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The (*ABC*) Conjecture and the radical index of integers

by

Paulo Ribenboim (Kingston, ON)

1. Introduction. In my papers [4], [5] I have derived many consequences of the (*ABC*) Conjecture concerning powerful numbers and almost powerful numbers. In the present paper I introduce the concept of power index of a non-zero integer and I extend and sharpen the earlier results. This time the statements are about integers with power index satisfying appropriate conditions.

(1.1) The *radical* of a non-zero integer n is, by definition,

$$\operatorname{rad}(n) = \prod_{p\mid n} p$$

(product of the distinct primes p which divide n). In particular $\operatorname{rad}(1) = \operatorname{rad}(-1) = 1$ and $\operatorname{rad}(-n) = \operatorname{rad}(n)$ for every $n \neq 0$.

(1.2) Let $k \geq 2$. A non-zero integer n is said to be *k-powerful* when the following property is satisfied: if p is a prime which divides n then p^k divides n.

If $2 \leq k < h$ every h-powerful number is also k-powerful. The integers $1, -1$ are k-powerful for every $k \geq 2$. A 2-powerful number is simply called a *powerful* number.

Let $k \geq 2$. Every non-zero integer n may be written in a unique way in the form $n = w_k(n)n'$, where $w_k(n)$ is a k-powerful number, $\gcd(w_k(n), n') = 1$ and if a prime p divides n' then p^k does not divide n'. The integer $w_k(n)$ is called the *k-powerful part* of n. If $k = 2$ we simply write $w(n)$ and call it the *powerful part* of n.

The next concept was introduced in [5].

(1.3) Let $k \geq 2$. A non-zero integer n is said to be *almost k-powerful* if $[\operatorname{rad}(n)]^k \leq |n|$.

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If $2 \leq k < h$ every almost h-powerful number is also almost k-powerful. Every k-powerful is almost k-powerful, but the converse is not true. For example, if $p < q$ are prime numbers then $p^{k-1}q^{k+1}$ is almost k-powerful but not k-powerful.

Now we introduce the radical index of an integer.

(1.4) Let n be an integer, $n \neq 0, 1, -1$. The *radical index* $\nu(n)$ of n is, by definition, given by the relation

$$[\mathrm{rad}(n)]^{\nu(n)} = |n|.$$

By convention the radical index of $1, -1$ is $\nu(1) = \nu(-1) = \infty$. So $\nu(n) \geq 1$ for every non-zero integer n.

If $|n| > 1$ then n is square-free if and only if $\nu(n) = 1$. Also, n is almost k-powerful (where $k \geq 2$) if and only if $\nu(n) \geq k$.

We gather below some easy statements about the radical index.

(1.5) (1) *Let $k \geq 1$ be an integer. Then $\nu(n^k) = k\nu(n)$ for all n with $|n| > 1$.*

(2) *For $|n| > 1$, $\nu(n)$ is either an integer or an irrational number.*

(3) *If p is a prime and $p \nmid n$ then*

$$\nu(np) = \nu(n) - (\nu(n) - 1)\frac{1}{1 + \log r/\log p},$$

where $r = \mathrm{rad}(n)$. So if $\nu(n) \neq 1$ then $\nu(np) < \nu(n)$.

(4) *If $s \mid r = \mathrm{rad}(n)$ then $\nu(ns) = \nu(n) + \log s/\log r$ and for $k \geq 1$:*

$$\nu(nr^k) = \nu(n) + k.$$

Proof. It suffices to prove the statements for positive integers m, n.

(1) This is trivial.

(2) If $\nu(n) = k/h$ (with positive integers k, h), we show that h divides k. We have $[\mathrm{rad}(n)]^{k/h} = n$ so $[\mathrm{rad}(n)]^k = n^h$. Let p be a prime dividing n, and let $e \geq 1$ be such that $p^e \mid n$, but $p^{e+1} \nmid n$. Then $p^k = p^{eh}$, so h divides k.

(3) We assume that $p \nmid r$. Then

$$\nu(np) = \frac{\log(np)}{\log(rp)} = \frac{\nu(n) + \log p/\log r}{1 + \log p/\log r}$$
$$= \nu(n) - \frac{[\nu(n) - 1](\log p/\log r)}{1 + \log p/\log r}.$$

So if $\nu(n) > 1$ then $\nu(np) < \nu(n)$.

(4) Let $s \mid r = \mathrm{rad}(n)$. Then

$$\nu(ns) = \frac{\log(ns)}{\log r} = \nu(n) + \frac{\log s}{\log r}.$$

It follows that $\nu(nr) = \nu(n) + 1$ and by induction on $k \geq 1$, $\nu(nr^k) = \nu(n) + k$. ∎

The next result is a consequence of the following well-known conjecture:

(1.6) CONJECTURE OF ALGEBRAIC INDEPENDENCE OF LOGARITHMS. *If $p_1, \ldots, p_k$ are distinct primes then the set $\{\log p_1, \ldots, \log p_k\}$ is algebraically independent over $\mathbb{Q}$.*

This conjecture is a special case of a more embracing conjecture formulated by Schanuel (see [1]).

