ACTA ARITHMETICA
XCVI.4 (2001)

The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes (II)

by

Hongze Li (Jinan)

1. Main results. The Goldbach conjecture is that every integer not less than 6 is a sum of two odd primes. The conjecture still remains open. Let $E(x)$ denote the number of positive even integers not exceeding x which cannot be written as a sum of two prime numbers. In 1975 Montgomery and Vaughan [15] proved that

$$E(x) \ll x^{1-\theta}$$

for some small computable constant $\theta > 0$. For the number θ, see [1]–[3]. In [5] the author proved that $E(x) \ll x^{0.921}$, recently in [6] the author improved the result to $E(x) \ll x^{0.914}$.

In 1951 and 1953, Linnik [8, 9] established the following "almost Goldbach" result.

Every large positive even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2, i.e.

$$N = p_1 + p_2 + 2^{\nu_1} + \ldots + 2^{\nu_k}. \tag{1.1}$$

Let $r_k''(N)$ denote the number of N in the form (1.1). In [10] Liu, Liu and Wang proved that for any $k \ge 54000$, there exists $N_k > 0$ depending on k only such that if $N \ge N_k$ is an even integer then

$$r_k''(N) \gg N(\log N)^{k-2}. \tag{1.2}$$

Recently in [7] the author improved the constant to $k \ge 25000$. In this paper we prove the following result.

Theorem 1. *For any integer $k \ge 1906$, there exists $N_k > 0$ depending on k only such that if $N \ge N_k$ is an even integer then*

$$r_k''(N) \gg N(\log N)^{k-2}.$$

2000 *Mathematics Subject Classification*: 11P32, 11P55.

Project supported by The National Natural Science Foundation of China.

Let $r'_k(n)$ denote the number of representations of an odd integer n in the form

$$n = p + 2^{\nu_1} + \ldots + 2^{\nu_k}. \tag{1.3}$$

The second purpose of this paper is to establish the following result.

THEOREM 2. *For any $\varepsilon > 0$, there exists a constant k_0 depending on ε only such that if $k \geq k_0$, $N \geq N_k$ then*

$$\sum_{2\nmid n\leq N} (r'_k(n) - 2(\log_2 N)^k (\log N)^{-1})^2 \leq \varepsilon 2N(\log_2 N)^{2k}(\log N)^{-2}.$$

In particular, for $\varepsilon = 0.9986, k_0 = 953$ is permissible.

In what follows, $L(s,\chi)$ denotes the Dirichlet L-function. ε denotes a positive constant which is arbitrarily small but not necessarily the same at each occurrence. A will be sufficiently large, $A < P$.

2. Some lemmas. Let N be a large integer,

$$\theta := \frac{1}{13}, \quad P := N^{\theta}, \quad T := P^{2.01}, \quad Q := P^{-1}N(\log N)^{-3}, \quad D = P^{1+\varepsilon}. \tag{2.1}$$

Let $A < q \leq P$, and let χ_q be a non-principal character mod q. Write $\alpha = 1 - \lambda/\log D$. Assume that

$$\alpha \leq \sigma \leq 1, \qquad |t| \leq D/q. \tag{2.2}$$

LEMMA 1. *Suppose P is sufficiently large. Then no function $L(s,\chi)$ with χ primitive modulo $q \leq P$, except for a possible exceptional one only, has a zero in the region*

$$\sigma \geq 1 - \frac{0.239}{\log P}, \qquad q(|t|+1) \leq P^{1+\varepsilon}.$$

If the exceptional function exists, denoted by $L(s,\widetilde{\chi})$, then $\widetilde{\chi}$ must be a real primitive character modulo $\widetilde{q}$, $\widetilde{q} \leq P$, and $L(s,\widetilde{\chi})$ has a real simple zero $\widetilde{\beta}$ which satisfies

$$1 - \frac{0.239}{\log P} \leq \widetilde{\beta} \leq 1 - \frac{c}{\widetilde{q}^{10^{-8}}}.$$

This is Lemma 2.3 of [5].

