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Quadratic minima and modular forms II

by

Barry Brent (Minneapolis, MN)

1. Introduction. Carl Ludwig Siegel showed in [S1] (English translation, [S2]) that the constant terms of certain level one negative-weight modular forms T_h are non-vanishing (Satz 2), and that this implies an upper bound on the least positive exponent of a non-zero Fourier coefficient for any level one entire modular form of weight h with a non-zero constant term. Level one theta functions fall into this category. Their Fourier coefficients code up representation numbers of quadratic forms. For positive even h, Siegel's result gives an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $n = 2h$ variables. This bound is sharper than Minkowski's for large n. (Mallows, Odlyzko and Sloane have improved Siegel's bound in [MOS].)

John Hsia [private communication to Glenn Stevens] suggested that Siegel's approach might be extended to higher levels. Following this hint, we constructed an analogue of T_h for $\Gamma_0(2)$, which we denote as $T_{2,h}$. To prove Satz 2, Siegel controlled the sign of the Fourier coefficients in the principal part of T_h. In [B] (henceforth, "part I"), following Siegel, we found upper bounds for the first positive exponent of a non-zero Fourier coefficient occurring in the expansion at infinity of an entire modular form with a non-zero constant term for $\Gamma_0(2)$ in the case $h \equiv 0 \pmod 4$. Siegel's method carried over intact.

In part I, we also stated that it was not clear that Siegel's method forces the non-vanishing of the $T_{2,h}$ constant terms when $h \equiv 2 \pmod 4$. But it turns out that we can tweak our definition of the $T_{2,h}$ and carry out Siegel's strategy.

Let us denote the vector space of entire modular forms of weight h for $\Gamma_0(N)$ as $M(N,h)$. In part I, we proved that the second non-zero Fourier coefficient of an element of $M(2,h)$ with non-zero constant term must have exponent at most $\dim M(2,h) = 1 + \lfloor h/4 \rfloor = r$ (say) if $h \equiv 0 \pmod 4$. This

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corresponds exactly to Siegel's bound for $f \in M(1,h)$. If $h \equiv 2 \pmod 4$, however, we only showed that the exponent is no more than $2r$.

In the second section, we prove that the exponent is at most r if $h \equiv 2 \pmod 4$. In the third section, we apply this result to the theory of quadratic forms.

We show that, if Q is an even positive-definite level two quadratic form in $v = 8u + 4$ variables, then Q represents a positive integer $2n \le 1 + v/4$. (In part I, we obtained the weaker bound $2 + v/2$. We also showed that if $v = 8u$, then Q represents an even positive integer $\le 2 + v/4$.)

2. Bounds for gaps in the Fourier expansions of entire modular forms. Section 2.1 is introductory. All but one of the results are stated without proof. The reader is referred to part I for details. In Section 2.2, we estimate the first positive exponent of a non-zero Fourier coefficient in the expansion of an entire modular form for $\Gamma_0(2)$ with a non-zero constant term.

2.1. *Some modular objects.* This section is a tour of the objects mentioned in the article. The main building blocks are Eisenstein series with known divisors and computable Fourier expansions.

As usual, we denote by $\Gamma_0(N)$ the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : c \equiv 0 \pmod N \right\}.$$

The vector space of entire modular forms of one variable in the upper half plane $\mathfrak{H}$ of weight h for $\Gamma_0(N)$ ("level N") and trivial character is denoted by $M(N,h)$. We have an inclusion lattice satisfying:

$$M(L,h) \subset M(N,h) \quad \text{if and only if} \quad L \mid N.$$

More particularly, any entire modular form for $\mathrm{SL}(2,\mathbb{Z})$ is also one for $\Gamma_0(2)$.

The dimension of $M(N,h)$ is denoted by $r(N,h)$, or r_h, or by r. For any positive even h,

$$r(2,h) = \left\lfloor \frac{h}{4} \right\rfloor + 1.$$

We write Δ for the weight 12, level one cusp form with Fourier series

$$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$$

and product expansion

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Here, τ is the Ramanujan function.

We describe some level two objects, using three special divisor sums:

$$\sigma^{\mathrm{odd}}(n) = \sum_{\substack{0<d\mid n\\ d\,\mathrm{odd}}} d, \qquad \sigma_k^{\mathrm{alt}}(n) = \sum_{0<d\mid n} (-1)^d d^k,$$

and

$$\sigma_{N,k}^*(n) = \sum_{\substack{0<d\mid n\\ N\nmid n/d}} d^k.$$

Let $E_{\gamma,2}$ denote the unique normalized form in the one-dimensional space $M(2,2)$ (i.e. the leading coefficient in the Fourier expansion of the form is a 1). The Fourier series is

$$E_{\gamma,2} = 1 + 24\sum_{n=1}^{\infty} \sigma^{\mathrm{odd}}(n) q^n. \tag{2.1}$$