(1.7) *If $\nu(n) = \nu(m)$ is not an integer then $|n| = |m|$.*

Proof. Without loss of generality we assume $m, n > 1$. Let $\nu(n) = \nu(m) = \alpha$, where α is not an integer. We show that $r = \mathrm{rad}(n)$ is equal to $s = \mathrm{rad}(m)$. Assume that p is a prime such that $p \mid r$ but $p \nmid s$. Let $n = p^e n'$ with $e \geq 1$, $p \nmid n'$. Then

$$\frac{e \log p + \log n'}{\log p + \log(r/p)} = \frac{\log m}{\log s}.$$

Hence

$$\log p (e \log s - \log m) = \log(r/p) \log m - \log n' \log s.$$

But $\nu(m) = \log m / \log s = \alpha$ is not an integer, so $e \log s - \log m \neq 0$. Hence $\log p$ belongs to the field generated by $\log p_1, \ldots, \log p_k$, where $p_1, \ldots, p_k$ are the prime factors of mn', so each $p_i \neq p$. This contradicts Conjecture (1.6). So $r \mid s$ and by symmetry $r = s$. Hence $\log m = \log n$ and $m = n$. ∎

The set $\{\nu(n) \mid n > 1\}$ is a countable subset of the set of real numbers $\alpha \geq 1$. It contains all integers $k \geq 1$. Moreover

(1.8) (1) *If $1 \leq \alpha < \beta$ there exists n such that $\alpha < \nu(n) < \beta$.*

(2) *For every $\alpha \geq 1$ there exists a sequence of positive integers $(n_i)_{i \geq 1}$ with $\nu(n_1) > \nu(n_2) > \ldots$ and $\lim_{i \to \infty} \nu(n_i) = \alpha$.*

(3) *For every $\alpha > 1$ there exists a sequence of positive integers $(n_i)_{i \geq 1}$ such that $\nu(n_1) < \nu(n_2) < \ldots$ and $\lim_{i \to \infty} \nu(n_i) = \alpha$.*

Proof. (1) Let $p \neq 2$ be a prime such that

$$\left(1 + \frac{\log p}{\log 2}\right)(\beta - \alpha) > 1.$$

So there exists an integer h such that

$$(\alpha - 1)\left(1 + \frac{\log p}{\log 2}\right) < h < (\beta - 1)\left(1 + \frac{\log p}{\log 2}\right).$$

So

$$\alpha < 1 + \frac{h}{1 + \log p / \log 2} = 1 + \frac{h \log 2}{\log(2p)} = \frac{\log(2^{h+1} p)}{\log(2p)} = \nu(2^{h+1} p) < \beta$$

(the last inequality checked in a similar way).

(2) and (3). These are trivial consequences of (1). ∎

We introduce the following notation. For all $\alpha > 1$ let $N_\alpha = \{n \mid \nu(n) \geq \alpha\}$. If k is an integer and $k \geq 2$ then N_k is the set of all almost k-powerful integers.

For the convenience of the reader, we state the (ABC) Conjecture as it will be used in this paper.

(1.9) The (ABC) Conjecture. *For every $\varepsilon > 0$, there exists $K > 0$, depending on ε, such that if A, B, C are non-zero coprime integers such that $A + B + C = 0$ then*

$$\max\{|A|, |B|, |C|\} < K[\mathrm{rad}(ABC)]^{1+\varepsilon}.$$

In the conjecture there is no suggestion of an explicit expression for K as a function of ε.

2. The equations $Ax+By+Cz=0$ and $Ax+By=C$. Let $0 < \delta \leq 1$, let R, S, T be positive coprime square-free integers. Let

$H = \{(x, y, z) \mid x, y, z$ are non-zero coprime integers, $1/\nu(x) + 1/\nu(y) + 1/\nu(z) < 1 - \delta$, there exist non-zero integers A, B, C such that $\mathrm{rad}(A) \mid R, \mathrm{rad}(B) \mid S, \mathrm{rad}(C) \mid T$ and $Ax + By + Cz = 0\}$.

(2.1) Theorem. *If the (ABC) Conjecture is true then H is a finite set.*

Proof. Let

$$H_1 = \{(x, y, z) \in H \mid |x| \geq |y|, |z|\},$$
$$H_2 = \{(x, y, z) \in H \mid |y| \geq |x|, |z|\},$$
$$H_3 = \{(x, y, z) \in H \mid |z| \geq |x|, |y|\}.$$

So

$$H = H_1 \cup H_2 \cup H_3.$$

We show that H_1 is a finite set. Let $(x, y, z) \in H_1$, so there exist $A, B, C \neq 0$ with $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax + By + Cz = 0$. We note that Ax, By, Cz are non-zero coprime integers.

Let $0 < \varepsilon < \delta/(1 - \delta)$. By the (ABC) Conjecture there exists $K > 0$, depending on ε, such that

$$|x| \leq |Ax| < Kr^{1+\varepsilon}$$

where

$$\begin{aligned} r &= \mathrm{rad}(Ax \cdot By \cdot Cz) \leq RST \cdot \mathrm{rad}(x) \cdot \mathrm{rad}(y) \cdot \mathrm{rad}(z) \\ &= RST \cdot |x|^{1/\nu(x)} |y|^{1/\nu(y)} |z|^{1/\nu(z)} \\ &\leq RST \cdot |x|^{1/\nu(x)+1/\nu(y)+1/\nu(z)} \leq RST|x|^{1-\delta}. \end{aligned}$$

Thus $|x| < K'|x|^{(1-\delta)(1+\varepsilon)}$ where $K' = K(RST)^{1+\varepsilon}$.

We note that $(1-\delta)(1+\varepsilon) < 1$. Therefore $|x|$ remains bounded and since $|y|, |z| \leq |x|$ then H_1 is a finite set. Similarly, H_2, H_3 are finite sets. ∎

We indicate two types of corollaries. First we spell out the case where the coefficients A, B, C are given. In the second corollary we discuss solutions in powerful numbers x, y, z.