For $q_1, q_2 \leq P$, we now consider the zeros of $L(s,\chi_{q_1})$ and $L(s,\chi_{q_2})$ for non-principal characters χ_{q_1} and χ_{q_2}. If $\varrho_1 = \beta_1 + i\gamma_1 = 1 - \lambda_1/\log P + i\gamma_1$ is a zero of $L(s,\chi_{q_1})$ satisfying $q_1(|\gamma_1|+1) \leq P^{1+\varepsilon}$, and $\varrho_2 = \beta_2 + i\gamma_2 = 1 - \lambda_2/\log P + i\gamma_2$ is a zero of $L(s,\chi_{q_2})$ satisfying $q_2(|\gamma_2|+1) \leq P^{1+\varepsilon}$, then we have the lower bounds for λ_2 given in Table 1.

Table 1. The lower bound for λ_2

λ_1	λ_2
0.24	0.444
0.26	0.418
0.28	0.393
0.30	0.37
0.32	0.349
0.334	0.334

If $[q_1, q_2] \le P^{\varepsilon}(q_1, q_2)$, then we have the following lower bounds for λ_2:

Table 2. The lower bound for λ_2

λ_1	λ_2	λ_1	λ_2
0.22	1.189	0.38	0.745
0.24	1.116	0.40	0.706
0.26	1.050	0.42	0.669
0.28	0.989	0.44	0.634
0.30	0.933	0.46	0.601
0.32	0.881	0.48	0.570
0.34	0.832	0.50	0.541
0.36	0.787	0.517	0.517

The entry 0.40, 0.706 indicates that $\lambda_2 \ge 0.706$ whenever $\lambda_1 \le 0.40$ (see [5]).

Let $S_{jq} = \{\chi_q : L(s, \chi_q)$ has only j zeros in the region (2.2)$\}$. Suppose $A < q_0 \le P$, define

$$N_1^*(\alpha, P) = N_1^*(\lambda, P) = \sum_{\substack{A<q\le P \\ [q,q_0]\le D^{\varepsilon}(q,q_0)}} \sum_{j\ge 1} \sideset{}{^*}\sum_{\chi\in S_{jq}} j, \tag{2.3}$$

$$N^*(\alpha, P) = N^*(\lambda, P) = \sum_{A<q\le P} \sum_{j\ge 1} \sideset{}{^*}\sum_{\chi\in S_{jq}} j \tag{2.4}$$

where $\sum^*$ indicates that the sum is over primitive characters.

Lemma 2. *Suppose* $A < q_0 \le P$, $0 < \lambda \le \varepsilon \log D$. *Then*

$$N_1^*(\alpha, P) = N_1^*(\lambda, P) \le \begin{cases} 4.356 C_1(\lambda) e^{4.064\lambda}, & 0.517 < \lambda \le 0.575, \\ 8.46 C_2(\lambda) e^{4.12\lambda}, & 0.575 < \lambda \le 0.618, \\ 14.3 C_3(\lambda) e^{4.5\lambda}, & 0.618 < \lambda \le 1, \\ 104.1 C_4(\lambda) e^{3.42\lambda}, & 1 < \lambda \le 5, \\ 268.6 e^{2.16\lambda}, & 5 < \lambda \le \varepsilon \log D. \end{cases}$$

$$N^*(\alpha, P) = N^*(\lambda, P) \le \begin{cases} 3.632C_5(\lambda)e^{5.2\lambda}, & 0.334 < \lambda \le 0.517, \\ 4.338C_6(\lambda)e^{4.82\lambda}, & 0.517 < \lambda \le 0.575, \\ 10.42C_7(\lambda)e^{4.5\lambda}, & 0.575 < \lambda \le 0.618, \\ 14.91C_8(\lambda)e^{5.2\lambda}, & 0.618 < \lambda \le 1, \\ 104.8C_9(\lambda)e^{4.16\lambda}, & 1 < \lambda \le 5, \\ 279.7e^{2.9\lambda}, & 5 < \lambda \le \varepsilon \log D, \end{cases}$$

