Magic $p$-dimensional cubes

by

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A magic $p$-dimensional cube of order $n$ is a $p$-dimensional matrix

$$M^p_n = |m(i_1, \ldots, i_p) : 1 \leq i_1, \ldots, i_p \leq n|,$$

containing natural numbers $1, \ldots, n^p$ such that the sum of the numbers along every row and every diagonal is the same.

By a row of $M^p_n$ we mean an $n$-tuple of elements $m(i_1, \ldots, i_p)$ which have identical coordinates at $p - 1$ places. A diagonal of $M^p_n$ is an $n$-tuple

$\{m(x, i_2, \ldots, i_p) : x = 1, \ldots, n, i_j = x \text{ or } i_j = 2p + 1 - x \text{ for all } 2 \leq j \leq p\}$.

A magic $p$-dimensional cube $M^p_n$ has $pn^p - 1$ rows and $2p - 1$ diagonals. The symbol $[x]$ denotes the integer part of $x$. A magic 1-dimensional cube $M^1_n$ of order $n$ is given by an arbitrary permutation of integers $1, \ldots, n$. Evidently, a magic $p$-dimensional cube of order 2 for $p \geq 2$ does not exist.

In [5] there is a construction of $M^3_2$ for every $n \neq 2$ and in [6] it is proved that a magic $p$-dimensional cube $M^p_n$ of order $n$ exists for every integer $p$ and $n \neq 2 \text{ (mod 4)}$. (The reader can find more information in [2, 3, 5, 6].) These results are improved in

**THEOREM.** A magic $p$-dimensional cube $M^p_n$ of order $n$ exists if and only if $p \geq 2$ and $n \neq 2$ or $p = 1$.

Before we begin the proof, we demonstrate a construction of a magic square $M^2_6$. The construction starts from four copies of a Latin square $U = |u(i_1, i_2) : 1 \leq i_1, i_2 \leq 3|$ of order 3 defined by the relation $u(i_1, i_2) \equiv (i_1 - i_2) \pmod{3}$. We insert these squares into a $6 \times 6$ table, so that Latin squares are symmetric about the lines going through the centres of two opposite sides. All elements of Latin squares are replaced by 0, 1, 2, 3 as shown in Figure A. On the left hand side in every cell there is an element of the Latin square, and its substitution on the right hand side.

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By multiplying all elements by 9 and adding elements of four copies of a magic square $M_3^2$ we obtain the magic square $M_6^2$ of order 6 which is shown in Figure B.

**Proof of the Theorem.** For $n \not\equiv 2 \pmod{4}$ the proof is in [6]. That paper gives the construction of $M_n^p$ for $n \equiv 1 \pmod{2}$ or $n \equiv 0 \pmod{4}$ and $p \geq 2$.

Let $n \equiv 2 \pmod{4}$, $n \not\equiv 2$ and $p \geq 2$ be two fixed natural numbers and let $m = n/2$. The construction of $M_n^p$ is described in 6 steps.

1. Let $D = |d(j, x) : 1 \leq j \leq m, 1 \leq x \leq 2^p|$ be a matrix defined by the following relations:

\[
d(1, x) = 2^{p-1} \cdot 2^{x \pmod{2}} - \left\lfloor \frac{x+1}{2} \right\rfloor,
\]

\[
d(2, x) = 2^{p-1} \cdot 2^{(x+1) \pmod{2}} - \left\lfloor \frac{x+1}{2} \right\rfloor,
\]

\[
d(3, x) = x + (-1)^x \left\lfloor \frac{x-1}{2^p-1} \right\rfloor [(p+1) \pmod{2}],
\]

\[
d(j, x) = \begin{cases} x-1, & j = 4, 6, 8, \ldots, m-1, \\ 2^p - x, & j = 5, 7, 9, \ldots, m. \end{cases}
\]

This definition yields the following facts which are crucial in our construction:

(a) for every $1 \leq x \leq 2^p$,

\[
\sum_{j=1}^{m} d(j, x) = \frac{n}{4}(2^p - 1) - \frac{1}{2} + \left\lfloor x + \frac{x-1}{2^p-1} \right\rfloor (p+1) \pmod{2},
\]

(b) $\{d(j, 1), \ldots, d(j, 2^p)\} = \{1, \ldots, 2^p\}$ for all $1 \leq j \leq m$,

(c) $d(1, x) + d(1, 2^p - x + 1) = m - 1$ for all $1 \leq x \leq 2^{p-1}$ (this is important only for $p \equiv 0 \pmod{2}$).