(2.2) Corollary. *Let A, B, C be non-zero coprime integers, let $0 < \delta < 1$. The set $H = \{(x,y,z) \mid x,y,z$ are non-zero coprime integers, $1/\nu(x) + 1/\nu(y) + 1/\nu(z) \le 1 - \delta$ and $Ax + By + Cz = 0\}$ is finite.*

The next corollary was proved in Ribenboim [4]:

(2.3) Corollary. *Let R, S, T be positive coprime square-free integers. Let $J = \{(x,y,z) \mid x,y,z$ are non-zero coprime integers, there exist integers $l,m,n \ge 2$ such that x is l-powerful, y is m-powerful, z is n-powerful, $1/l + 1/m + 1/n < 1$ and there exist non-zero integers A, B, C such that $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax + By + Cz = 0\}$. Then J is a finite set.*

Proof. It is easy to see that there exists $\mu < 1$ such that if $1/l + 1/m + 1/n < 1$ then $1/l + 1/m + 1/n \le \mu$ (see Ribenboim [4] for details). Let $\delta = 1 - \mu$. Next, with the above notations, $[\mathrm{rad}(x)]^l \le |x|$, so $l \le \nu(x)$, similarly, $m \le \nu(y)$ and $n \le \nu(z)$. Thus $J \subseteq H$, hence J is a finite set. ■

We may give an equivalent formulation of the above corollary in terms of the following set J^*.

Let J^* be the set of all (x,y,z,l,m,n) where x,y,z,l,m,n are as indicated in the definition of J. The mapping $(x,y,z,l,m,n) \in J^* \mapsto (x,y,z) \in J$ is surjective. Now, we observe that for each non-zero integer t there are at most finitely many $k \ge 2$ such that t is k-powerful. Therefore J is a finite set if and only if J^* is a finite set.

We prove now a similar result for the equation $Ax + By = C$.

Let R, S, T be positive square-free coprime integers, let $0 < \delta < 1$ and let $d \ge 1$. Consider the set $H' = \{(x,y) \mid x,y$ are non-zero integers, $\gcd(x,y) \mid d$, $1/\nu(x) + 1/\nu(y) < 1 - \delta$ and there exist non-zero integers A, B, C such that $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax + By = C\}$.

(2.4) Theorem. *If the (ABC) Conjecture is true then H' is a finite set.*

Proof. Let $H'_1 = \{(x,y) \in H' \mid |x| \ge |y|\}$, and $H'_2 = \{(x,y) \in H' \mid |y| \ge |x|\}$, so $H' = H'_1 \cup H'_2$. We show that H'_1 is a finite set. Let $(x,y) \in H'_1$ and let $Ax + By = C$ with A, B, C as indicated. Let $e = \gcd(x,y)$ so $e \mid d$ and $A\frac{x}{e}$, $B\frac{y}{e}$, $\frac{C}{e}$ are non-zero coprime integers.

Let $0 < \varepsilon < \delta/(1-\delta)$; by the (ABC) Conjecture there exists $K > 0$, depending on ε, such that

$$\frac{|x|}{d} \le \frac{|Ax|}{e} < Kr^{1+\varepsilon}$$

where

$$r = \mathrm{rad}\left(\frac{Ax}{e}\cdot\frac{By}{e}\cdot\frac{C}{e}\right) \le RST\,\mathrm{rad}(x)\cdot\mathrm{rad}(y)$$
$$= RST|x|^{1/\nu(x)}\cdot|y|^{1/\nu(y)} \le RST|x|^{1/\nu(x)+1/\nu(y)} \le RST|x|^{1-\delta}.$$

Let $K' = Kd(RST)^{1+\varepsilon}$ so $|x| < K'|x|^{(1-\delta)(1+\varepsilon)}$.

But $(1-\delta)(1+\varepsilon) < 1$, so $|x|$ remains bounded; since $|y| \le |x|$ then H_1' is a finite set. Similarly, H_2' is a finite set. ∎

We give now an immediate corollary.

(2.5) Corollary. *Let $0 < \delta < 1$, let A, B, C be non-zero coprime integers. Then the set $J = \{(x,y) \mid x,y \ne 0,\ 1/\nu(x)+1/\nu(y) \le 1-\delta$ and $Ax+By = C\}$ is finite.*

Proof. For each divisor d of C, the set $J_d = \{(x,y)\in J \mid \gcd(x,y) = d\}$ is finite. So J is also finite. ∎

(2.6) Corollary. *Let $\alpha,\beta > 1$ be such that $1/\alpha+1/\beta < 1$, let T be a square-free positive integer and let $d \ge 1$. Then the sets $I_\pm = \{x \in N_\alpha \mid$ there exists $C \ne 0$ such that $\mathrm{rad}(C)\,|\,T,\ \gcd(x,C)\,|\,d$ and $x \pm C \in N_\beta\}$ are finite.*

Proof. We consider the sets $H_\pm' = \{(x,y) \mid x\in N_\alpha, y \in N_\beta, \mathrm{rad}(C)\,|\,T$, $\gcd(x,C) \mid d$ and $x\pm C = y\}$. We have $\nu(x)\ge\alpha$, $\nu(y)\ge\beta$; taking $A=1$, $B=-1$, since $\gcd(x,y)\,|\,d$ by the theorem the sets $H_\pm'$ are finite. Therefore, $I_\pm$ are finite sets. ∎

We proved the following very special case in [4]:

(2.7) Corollary. *Let T be a square-free positive integer, let $d \ge 1$. Then there exist only finitely many 3-powerful (resp. powerful) integers x such that there exists $C \ne 0, x$ with $\mathrm{rad}(C)\,|\,T$, $\gcd(x,C)\,|\,d$ and such that $x\pm C$ is a powerful (resp. a 3-powerful) integer. In particular there are only finitely many 3-powerful (resp. powerful) integers x such that $x \pm 1$ is a powerful (resp. a 3-powerful) integer.*

No proof of this last assertion is known, without appealing to the (ABC) Conjecture.