where

$$C_1(\lambda) = \lambda^{-1}\left(1 - e^{-4.064\lambda}\frac{e^{2.808\lambda} - e^{1.76\lambda}}{1.048\lambda}\right),$$
$$C_2(\lambda) = \lambda^{-1}\left(1 - e^{-4.12\lambda}\frac{e^{2.855\lambda} - e^{1.78\lambda}}{1.075\lambda}\right),$$
$$C_3(\lambda) = \lambda^{-1}\left(1 - e^{-4.5\lambda}\frac{e^{3.198\lambda} - e^{2.013\lambda}}{1.185\lambda}\right),$$
$$C_4(\lambda) = \lambda^{-1}\left(1 - e^{-3.42\lambda}\frac{e^{2.358\lambda} - e^{1.64\lambda}}{0.718\lambda}\right),$$
$$C_5(\lambda) = \lambda^{-1}\left(1 - e^{-5.2\lambda}\frac{e^{3.866\lambda} - e^{2.668\lambda}}{1.198\lambda}\right),$$
$$C_6(\lambda) = \lambda^{-1}\left(1 - e^{-4.82\lambda}\frac{e^{3.565\lambda} - e^{2.51\lambda}}{1.055\lambda}\right),$$
$$C_7(\lambda) = \lambda^{-1}\left(1 - e^{-4.5\lambda}\frac{e^{3.32\lambda} - e^{2.36\lambda}}{0.96\lambda}\right),$$
$$C_8(\lambda) = \lambda^{-1}\left(1 - e^{-5.2\lambda}\frac{e^{3.928\lambda} - e^{2.7312\lambda}}{1.1968\lambda}\right),$$
$$C_9(\lambda) = \lambda^{-1}\left(1 - e^{-4.16\lambda}\frac{e^{3.104\lambda} - e^{2.38\lambda}}{0.724\lambda}\right).$$

This is Lemma 6 of [6].

3. The major arcs. By Dirichlet's lemma on rational approximations, each $\alpha \in [Q^{-1}, 1 + Q^{-1}]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \le (qQ)^{-1}, \tag{3.1}$$

for some positive integers a, q with $1 \le a \le q, (a, q) = 1$ and $q \le Q$. We denote by $I(a, q)$ the set of α satisfying (3.1), and put

$$E_1 = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} I(a, q), \qquad E_2 = [Q^{-1}, 1 + Q^{-1}] - E_1.$$

When $q \le P$ we call $I(a, q)$ a major arc. By (2.1), all major arcs are mutually disjoint. Let $e(\alpha) = \exp(i2\pi\alpha)$ and $S(\alpha) = \sum_{N^{1-\varepsilon} < p \le N} e(p\alpha)$.

Let $\sigma(n)$ denote the singular series in the Goldbach problem, i.e.

$$\sigma(n) := \prod_{p|n}(1+(p-1)^{-1})\prod_{p\nmid n}(1-(p-1)^{-2}) \gg 1$$

for even n. Let

$$J(n) := \sum_{\substack{1<n_1,n_2\le N\\ n_1-n_2=n}} (\log n_1 \log n_2)^{-1}.$$

THEOREM 3. *Let n with $|n| \le N^2$ be a non-zero integer, and let P, Q satisfy* (2.1). *Then for even n we have*

$$\int_{E_1} |S(\alpha)|^2 e(n\alpha)\,d\alpha = \sigma(n)J(n) + R,$$

where

$$|R| \le \sigma(n)N(\log N)^{-2}\{0.11943387 + O(\widetilde{q}\phi((n,\widetilde{q}))/\phi^2(\widetilde{q}))\},$$

the O term occurring only when there exists $\widetilde{\beta}$ in Lemma 1.