$E_{\gamma,2}$ has a $\frac{1}{2}$-order zero at points of $\mathfrak{H}$ which are $\Gamma_0(2)$-equivalent to $-\frac{1}{2}+\frac{1}{2}i=\gamma$ (say). The vector space $M(2,4)$ is spanned by two forms $E_{0,4}$ and $E_{\infty,4}$, which vanish with order one at the $\Gamma_0(2)$-inequivalent zero and infinity cusps, respectively. They have Fourier expansions

$$E_{0,4} = 1 + 16\sum_{n=1}^{\infty} \sigma_3^{\mathrm{alt}}(n) q^n \tag{2.2}$$

and

$$E_{\infty,4} = \sum_{n=1}^{\infty} \sigma_{2,3}^*(n) q^n. \tag{2.3}$$

The following lemma was Proposition 2.3 in part I:

LEMMA 2.1. *The modular form $E_{\infty,4}\in M(2,4)$ has the following product decomposition in the variable q*:

$$E_{\infty,4}(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\setminus 2\mathbb{Z}} (1-q^n)^{-8}.$$

We do not need it, but it is also easy to show that

$$E_{0,4}(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\setminus 2\mathbb{Z}} (1-q^n)^{16}.$$

To exploit Lemma 2.1, we need a result due essentially to Euler (Lemma 2.11 of part I, quoted from [A], Theorem 14.8, valid when the infinite series is absolutely convergent):

LEMMA 2.2. *For a given set A and a given arithmetical function f, the numbers $p_{A,f}(n)$ defined by the equation*

$$\prod_{n\in A} (1-x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n) x^n$$

satisfy the recursion formula

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k) p_{A,f}(n-k),$$

where $p_{A,f}(0) = 1$ *and*

$$f_A(k) = \sum_{\substack{d|k \\ d\in A}} f(d).$$

Next, we construct a level two analogue of the level one Klein invariant j:

$$j_2 = E_{\gamma,2}^2 E_{\infty,4}^{-1}.$$

The function j_2 is analogous to j because it is modular (weight zero) for $\Gamma_0(2)$, holomorphic on the upper half plane, has a simple pole at infinity, generates the field of $\Gamma_0(2)$-modular functions, and defines a bijection of a $\Gamma_0(2)$ fundamental set with $\mathbb{C}$.

The following lemma was Proposition 2.7 in part I:

LEMMA 2.3. *For* $z \in \mathfrak{H}$,

$$\frac{d}{dz} j_2(z) = -2\pi i E_{\gamma,2}(z) E_{0,4}(z) E_{\infty,4}(z)^{-1}.$$

We introduce a level two analogue of Siegel's T_h for $h \equiv 2 \pmod 4$. For $r = r(2,h)$, we set

$$t_h = E_{0,4} E_{\infty,4}^{-r}.$$

The t_h are a useful replacement for the functions

$$T_{2,h} = E_{\gamma,2}^2 E_{0,4} E_{\infty,4}^{-1-r}$$

we defined in part I for the same h.

Finally, for $h \equiv 2 \pmod 4$ and $f \in M(2,h)$, let

$$w_2(f) = E_{\gamma,2}^{-1} E_{\infty,4}^{1-r} f.$$

This replaces the part I function $W_2(f) = E_{\gamma,2} E_{\infty,4}^{-(h+2)/4} f$ defined on the same f.

2.2. *A structural result on the q series of entire level two modular forms.* We establish a sequence of propositions mimicking the argument of [S2], pp. 249–254. (Siegel's proof is also sketched on p. 263 of part I.)

PROPOSITION 2.1. *The map* w_2 *is a vector space isomorphism from* $M(2,h)$ *onto the space of polynomials in* j_2 *of degree less than* r.

Proof. No non-trivial polynomial in j_2 can vanish almost everywhere, so the modular forms $j_2^d E_{\gamma,2} E_{\infty,4}^{r-1}$, $d = 0, 1, \ldots, r-1$, are a basis for $M(2,h)$, and we have

$$w_2(j_2^d E_{\gamma,2} E_{\infty,4}^{r-1}) = j_2^d.$$

The map w_2 is clearly linear and 1-to-1. ∎

PROPOSITION 2.2. *For $f \in M(2,h)$, the constant term in the Fourier expansion at infinity of $t_h f$ is zero.*

Proof. By applying Lemma 2.3, we see that

$$w_2(f)\frac{d}{dz}j_2 = -2\pi i t_h f.$$

Thus, $t_h f$ is the derivative of a polynomial in j_2, so it can be expressed in a neighborhood of infinity as the derivative with respect to z of a power series in the variable $q = \exp(2\pi i z)$. This derivative is a power series in q with vanishing constant term. ∎

PROPOSITION 2.3. *For $h \equiv 2 \pmod 4$, the constant term in the Fourier expansion at infinity of t_h is non-zero.*