Here is another consequence of (2.6):

(2.8) Corollary. *Let T be a positive square-free integer, let $d \ge 1$ and let $\alpha,\beta > 1$ such that $1/(2\alpha)+1/\beta < 1$. Then the set $L' = \{x \in N_\alpha \mid$ there exists $C \ne 0$, $\mathrm{rad}(C)\,|\,T$, $\gcd(x,C)\,|\,d$ and both $x+C$ and $x-C \in N_\beta\}$ is finite.*

P r o o f. Let $x \in L'$. Then $\nu(x^2) \geq 2\alpha$, also $\mathrm{rad}(C^2) = \mathrm{rad}(C) \mid T$ and $x^2 - C^2 \in N_\beta$: indeed,

$$\mathrm{rad}(x^2-C^2) \leq \mathrm{rad}(x-C)\cdot\mathrm{rad}(x+C) \leq (x-C)^{1/\beta}(x+C)^{1/\beta} = (x^2-C^2)^{1/\beta}.$$

Since $1/(2\alpha)+1/\beta < 1$ this shows that $x^2 \in I_-$ as defined in (2.6). Therefore L' is a finite set. ∎

The following special case was proved in [4]:

(2.9) COROLLARY. *Let* T *be a positive square-free integer*, *let* $d \geq 1$. *Then there exist only finitely many triples* $(x - c, x, x + c)$ *of powerful numbers*, *where* $0 < c < x$, $\gcd(x, c) \mid d$, $\mathrm{rad}(c) \mid T$.

More special statements are the following:

(2.10) WEAKER ERDŐS' CONJECTURE. *There are at most finitely many triples of consecutive powerful numbers.*

(2.11) ERDŐS' CONJECTURE. *Three consecutive integers cannot all be powerful.*

The weaker Erdős' Conjecture has not been proved without appealing to the (ABC) Conjecture, while Erdős' Conjecture has not been proved to be a consequence of the (ABC) Conjecture.

Furthermore, we note the following corollary.

Let $0 < \delta < 1$, let $S = \{s_1, s_2, \ldots\}$, where each s_i is a real number, $s_i > 1$ and $1/s_i + 1/s_{i+1} \leq 1 - \delta$.

(2.12) COROLLARY. *Let* $1 < n_1 < n_2 < \ldots$ *be a sequence of integers such that* $\nu(n_i) \geq s_i$ *for all* $i \geq 1$. *Then*

$$\lim_{i\to\infty} (n_{i+1} - n_i) = \infty.$$

P r o o f. By (2.5), for every $k \geq 1$ the set $I = \{(x, y) \mid x, y > 0,\ 1/\nu(x) + 1/\nu(y) \leq 1 - \delta \text{ and } x - y \leq k\}$ is finite.

In particular, the subset $\{(n_{i+1}, n_i) \mid n_{i+1} - n_i \leq k\}$ is finite. This proves that $\lim_{i\to\infty}(n_{i+1} - n_i) = \infty$. ∎

A special case is the following:

(2.13) COROLLARY. *Let* $S = \{s_1, s_2, \ldots\}$ *where each* s_i *is an integer*, $s_i \geq 2$ *but two consecutive integers* s_i, s_{i+1} (*for any* $i \geq 1$) *are not both equal to* 2. *Let* $1 < n_1 < n_2 < \ldots$ *be a sequence of integers such that* n_i *is* s_i*-powerful. Then* $\lim_{i\to\infty}(n_{i+1} - n_i) = \infty$.

The following conjecture was formulated by Pillai (see Ribenboim [3]):

(2.14) PILLAI'S CONJECTURE. *Let* $1 < n_1 < n_2 < \ldots$ *be the sequence of all integers which are proper powers. Then* $\lim_{i\to\infty}(n_{i+1} - n_i) = \infty$.

Pillai's Conjecture is included in the last corollary, after remarking that there are only finitely many squares with bounded difference.

Let R, S, T be positive square-free coprime integers, let $M \geq 1$, let $H = \{y > 0 \mid$ there exist $x > 0$, non-zero integers A, B, C, such that $\gcd(x, y) = 1$, $\nu(x) \geq 2$, $x/y^2 \leq M$, $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax + By^2 = C\}$.

We note that if $|B/A|, |C/A| \leq M/2$ then $x/y^2 \leq M$. Indeed

$$\left|\frac{x}{y^2} + \frac{B}{A}\right| = \left|\frac{C}{A}\right| \cdot \frac{1}{y^2} \leq \left|\frac{C}{A}\right|,$$

so

$$\left|\frac{y}{x^2}\right| \leq \left|\frac{B}{A}\right| + \left|\frac{C}{A}\right| \leq M.$$

(2.15) THEOREM. *Let R, S, T, M and H be as indicated above. We assume that the (ABC) Conjecture is true.*

(1) *For every $\varepsilon > 0$ and $\alpha > 1$ there exists $K > 0$ (depending on R, S, T, M, ε, α) such that if $y \in H$, $y > K$ and $y_1 \mid y$, with $y_1 \in N_\alpha$, then $y_1 < y^\varepsilon$.*

(2) *The set of integers $y \in H$ having a factor $y_1 \in N_\alpha$ such that $y_1 > y^\varepsilon$, is finite.*

Proof. (1) Let

$$\delta = \frac{(1 - 1/\alpha)\varepsilon}{4 + (1 + 1/\alpha)\varepsilon};$$

we observe that $0 < \delta < (\alpha - 1)/(\alpha + 1)$.