Let

$$r_0(n) = \int_{E_1} |S(\alpha)|^2 e(n\alpha)\,d\alpha, \qquad S(\lambda,\chi) = \sum_{N^{1-\varepsilon}<p\le N} \chi(p)e(p\lambda),$$

$$T(\lambda) = \sum_{N^{1-\varepsilon}<m\le N} e(m\lambda)/\log m, \qquad \widetilde{T}(\lambda) = -\sum_{N^{1-\varepsilon}<m\le N} m^{\widetilde{\beta}-1}e(m\lambda)/\log m,$$

and

$$\text{(3.2)}\qquad \begin{cases} S(\lambda,\chi_q^0) = T(\lambda) + W(\lambda,\chi_q^0), & \\ S(\lambda,\chi_q^0\widetilde{\chi}) = \widetilde{T}(\lambda) + W(\lambda,\chi_q^0\widetilde{\chi}) & \text{if } \widetilde{q}\,|\,q,\\ S(\lambda,\chi_q) = W(\lambda,\chi_q) & \text{otherwise,}\end{cases}$$

$$G(m,\chi) = \sum_{h=1}^{q} \chi(h)e\Big(\frac{mh}{q}\Big), \qquad \tau(\chi) = G(1,\chi), \qquad C_q(m) = \sum_{\substack{h\le q\\ (h,q)=1}} e\Big(\frac{mh}{q}\Big).$$

As in (4.7) of [11], we have

$$\text{(3.3)}\qquad r_0(n) = \sum_{j=1}^{9} r_j(n) + O(P^2(\log N)^3).$$

For the definitions of $r_j(n)$, see [11].

LEMMA 3.

$$r_1(n) = \sigma(n)J(n) + O(N(\log N)^{-3}),$$

$$r_2(n), r_3(n) \ll \frac{\widetilde{q}}{\phi^2(\widetilde{q})}\cdot\frac{N}{(\log N)^2}\sigma(n), \qquad r_4(n) \ll \frac{\widetilde{q}\phi((n,\widetilde{q}))}{\phi^2(\widetilde{q})}\cdot\frac{N}{(\log N)^2}\sigma(n),$$

$$r_5(n), r_6(n) \ll \frac{N}{(\log N)^6}\sigma(n).$$

Proof. Apply Lemmas 14 and 16 of [11] (note that $W \ll 1$).

LEMMA 4. *Let* χ_i *be primitive characters* $(mod\ r_i)$, $i = 1, 2$, $r = [r_1, r_2]$. *Then for* $m \neq 0$,

$$\sum_{\substack{q \le P \\ r \mid q}} \phi(q)^{-2} |G(m, \overline{\chi}_1\overline{\chi}_2\chi_0)\tau(\chi_1\chi_0)\tau(\chi_2\chi_0)| \le 2.140782\sigma(m),$$

$$\sum_{\substack{q \le P \\ r \mid q}} \phi(q)^{-2} |G(m, \overline{\chi}_1\overline{\chi}_2\chi_0)\tau(\chi_1\chi_0)\tau(\chi_2\chi_0)| \ll (r_1, r_2) r^{-1} \sigma(m) \log P.$$

Proof. This lemma is similar to Lemma 5.5 of [15], but our proof is similar to that of Lemma 5.2 of [14]. Define

$$\begin{aligned} Z(q, \chi_1, \chi_2) &:= \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{hm}{q}\right) \prod_{j=1}^{2} G(h, \chi_j\chi_0) \\ &= |G(m, \overline{\chi}_1\overline{\chi}_2\chi_0)\tau(\chi_1\chi_0)\tau(\chi_2\chi_0)|, \\ A(q) &:= \phi(q)^{-2} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} e\left(\frac{hm}{q}\right) \prod_{j=1}^{2} G(h, \chi_0). \end{aligned}$$

By Lemma 4.1 of [13], we know $A(q)$ is multiplicative. For any prime p, let

$$s(p) := 1 + A(p).$$

Since $A(p) = 1/(p-1)$ if $p \mid m$ and $A(p) = -1/(p-1)^2$ if $p \nmid m$, similarly to Lemma 4.6 of [13] and Lemma 5.2 of [14], the first inequality holds. By the proof of Lemma 5.5 of [15], the second inequality holds.