2. Let $\sigma$ be a permutation of the set $\{1, \ldots, 2^p\}$ which satisfies:

(i) if the number of ones in the binary representation of the number $k-1$ is even (odd) then $\sigma(k)$ is an even (odd) number for every $k = 1, \ldots, 2^{p-1}$,

(ii) if $k \leq 2^{p-1}$ then $\sigma(k) \leq 2^{p-1}$,

(iii) if $2^{p-1} < k \leq 2^p$ then $\sigma(k) = 2^p - \sigma(2^p - k + 1)$.
3. Let \( U = |u(i_1, \ldots, i_p): 1 \leq i_1, \ldots, i_p \leq m| \) be a \( p \)-dimensional matrix defined by

\[
u(i_1, \ldots, i_p) = \left( \sum_{x=1}^{p} (-1)^{x+1} i_x \right) \pmod{m}.
\]

Every row of \( U \) is the set \{0, 1, \ldots, m-1\}. If \( p \equiv 1 \pmod{2} \) then the diagonal \( \{u(i, \ldots, i): i = 1, \ldots, m\} \) of \( U \) is the set \{0, 1, \ldots, m-1\}. If \( p \equiv 0 \pmod{2} \) then it is \{0, 0, \ldots, 0\}.

4. Let \( V(k) = |v(k)(i_1, \ldots, i_p): 1 \leq i_1, \ldots, i_p \leq m|, 1 \leq k \leq 2^p, \) be \( p \)-dimensional matrices defined by

\[
\text{if } u(i_1, \ldots, i_p) = q \text{ then } v(k)(i_1, \ldots, i_p) = d(q, \sigma(k)).
\]

5. Let \( M(k) = |m(k)(i_1, \ldots, i_p): 1 \leq i_1, \ldots, i_p \leq m|, 1 \leq k \leq 2^p, \) be \( p \)-dimensional matrices defined by

\[
m(k)(i_1, \ldots, i_p) = v(k)(i_1, \ldots, i_p)m^p + m(i_1, \ldots, i_p),
\]

where \( m(i_1, \ldots, i_p) \) is the element of \( M_m^p \) which is constructed in [6].

Because \( v(j)(i_1, \ldots, i_p) \neq v(k)(i_1, \ldots, i_p) \) for all \( j \neq k \) and from the previous relation it follows that:

(a) no two elements \( m(k)(i_1, \ldots, i_p) \) with different coordinates or indices are equal,

(b) the row sum of \( M(k) \) for fixed \( k \) is the same, i.e.

\[
\left\lfloor \frac{m}{2}(2^p - 1) + \frac{(-1)^\omega}{2} \right\rfloor m^p + \frac{m(m^p + 1)}{2}, \quad \text{where } \omega = 1 \text{ or } 2,
\]

(c) if \( p \equiv 1 \pmod{2} \) then \( \sum_{i=1}^{m} m(k)(i, \ldots, i) \) is equal to the row sum of \( M(k) \), if \( p \equiv 0 \pmod{2} \) then

\[
\sum_{i=1}^{m} m(k)(i, \ldots, i) = d(1, \sigma(k))m^{p+1} + m(m^p + 1)/2.
\]

6. We define a magic \( p \)-dimensional cube \( M_n^p = |m(i_1, \ldots, i_p): 1 \leq i_1, \ldots, i_p \leq n| \) of order \( n \equiv 2 \pmod{4} \) by

\[
m(i_1, \ldots, i_p) = m(k)(i_1^*, \ldots, i_p^*),
\]

where \( i_j^* = \min\{i_j, n + 1 - i_j\} \) and \( k = \sum_{x=1}^{p} \lfloor \frac{i_x-1}{m} \rfloor 2^{x-1} + 1. \)

From the definition of \( M_n^p \) we get:

(a) every row of \( M_n^p \) consists of one row of \( M(j) \) and one row of \( M(k) \) which have different row sums,

(b) every diagonal of \( M_n^p \) consists of \( m(k)(i, \ldots, i), m(2^p+1-k)(i, \ldots, i), \)

\( i = 1, \ldots, m. \) If \( p \equiv 1 \pmod{2} \) then \( M(k) \) and \( M(2^p-k+1) \) have different row sums. If \( p \equiv 0 \pmod{2} \) then the row sums of \( M(k) \) and \( M(2^p-k+1) \) are the same.
It is easy to see that $M^p_n$, which is a union of $2^p$ matrices $M_{(k)}$, satisfies the conditions for a magic $p$-dimensional cube.

**Remark 1.** Magic squares have fascinated people for centuries. Mathematicians have studied many properties of magic squares and formulated problems which have not been solved. (See [1].) We can formulate similar problems for magic cubes, too.

**Remark 2.** Another “magic” $p$-dimensional cube was studied by J. Ivančo. In [4] it is proved that if $4 \leq p \equiv 0 \pmod{2}$ then the edges of a $p$-dimensional cube can be labelled by integers $1, 2, \ldots, 2^{p-1}p$ in such a way that the sum of the labels of edges incident to each vertex is the same.

**References**


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