Let $y \in H$, so there exist x, A, B, C, as indicated in the definition of H. We note that Ax, By, C are coprime. By the (ABC) Conjecture there exists $K_1 > 0$ such that $|B|y^2 < K_1 r^{1+\delta}$ where

$$\begin{aligned} r &= \mathrm{rad}(Ax \cdot By^2 \cdot C) \leq \mathrm{rad}(ABC) \cdot \mathrm{rad}(x) \cdot \mathrm{rad}(y^2) \\ &\leq RSTx^{1/\nu(x)}\,\mathrm{rad}(y). \end{aligned}$$

By hypothesis, $x = \frac{x}{y^2} \cdot y^2 \leq My^2$. Therefore

$$y^2 \leq |B|y^2 < K_2[y^{2/\nu(x)}\,\mathrm{rad}(y)]^{1+\delta}$$

where $K_2 = [RSTM]^{1+\delta}$.

Let $y = y_1 y_2$ with $y_1 \in N_\alpha$, so $\nu(y_1) \geq \alpha$. We have $\mathrm{rad}(y) \leq \mathrm{rad}(y_1) \cdot y_2 \leq y_1^{1/\alpha} \cdot y_2$. Hence

$$y_1^2 y_2^2 < K_2[y_1^{1/\nu(x)+1/\alpha} y_2^{1/\nu(x)+1}]^{1+\delta} \leq K_2[y_1^{1+1/\alpha} y_2^2]^{1+\delta}.$$

Let $e = 2 - (1 + 1/\alpha)(1 + \delta)$ so $e > 0$ since $\delta < (\alpha - 1)/(\alpha + 1)$. Thus $y_1^e < K_2 y_2^{2\delta}$. Let $K_3 = K_2^{1/e}$, so $y_1 < K_3 y^{2\delta/e}$.

A simple calculation shows that $2\delta/e < \varepsilon$. Let $f = 1/(\varepsilon - 2\delta/e)$ and $K = K_3^f$ so K depends on R, S, T, M, ε, α. If $y > K$ then $y^{1/f} > K_3$. So $y_1 < y^\varepsilon$.

(2) This follows at once from (1). ∎

The following corollary was proved by Ribenboim and Walsh [6]:

(2.16) COROLLARY. *Let A, B, C be non-zero coprime integers. For every $\varepsilon > 0$ there exist only finitely many integers $y > 0$ with the following properties*:

(1) *There exists $x > 0$ with $\gcd(x, y) = 1$ such that $Ax^2 + By^2 = C$.*
(2) *The powerful part $w(y)$ of y satisfies $w(y) > y^\varepsilon$.*

In particular there are only finitely many powerful integers y satisfying (1).

Proof. We apply the theorem, replacing x by x^2, so $\nu(x^2) \geq 2$; we have x^2/y^2 bounded and $y_1 = w(y) \in N_2$. If y satisfies (1) and it is powerful then $w(y) = y > y^\varepsilon$, so y belongs to the finite set of integers, which also satisfy (2). ∎

3. Values of polynomials. To begin we mention the following conjecture of Langevin [2]:

(3.1) LANGEVIN'S CONJECTURE (L). *Let $f \in \mathbb{Z}[X]$ with degree $d \geq 2$ and no multiple root. For every $\varepsilon > 0$ there exist n_0, $K > 0$ (depending on ε, f) such that if $x > n_0$ then $f(x) \neq 0$ and $\operatorname{rad}(f(x)) \geq Kx^{d-1-\varepsilon}$.*

Langevin proved:

(3.2) THEOREM. *If the (ABC) Conjecture is true then the Conjecture* (L) *is true.*

Let f be a primitive polynomial of $\mathbb{Z}[X]$ with degree $d \geq 2$. Let $g \in \mathbb{Z}[X]$ be the product of all primitive irreducible polynomials $p \in \mathbb{Z}[X]$ such that $p \mid f$ but $p^2 \nmid f$. Let $\deg(g) = e$. We assume that $e \geq 2$; let $\alpha > d/(e-1) > 1$. Let R be a positive square-free integer and let

$$T = \{x > 0 \mid \text{there exist } a > 0 \text{ such that } \operatorname{rad}(a) \mid R, \text{ and } z \in N_\alpha \text{ such that } f(x) = az\}.$$

(3.3) THEOREM. *If the Conjecture* (L) *is true then T is a finite set.*

Proof. Let $0 < \varepsilon < (\alpha e - d - \alpha)/(\alpha + 1)$. There exists $n_0 > 0$ such that if $x > n_0$ then $|f(x)| < x^{d+\varepsilon}$. By the Conjecture (L) there exist $n_1 > n_0$ and $K > 0$ such that if $x > n_1$ then $f(x) \neq 0$ and $\operatorname{rad}(g(x)) > Kx^{e-1-\varepsilon}$.

Let $x \in T$, $x > n_1$. Since $g(x)$ divides $f(x)$ then

$$\begin{aligned}\operatorname{rad}(g(x)) &\le \operatorname{rad}(f(x)) = \operatorname{rad}(az) \le R\operatorname{rad}(z)\\ &= Rz^{1/\nu(z)} \le R^{1-1/\nu(z)}(az)^{1/\nu(z)} = R(f(x))^{1/\nu(z)}\\ &\le Rx^{(d+\varepsilon)/\nu(z)} \le Rx^{(d+\varepsilon)/\alpha}.\end{aligned}$$

So

$$x^{e-1-\varepsilon-(d+\varepsilon)/\alpha} < R/K.$$

But

$$\alpha e - \alpha - \alpha\varepsilon - d - \varepsilon = \alpha e - (d+\alpha) - (\alpha+1)\varepsilon > 0.$$

This shows that x remains bounded so T is a finite set. ■

The following special case was proved by Walsh [9]:

(3.4) *Let $f \in \mathbb{Z}[X]$ be a polynomial without multiple roots. If* $\deg(f) \ge 3$ *then the set* $\{x > 0 \mid f(x)$ *is powerful*$\}$ *is finite. If* $\deg(f) = 2$ *then the set* $\{x > 0 \mid f(x)$ *is* 3*-powerful*$\}$ *is finite.*

Proof. We apply the theorem with $R = 1$ and $d = e$. If $d \ge 3$ then $\alpha = 2$ satisfies the required condition; if $d = 2$ then $\alpha = 3$ satisfies the condition. ■

We shall require the following facts about the location of zeros of polynomials.