Let

$$W(\chi_d) = \Big(\int_{-1/(dQ)}^{1/(dQ)} |W(\lambda, \chi_d)|^2 \, d\lambda \Big)^{1/2}, \tag{3.4}$$

$$W(P) = \sum_{d \le P} \sum_{\chi_d}{}^{*} W(\chi_d), \tag{3.5}$$

where $*$ indicates that the sum is over for primitive characters χ_d; and

$$W(P, \widetilde{q}) = \sum_{\substack{d \le P \\ [d, \widetilde{q}] \le D^{\varepsilon}(d, \widetilde{q})}} \sum_{\chi_d}{}^{*} W(\chi_d), \tag{3.6}$$

$$W'(P) = \max \sum_{\substack{d \le P \\ [d_1, d] \le D^{\varepsilon}(d_1, d)}} \sum_{\chi_d}{}^{*} W(\chi_d). \tag{3.7}$$

Here the max is over $A < d_1 \le P$.

Similarly to Section III of [2] we have

$$W(\chi_d) \le (1 + 2\cdot 10^{-5})(N^{1/2}/\log N) \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{1+\varepsilon}d^{-1}}} N^{(1-\varepsilon)(\beta-1)} + O\Big(N^{1/2-\varepsilon} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{1.01}d^{-1}}} N^{\beta-1}\Big) + O\Big(P^{1/2-0.01\theta} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{2.01}}} N^{\beta-1}\Big) + O(N^{1/2-1.01\theta+\varepsilon}d^{-1}), \tag{3.8}$$

where $\sum'$ indicates that the sum does not contain the exceptional zero $\widetilde{\beta}$.

By the same methods as used in [1] we have

$$\begin{gathered} \sum_{\substack{d\le P\\ [d_1,d]\le D^{\varepsilon}(d_1,d)}} \sideset{}{^*}\sum_{\chi_d} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{2.01}}} N^{\beta-1} \ll N^{0.7\varepsilon},\\ \sum_{d\le P} \sideset{}{^*}\sum_{\chi_d} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{2.01}}} N^{\beta-1} \ll N^{0.7\varepsilon}. \end{gathered} \tag{3.9}$$

Let

$$\begin{aligned} I_1 &= \sum_{\substack{d\le P\\ [d_1,d]\le D^{\varepsilon}(d_1,d)}} \sideset{}{^*}\sum_{\chi_d} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{1+\varepsilon}d^{-1}}} N^{(1-\varepsilon)(\beta-1)},\\ I_2 &= \sum_{d\le P} \sideset{}{^*}\sum_{\chi_d} \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le P^{1+\varepsilon}d^{-1}}} N^{(1-\varepsilon)(\beta-1)}. \end{aligned} \tag{3.10}$$

Suppose $\varrho_{\chi_d} = \beta_{\chi_d} + i\gamma_{\chi_d}, |\gamma_{\chi_d}| \le P^{1+\varepsilon}d^{-1}$, is a zero of $L(s,\chi_d)$. Let $\mathcal{L} = (1+\varepsilon)\log P$.

1) If $1 - 0.24/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.239/\mathcal{L}$, then by Lemma 1 and Tables 1 and 2 we have

$$I_1 \le 2e^{-0.239/(\theta+\varepsilon)} + \frac{1}{\theta+\varepsilon}\int_{1.116}^{\infty} e^{-(1-\varepsilon)t/(\theta+\varepsilon)} N_1^*(t,P)\,dt \le 0.091628,$$

$$I_2 \le 2e^{-0.239/(\theta+\varepsilon)} + \frac{1}{\theta+\varepsilon}\int_{0.444}^{\infty} e^{-(1-\varepsilon)t/(\theta+\varepsilon)} N^*(t,P)\,dt \le 0.482901.$$

2) $1 - 0.26/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.24/\mathcal{L} \Rightarrow I_1 \le 0.092516,\ I_2 \le 0.537213.$
3) $1 - 0.28/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.26/\mathcal{L} \Rightarrow I_1 \le 0.075429,\ I_2 \le 0.5834782.$
4) $1 - 0.30/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.28/\mathcal{L} \Rightarrow I_1 \le 0.0624122,\ I_2 \le 0.6431567.$