Proof. Lemmas 2.1 and 2.3 imply that, for fixed s,

$$E_{\infty,4}^{-s} = q^{-s}\sum_{n=0}^{\infty} R(n)q^n, \tag{2.4}$$

where $R(0) = 1$ and $n > 0$ implies that

$$R(n) = \frac{8s}{n}\sum_{a=1}^{n}\sigma_1^{\mathrm{alt}}(a)R(n-a). \tag{2.5}$$

The divisor functions $\sigma_k^{\mathrm{alt}}(n)$, k odd, alternate sign, so the alternation of the sign of $R(n)$ follows by an easy induction argument from (2.5). To be specific, $R(n) = U_n(-1)^n$ for some $U_n > 0$. Thus we may write

$$E_{\infty,4}^{-r} = U_0(-1)^0 q^{-r} + U_1(-1)^1 q^{1-r} + \ldots + U_{r-1}(-1)^{r-1}q^{-1} + U_r(-1)^r + \ldots$$

On the other hand, the Fourier coefficient of q^n, $n \geq 0$, in the expansion of $E_{0,4}$ is $W_n(-1)^n$ for positive W_n, by (2.2). Thus the constant term of $t_h = E_{0,4}E_{\infty,4}^{-r}$ is

$$\sum_{m=-r}^{0} U_m(-1)^m W_{r-m}(-1)^{r-m} \neq 0. \ ∎$$

What follows is our main theorem on modular forms.

THEOREM 2.1. *Suppose $f \in M(2,h)$ with Fourier expansion at infinity*

$$f(z) = \sum_{n=0}^{\infty} A_n q^n, \qquad A_0 \neq 0.$$

Then some $A_n \neq 0, 1 \leq n \leq r(2,h)$.

Proof. First suppose that $h \equiv 2 \pmod 4$. We denote the coefficient of q^n in the Fourier expansion of f at infinity as $c_n[f]$. The normalized meromorphic form t_h has a Fourier series of the form

$$t_h = C_{h,-r}q^{-r} + \ldots + C_{h,0} + \ldots,$$

with $C_{h,-r} = 1$. By Proposition 2.2,

$$0 = c_0[t_h f] = C_{h,0}A_0 + \ldots + C_{h,-r}A_r.$$

By hypothesis, $A_0 \neq 0$. By Proposition 2.3, $C_{h,0} \neq 0$, so

$$A_0 = -(C_{h,0})^{-1}(C_{h,-1}A_1 + \ldots + C_{h,-r}A_r).$$

It follows that one of the A_n $(n = 1, \ldots, r)$ is non-zero.

To complete the proof, we point out that the claim was proved for $h \equiv 0 \pmod 4$ in Theorem 2.12 of part I by the same sort of argument. ∎

Remark This result is slightly better than the bound $n \leq r + 1$ of Conjecture 6.2, part I for $h \equiv 2 \pmod 4$. The reason is our different choice of a level two T_h analogue.

3. Quadratic minima

3.1. *Quadratic forms and modular forms.* For even v, we write $\mathbf{x} = {}^t(x_1, \ldots, x_v)$, so that $\mathbf{x}$ is a column vector. Let A be an v by v square symmetric matrix with integer entries, even entries on the diagonal, and positive eigenvalues. Then $Q_A(\mathbf{x}) = {}^t\mathbf{x}A\mathbf{x}$ is a homogeneous second degree polynomial in the x_i. We refer to Q_A as the even positive-definite quadratic form associated with A. If $\mathbf{x} \in \mathbb{Z}^v$, then $Q_A(\mathbf{x})$ is a non-negative even number, which is zero only if $\mathbf{x}$ is the zero vector. The level of Q_A is the smallest positive integer N such that NA^{-1} also has integer entries and even entries on the diagonal. Let $\#Q_A^{-1}(n)$ denote the cardinality of the inverse image in $\mathbb{Z}^v$ of an integer n under the quadratic form Q_A.

The following specialization of known results was Proposition 5.1 of part I.

Lemma 3.1. *Suppose that Q_A is a level two quadratic form. Then the function $\Theta_A : \mathfrak{H} \to \mathbb{C}$ satisfying*

$$\Theta_A(z) = \sum_{n=0}^{\infty} \#Q_A^{-1}(2n)q^n \tag{3.1}$$

lies in $M(2, v/2)$.

Since $M(2, h)$ is non-trivial only for even h, it also follows that $4 \mid v$.

3.2. *Quadratic minima.* In this section we apply Theorem 2.1 to the problem of quadratic minima.

Theorem 3.1. *If Q is a level two even positive-definite quadratic form in v variables, $8 \mid v$, then Q represents a positive integer $2n \leq 2 + v/4$. If $v \equiv 4 \pmod 8$, then Q represents a positive integer $2n \leq 1 + v/4$.*

Proof. The claim for $8\,|\,v$ was included in Theorem 5.2 of part I. Suppose $v = 8u + 4$. Let A be the matrix associated with Q, so that $Q = Q_A$. Then $\Theta_A \in M(2, 4u+2)$, and $\#Q_A^{-1}(2n) \neq 0$ for some $n, 1 \le n \le r(2, 4u+2) = 1 + u$. Thus Q represents an integer $2n \le 2 + 2u = 1 + v/4$. ∎

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Department of Mathematics
University of Minnesota
Vincent Hall
206 Church Street S.E.
Minneapolis, MN 55455, U.S.A.
E-mail: barryb@primenet.com

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