If $f \in \mathbb{Z}[X]$, the *height* of f, denoted by $H(f)$, is the maximum of the absolute values of its coefficients. The *length* of f, denoted by $L(f)$, is the number of its non-zero monomials.

(3.5) *If $f(x) = 0$ then $|x| < H(f) + 1$.*

Proof. We may assume that $|x| > 1$. Let $f(X) = a_0X^m + \ldots + a_{m-1}X + a_m$, with $m \ge 1$, $a_0 \ne 0$. If $f(x) = 0$ then

$$\begin{aligned}|x^m| \le |a_0x^m| &= |a_1x^{m-1} + \ldots + a_{m-1}x + a_m|\\ &\le H(f)[|x|^{m-1} + \ldots + |x|^{m-L(f)}]\\ &= H(f)|x|^{m-L(f)} \cdot \frac{|x|^{L(f)} - 1}{|x| - 1},\end{aligned}$$

hence

$$|x|^{L(f)} \le H(f)\frac{|x|^{L(f)} - 1}{|x| - 1}.$$

Hence

$$|x| - 1 \le H(f)\frac{|x|^{L(f)} - 1}{|x|^{L(f)}} < H(f). \ ■$$

Now we consider certain families of polynomials. Let $A \geq 1$ and $\alpha > 1$. Let $F_{A,\alpha}$ be the set of all $f(X) = aX^m + g(X) \in \mathbb{Z}[X]$, with $H(f) \leq A$, $a \geq 1$, $\deg(g) = k \geq 1$ and $m - 1 \geq \alpha(k+2)$.

(3.6) THEOREM. *Assuming that the* (*ABC*) *Conjecture is true*, *given* A, α *as above and given* $\gamma > \alpha/(\alpha-1)$ *the set* $V = \{f(x) \mid f \in F_{A,\alpha}$, $|x| \geq A+1$ *and* $f(x) \in N_\gamma\}$ *is finite.*

Proof. Let $f(x) \in N_\gamma$. Since $|x| \geq A+1$, $H(f) \geq H(g)$, then $f(x) \neq 0$ and $g(x) \neq 0$. Let

$$d = \gcd(f(x), ax^m, g(x)).$$

Let $0 < \varepsilon < (\alpha\gamma - \alpha - \gamma)/(\alpha+\gamma)$. By the (*ABC*) Conjecture there exists $K > 0$ such that

$$\frac{|f(x)|}{d^{1+\varepsilon}} \leq \frac{|f(x)|}{d} < Kr^{1+\varepsilon}$$

where

$$\begin{aligned} r &= \operatorname{rad}\left(\frac{f(x)}{d} \cdot \frac{ax^m}{d} \cdot \frac{g(x)}{d}\right) \\ &\leq \operatorname{rad}(f(x)) \cdot A|x| \cdot \frac{|g(x)|}{d} \leq |f(x)|^{1/\gamma} A|x| \cdot |g(x)|. \end{aligned}$$

We have

$$|g(x)| \leq A(|x|^k + \ldots + |x| + 1) = A\frac{|x|^{k+1} - 1}{|x| - 1} < |x|^{k+1}.$$

Next

$$\begin{aligned} |f(x)| &= |ax^m + g(x)| \geq a|x|^m - |g(x)| \\ &> (A+1)|x|^{m-1} - |x|^{k+1} \geq A|x|^{m-1} \geq |x|^{m-1}, \end{aligned}$$

since $m-1 \geq k+1$. Hence $|x| \leq |f(x)|^{1/(m-1)}$ and $|g(x)| \leq |f(x)|^{(k+1)/(m-1)}$.

Therefore

$$|f(x)| < K'|f(x)|^\beta$$

where

$$K' = K \cdot A^{1+\varepsilon}$$

and

$$\begin{aligned} \beta &= \left(\frac{1}{\gamma} + \frac{1}{m-1} + \frac{k+1}{m-1}\right)(1+\varepsilon) \\ &= \left(\frac{1}{\gamma} + \frac{k+2}{m-1}\right)(1+\varepsilon) \leq \left(\frac{1}{\gamma} + \frac{1}{\alpha}\right)(1+\varepsilon) < 1. \end{aligned}$$

The upper bound for β is independent of x, f. Thus $|f(x)|$ remains bounded, showing that the set V is finite. ∎

We illustrate with some corollaries.

(3.7) COROLLARY. *The set* $\{x^m \pm x^k \pm 1 \in N_{3/2} \mid x \geq 1, k \geq 1, 4m \geq 13k + 30\}$ *is finite.*

Proof. Let $A = 1$, $\alpha = 13/4$. Then $m - 1 \geq \alpha(k+2)$, so $f(X) = X^m \pm X^k \pm 1 \in F_{A,\alpha}$. Since $\gamma = 3/2 > \alpha/(\alpha - 1)$, the set $\{x^m \pm x^k \pm 1 \in N_{3/2} \mid x \geq 1,\ k \geq 1,\ 4m \geq 13k + 30\}$ is finite. ∎

In particular $\{x^m \pm x \pm 1 \in N_{3/2} \mid x \geq 1,\ m \geq 11\}$ is finite. Also $\{x^m \pm x^2 \pm 1 \in N_{3/2} \mid x \geq 1,\ m \geq 14\}$ is finite.