5) $1 - 0.32/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.30/\mathcal{L} \Rightarrow I_1 \le 0.0543097,\ I_2 \le 0.714270.$
6) $1 - 0.34/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.32/\mathcal{L} \Rightarrow I_1 \le 0.0509092,\ I_2 \le 0.774367.$
7) $1 - 0.36/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.34/\mathcal{L} \Rightarrow I_1 \le 0.0520594,\ I_2 \le 0.7143177.$
8) $1 - 0.38/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.36/\mathcal{L} \Rightarrow I_1 \le 0.0581037,\ I_2 \le 0.628356.$
9) $1 - 0.40/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.38/\mathcal{L} \Rightarrow I_1 \le 0.0694366,\ I_2 \le 0.5560776.$
10) $1 - 0.42/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.40/\mathcal{L} \Rightarrow I_1 \le 0.0871545,\ I_2 \le 0.4952959.$
11) $1 - 0.44/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.42/\mathcal{L} \Rightarrow I_1 \le 0.1123271,\ I_2 \le 0.4441753.$
12) $1 - 0.46/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.44/\mathcal{L} \Rightarrow I_1 \le 0.1354119,\ I_2 \le 0.40117443.$
13) $1 - 0.48/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.46/\mathcal{L} \Rightarrow I_1 \le 0.152843,\ I_2 \le 0.364999.$
14) $1 - 0.50/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.48/\mathcal{L} \Rightarrow I_1 \le 0.164587,\ I_2 \le 0.334561.$
15) $1 - 0.517/\mathcal{L} \le \beta_{\chi_d} \le 1 - 0.50/\mathcal{L} \Rightarrow I_1 \le 0.1774831,\ I_2 \le 0.3089471.$
16) $1 - 0.517/\mathcal{L} \ge \beta_{\chi_d} \Rightarrow I_1 \le 0.1774831,\ I_2 \le 0.3089471.$

Hence in all cases we have

$$I_1 I_2 \le 0.0557876. \tag{3.11}$$

Now we suppose that the exceptional primitive real character $\widetilde{\chi} \pmod{\widetilde{q}}$ exists, and the unique exceptional real zero $\widetilde{\beta}$ of $L(s, \widetilde{\chi})$ satisfies the condition $\widetilde{\delta}(\theta + \varepsilon) \log x \le 0.239$ where $\widetilde{\delta} = 1 - \widetilde{\beta}$. In this case, as above we have

$$I_1 \le 0.00215731, \qquad I_2 \le 0.39343082. \tag{3.12}$$

Hence we have

$$I_1 I_2 \le 0.00084876. \tag{3.13}$$

By the definitions of $r_7(n), r_8(n), r_9(n)$, just as for $D_{16}(n), D_{13}(n)$ in [1]–[3], by Cauchy's inequality we have

$$|r_7(n)| \le \sum_{r \le P} \sideset{}{^*}\sum_{\chi \,(\mathrm{mod}\, r)} \sum_{\substack{q \le P \\ [\widetilde{q}, r] \mid q}} \frac{1}{\phi^2(q)} |Z(q, \widetilde{\chi}, \chi)| \Big\{ \int_{-1/(qQ)}^{1/(qQ)} |\widetilde{T}(z)|^2 \, dz \Big\}^{1/2} W(\chi).$$

Since

$$\int_{-1/(qQ)}^{1/(qQ)} |\widetilde{T}(z)|^2 \, dz \le \int_{-1}^{1} |\widetilde{T}(z)|^2 \, dz \le N \log^{-2} N,$$

by Lemma 4, (3.8) and (3.12) we have

$$\begin{aligned} |r_7(n)| &\le 2.140782(1+\varepsilon)\sigma(n) N \log^{-2} N (1 + 2 \cdot 10^{-5}) W(P, \widetilde{q}) \\ &\le 0.0046185 \sigma(n) N \log^{-2} N. \end{aligned}$$

