, Springer, Berlin, 1999.
- [13] A. Nickel, *On the equivariant Tamagawa number conjecture in tame CM-extensions*, Math. Z., to appear.
- [14] D. S. Rim, *An exact sequence in Galois cohomology*, Proc. Amer. Math. Soc. 16 (1965), 837–840.
- [15] J. Ritter and A. Weiss, *A Tate sequence for global units*, Compos. Math. 102 (1996), 147–178.
- [16] —, —, *Cohomology of units and L -values at zero*, J. Amer. Math. Soc. 10 (1997), 513–552.
- [17] A. Scholz, *Totale Normenreste, die keine Normen sind, als Erzeuger nichtabelscher Körpererweiterungen, II*, J. Reine Angew. Math. 182 (1940), 217–234.
- [18] J.-P. Serre, *Local Fields*, Grad. Texts in Math. 67, Springer, New York, 1979.
- [19] J. Tate, *The cohomology groups of tori in finite Galois extensions of number fields*, Nagoya Math. J. 27 (1966), 709–719.
- [20] —, *Global class field theory*, in: Algebraic Number Theory (Brighton, 1965), Thompson, Washington, DC, 1967, 162–203.
- [21] —, *Les conjectures de Stark sur les fonctions L d’Artin en $s = 0$* , Progr. Math. 47, Birkhäuser Boston, Boston, MA, 1984.
- [22] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge Stud. Adv. Math. 38, Cambridge Univ. Press, Cambridge, 1994.

Paul Richard Buckingham
 Department of Mathematical and Statistical Sciences, 632 CAB
 University of Alberta
 Edmonton, AB T6G 2G1, Canada
 E-mail: p.r.buckingham@ualberta.ca

*Received on 23.10.2010
 and in revised form on 20.1.2011*

(6525)