Similarly

$$|r_8(n)| \le 0.0046185 \sigma(n) N \log^{-2} N.$$

For $r_9(n)$, by the definition and Cauchy's inequality

$$|r_9(n)| \le \sum_{r_1 \le P} \sideset{}{^*}\sum_{\chi \pmod{r_1}} \sum_{r_2 \le P} \sideset{}{^*}\sum_{\chi \pmod{r_2}} \sum_{\substack{q \le P \\ [r_1, r_2] | q}} \phi(q)^{-2} |Z(q, \widetilde{\chi}, \chi)| W(\chi_1) W(\chi_2).$$

By Lemma 4, (3.8) and (3.11)

$$\begin{aligned} |r_9(n)| &\le 2.140782(1+\varepsilon)\sigma(n) N \log^{-2} N (1 + 2 \cdot 10^{-5}) W(P) W'(P) \\ &\le 0.11943387 \sigma(n) N \log^{-2} N. \end{aligned}$$

When $\widetilde{\beta}$ does not exist, then there is no $r_7(n), r_8(n)$. By Lemma 4, (3.8) and (3.13),

$$\begin{aligned} |r_9(n)| &\le 2.140782(1+\varepsilon)\sigma(n) N \log^{-2} N (1 + 2 \cdot 10^{-5}) W(P) W'(P) \\ &\le 0.0018171 \sigma(n) N \log^{-2} N. \end{aligned}$$

By Lemma 3 and (3.3), Theorem 3 follows.

4. Proof of Theorems 1 and 2. In this section we let $L := \log_2 N$, and let $r_{k,k}(n)$ denote the number of representations of n in the form

$$n = 2^{\nu_1} + \ldots + 2^{\nu_k} - 2^{\mu_1} - \ldots - 2^{\mu_k}$$

with $1 \le \nu_i, \mu_i \le L$.

LEMMA 5. *For $k \ge 2$ and $\varepsilon > 0$, there exists a positive constant $N(k, \varepsilon)$ such that when $N \ge N(k, \varepsilon)$ we have*

$$\Big| \sum_{m \ne 0} r_{k,k}(m) \sigma(m) - 2L^{2k} \Big| \le 2L^{2k} \{H(k) + \varepsilon\},$$

where

$$H(k) := \min_{9 \le E \le L} \left\{ 1.7811 \left(1 - \frac{1}{E \csc^2(\pi/8)}\right)^{2k} \log E + 2.3270 \cdot \frac{1 + \log E}{E} \right\}.$$

This is Lemma 7 of [10].

Let

$$G(\alpha) = \sum_{\nu \le L} e(2^\nu \alpha).$$

LEMMA 6. *We have*

$$\int_0^1 |S(\alpha) G(\alpha)|^2 \, d\alpha \le \frac{2}{\log^2 2} CN$$

where $C < 8.23382$.

P r o o f. The proof is the same as that of Lemma 4 of [12]. Let

$$s(N)=\int_0^1 |S(\alpha)G(\alpha)|^2\,d\alpha.$$

Since for fixed $l\ge 1$,

$$|\{m_j\le L:|m_2-m_1|=l\}|\le 2(L-l),$$

instead of (3.7) of [12] we have

$$s(N)<2C_0C_2\frac{N}{\log^2 N}\sum_{1\le l\le L}(L-l)g(2^l-1)+\left(\frac{1}{\log 2}+\varepsilon\right)N.$$

By the proof of Lemma 4 of [12] we have

$$\sum_{1\le l\le y} g(2^l-1)\le\left(1.1160+\frac{1.4818(1+\log 10)}{10}\right)y<1.605378y.$$

Consequently, by Lemma 2.6 of [16],

$$\sum_{1\le l\le L}(L-l)g(2^l-1)\le 0.802689L^2.$$

Hence

$$\begin{aligned}s(N)&\le\left(\frac{2}{\log^2 2}0.6602\cdot 7.8342\cdot 1.8998\cdot 0.802689+\frac{1}{\log 2}+\varepsilon\right)N\\&<\frac{2}{\log^2 2}8.23382N.\end{aligned}$$

The proof of Lemma 6 is complete.

Define

$$\begin{aligned}\Theta:=\Theta(\eta):=&\frac{1}{\log 2}\eta\csc^2(\pi/8)\log\frac{1}{\eta\csc^2(\pi/8)}\\&+\frac{1}{\log 2}(1-\eta\csc^2(\pi/8))\log\frac{1}{1-\eta\csc^2(\pi/8)}.\end{aligned}$$

LEMMA 7. *Let $\eta=1/725$. Then for $k\ge 2$ and $\varepsilon>0$, there exists a positive constant $N(k,\varepsilon)$ such that when $N\ge N(k,\varepsilon)$ we have*

$$\sum_{m\le N}(r_k'(m))^2\le\frac{2NL^{2k}}{\log^2 N}\{1.11943387(1+H(k))+8.23382(1-\eta)^{2k-2}+\varepsilon\}.$$

P r o o f. As in Lemma 10 of [10] (note that $\Theta(\eta)<1/13$), by Lemmas 5–7 and Theorem 3 the lemma follows.

Proof of Theorems 1 and 2. For $k=953$, choose $E=52$. We have $H(953)<0.254146$. Note that

$$0.11943387 + 1.11943387 \cdot 0.254146 + 8.23382\left(1 - \frac{1}{725}\right)^{1904} < 0.9986.$$

Theorem 1 and Theorem 2 can now be proved in the same way as Theorem 1 and Theorem 2 in Section 7 of [11].

References

[1] J. R. Chen, *The exceptional set of Goldbach numbers (II)*, Sci. Sinica 26 (1983), 714–731.

[2] J. R. Chen and J. M. Liu, *The exceptional set of Goldbach numbers (III)*, Chinese Quart. J. Math. 4 (1989), 1–15.

[3] J. R. Chen and C. D. Pan, *The exceptional set of Goldbach numbers*, Sci. Sinica 23 (1980), 416–430.

[4] H. Z. Li, *Zero-free regions for Dirichlet L-functions*, Quart. J. Math. Oxford 50 (1999), 13–23.

[5] —, *The exceptional set of Goldbach numbers*, ibid., 471–482.

[6] —, *The exceptional set of Goldbach numbers (II)*, Acta Arith. 92 (2000), 71–88.

[7] —, *The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes*, ibid., 229–237.

[8] Yu. V. Linnik, *Prime numbers and powers of two*, Trudy Mat. Inst. Steklov. 38 (1951), 151–169 (in Russian).

[9] —, *Addition of prime numbers and powers of one and the same number*, Mat. Sb. (N.S.) 32 (1953), 3–60 (in Russian).

[10] J. Y. Liu, M.-C. Liu and T. Z. Wang, *The number of powers of 2 in a representation of large even integers (I)*, Sci. China Ser. A 41 (1998), 386–398.

[11] —, —, —, *The number of powers of 2 in a representation of large even integers (II)*, ibid., 1255–1271.

[12] —, —, —, *On the almost Goldbach problem of Linnik*, J. Théor. Nombres Bordeaux 11 (1999), 133–147.

[13] M.-C. Liu and K. M. Tsang, *Small prime solutions of linear equations*, in: Théorie des Nombres, Koninck & Levesque (eds.), de Gruyter, 1989, 595–624.

[14] M.-C. Liu and T. Z. Wang, *A numerical bound for small prime solutions of some ternary linear equations*, Acta Arith. 86 (1998), 343–383.

[15] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach's problem*, ibid. 27 (1975), 353–370.

[16] R. C. Vaughan, *The Hardy–Littlewood Method*, Cambridge Univ. Press, 1981.

Department of Mathematics
Shandong University
Jinan Shandong 250100
P.R. China
E-mail: lihz@sdu.edu.cn

Received on 11.1.2000
and in revised form 18.4.2000 (3737)