The following special case is already in [4]:

(3.8) (1) *The set* $\{x^m \pm x \pm 1$ *powerful* $\mid x \geq 1,\ 4m \geq 9k + 22\}$ *is finite.*
(2) *The set* $\{x^m \pm x^k \pm 1$ *3-powerful* $\mid x \geq 1,\ 8m \geq 13k + 34\}$ *is finite.*

Proof. (1) We take $\alpha = 9/4$, so $\gamma = 2 > \alpha/(\alpha - 1)$. We have $m - 1 \geq (9/4)(k+2)$; by the theorem, there are only finitely many powerful numbers $x^m \pm x^k \pm 1$, with $4m \geq 9k + 22$.

(2) We take $\alpha = 13/8$ so $\gamma = 3 > \alpha/(\alpha - 1)$. Then $m - 1 \geq (13/8)(k+2)$. By the theorem there exist only finitely many 3-powerful numbers $x^m \pm x^k \pm 1$ with $8m \geq 13k + 34$. ∎

In particular, $\{x^m \pm x \pm 1$ powerful $\mid m \geq 8\}$ is finite, and $\{x^m \pm x \pm 1$ which are 3-powerful $\mid m \geq 6\}$ is finite. The set $\{x^m \pm x^2 \pm 1$ powerful $\mid m \geq 10\}$ is finite. The set $\{x^m \pm x^2 \pm 1$ which are 3-powerful $\mid m \geq 8\}$ is finite.

4. Consequences for binary recurrences. In this section we give some consequences of the (ABC) Conjecture for binary recurrences.

Let P, Q be non-zero integers with $P > 0$ and assume that $D = P^2 - 4Q \neq 0$. Let $U_n = U_n(P,Q)$ and $V_n = V_n(P,Q)$ be defined as follows:

$$U_0 = 0, \quad U_1 = 1, \quad U_n = PU_{n-1} - QU_{n-2} \quad \text{for } n \geq 2,$$

and

$$V_0 = 2, \quad V_1 = P, \quad V_n = PV_{n-1} - QV_{n-2} \quad \text{for } n \geq 2.$$

The roots of $f(X) = X^2 - PX + Q$ are $\alpha = (P + \sqrt{D})/2$, $\beta = (P - \sqrt{D})/2$, so $\alpha + \beta = P$, $\alpha\beta = Q$, $\alpha - \beta = \sqrt{D}$.

We note the following well-known facts:

(4.1) $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$, $V_n = \alpha^n + \beta^n$ *and* $V_n^2 - DU_n^2 = 4Q^n$ *for each* $n \geq 0$.

In the special case when $P = Q + 1$ (so $Q \neq \pm 1$, since $P, Q, D \neq 0$) we have $D = (Q-1)^2$, $\alpha = Q$, $\beta = 1$, $U_n = (Q^n - 1)/(Q - 1)$, $V_n = Q^n + 1$.

We shall henceforth assume that $\gcd(P,Q) = 1$ and $D > 0$. Then α, β are real numbers, $\alpha > \beta$, $\alpha > 0$ (since $P > 0$). Moreover $\beta > 0$ if and only if $Q > 0$.

(4.2) LEMMA. (1) *If* $Q > 0$ *then*

$$D < V_n^2/U_n^2 \le P^2.$$

(2) *If* $Q < 0$ *then*

$$P^2 \le V_n^2/U_n^2 \le D^2/P^2.$$

Proof. (1) Let $Q > 0$ so $0 < \beta < \alpha$ and

$$\frac{V_n}{U_n} = (\alpha - \beta)\frac{\alpha^n + \beta^n}{\alpha^n - \beta^n} = (\alpha - \beta)\frac{(\alpha/\beta)^n + 1}{(\alpha/\beta)^n - 1}.$$

Let $F(t) = (\xi^t + 1)/(\xi^t - 1)$ where $t > 0$, $\xi = \alpha/\beta > 1$. Then $F'(t) < 0$ so

$$\frac{V_n}{U_n} \le (\alpha - \beta)\frac{\alpha/\beta + 1}{\alpha/\beta - 1} = \alpha + \beta = P$$

and

$$\frac{V_n}{U_n} > (\alpha - \beta) \cdot \lim_{k\to 0} \frac{1 + (\beta/\alpha)^k}{1 - (\beta/\alpha)^k} = \alpha - \beta = \sqrt{D}.$$

Thus

$$D < V_n^2/U_n^2 \le P^2.$$

(2) Let $Q < 0$ so $\beta < 0$; from $0 < P = \alpha + \beta$ it follows that $|\beta| < \alpha$. Then

$$\frac{V_n}{U_n} = (\alpha + |\beta|)\frac{(\alpha/|\beta|)^n + (-1)^n}{(\alpha/|\beta|)^n - (-1)^n}.$$

If n is even, we obtain as before (with $\xi = \alpha/|\beta| > 1$)

$$\frac{V_n}{U_n} \le (\alpha + |\beta|)\frac{\alpha/|\beta| + 1}{\alpha/|\beta| - 1} = \frac{D}{P}$$

and

$$\frac{V_n}{U_n} > (\alpha + |\beta|) \cdot \lim_{k\to\infty} \frac{1 + (|\beta|/\alpha)^k}{1 - (|\beta|/\alpha)^k} = \alpha - \beta = \sqrt{D},$$

so

$$D < V_n^2/U_n^2 \le D^2/P^2.$$

If n is odd, we obtain as before

$$\frac{V_n}{U_n} \ge (\alpha + |\beta|)\frac{\alpha/|\beta| - 1}{\alpha/|\beta| + 1} = \alpha - |\beta| = P,$$

while

$$\frac{V_n}{U_n} < (\alpha + |\beta|) \cdot \lim_{k\to\infty} \frac{1 - (|\beta|/\alpha)^k}{1 + (|\beta|/\alpha)^k} = \alpha - \beta = \sqrt{D},$$

so

$$P^2 \le V_n^2/U_n^2 < D^2/P^2.$$

Hence for all $n \ge 1$,

$$P^2 \le V_n^2/U_n^2 < D^2/P^2. \blacksquare$$

(4.3) THEOREM. *Let P, Q be as before, let $\varepsilon > 0$ and $\alpha > 1$. Assuming that the* (*ABC*) *Conjecture is true, we have*:

(1) *The set $G = \{U_n \mid n \geq 1$ and there exists $u \in N_\alpha$ such that $u \mid U_n$ and $u > U_n^\varepsilon\}$ is finite.*

(2) *The set $H = \{V_n \mid n \geq 1$ and there exists $v \in N_\alpha$ such that $v \mid V_n$ and $v > V_n^\varepsilon\}$ is finite.*

Proof. To begin we recall that $d_n = \gcd(U_n, V_n) = 1$ or 2 (for $n \geq 1$). Let $D_i = \{n \geq 1 \mid d_n = i\}$ for $i = 1, 2$.

(1) We consider two cases:

FIRST CASE: $n \in D_1$. If P is even, let $Z_n = V_n/2$, $\Delta = D/4$, $E = 1$. If P is odd, let $Z_n = V_n$, $\Delta = D$, $E = 4$. Then in both cases

$$Z_n^2 - \Delta U_n^2 = EQ^n.$$

We shall apply Theorem (2.15) with $R = 1$, $S = \mathrm{rad}(\Delta)$, $T = \mathrm{rad}(EQ)$. In both cases R, S, T are positive coprime square-free integers.

Let $G^{(1)} = \{U_n \in G \mid d_n = 1\}$. If $U_n \in G^{(1)}$ then by Lemma (4.2),

$$\frac{V_n^2}{U_n^2} \leq \begin{cases} P^2 & \text{if } Q > 0, \\ D^2/P^2 & \text{if } Q < 0, \end{cases}$$

so there exists $M_1 \geq 1$ such that $Z_n^2/U_n^2 \leq M_1$. Then $G^{(1)}$ is contained in the set $Y_1 = \{y > 0 \mid$ there exists $x > 0$ with $\gcd(x, y) = 1$, $x^2/y^2 \leq M_1$ and there exist non-zero integers A, B, C with $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax^2 + By^2 = C\}$. Indeed, for each $y = U_n$ we take $x = Z_n$, $A = 1$, $B = \Delta$, $C = EQ^n$. By (2.15), Y_1 is a finite set, so $G^{(1)}$ is also a finite set.

SECOND CASE: $n \in D_2$. Let $V_n' = V_n/2$, $U_n' = U_n/2$, so $V_n'^2 - DU_n'^2 = Q^n$.

Let $R = 1$, $S = \mathrm{rad}(D)$, $T = \mathrm{rad}(Q)$ so R, S, T are positive coprime square-free integers. Let $G^{(2)} = \{U_n \in G \mid d_n = 2\}$. If $U_n \in G^{(2)}$ then by Lemma (4.2),

$$\frac{V_n'^2}{U_n'^2} = \frac{V_n^2}{U_n^2} \leq \begin{cases} P^2 & \text{if } Q > 0, \\ D^2/P^2 & \text{if } Q < 0, \end{cases}$$

so there exists $M_2 \geq 1$ such that $V_n'^2/U_n'^2 \leq M_2$. Therefore the set $\{U_n/2 \mid U_n \in G^{(2)}\}$ is contained in $Y_2 = \{y > 0 \mid$ there exists $x > 0$ with $\gcd(x, y) = 1$, $x^2/y^2 \leq M_2$ and there exist non-zero integers A, B, C such that $\mathrm{rad}(A) \mid R$, $\mathrm{rad}(B) \mid S$, $\mathrm{rad}(C) \mid T$ and $Ax^2 + By^2 = C\}$. Indeed, for $y = U_n/2$ we take $x = V_n/2$, $A = 1$, $B = D$, $C = Q^n$. By Theorem (2.15) the set Y_2 is finite, so $G^{(2)}$ is also finite.

(2) The proof is similar. We require that by Lemma (4.2),

$$\frac{U_n^2}{V_n^2} \leq \begin{cases} 1/D & \text{if } Q > 0, \\ 1/P^2 & \text{if } Q < 0, \end{cases}$$

so in both cases $U_n^2/V_n^2 \leq 1$. ■

(4.4) COROLLARY. *Let P, Q be as above.*

(1) *For each $\alpha > 1$ there are only finitely many terms $U_n \in N_\alpha$ and $V_n \in N_\alpha$.*

(2) *There are only finitely many terms U_n, V_n which are powerful.*

The above statement (2) is already in the paper [6] by Ribenboim and Walsh.

The next result concerns families of binary recurrences.

(4.5) THEOREM. *Let R be a positive square-free integer, let $\alpha > 1$. If the (ABC) Conjecture is true, the set $S_\alpha = \{(x,m) \mid x \geq 2,\ m \geq 3 + 1/(\alpha-1),\ (x^m-1)/(x-1) = az,$ where $\mathrm{rad}(a) \mid R,\ z \in N_\alpha\}$ is finite.*

Proof. Let $0 < \varepsilon < (\alpha-1)^2/(2\alpha^2-1)$. If $(x,m) \in S_\alpha$ we write

$$x^m = \frac{x^m-1}{x-1}(x-1)+1.$$

By the (ABC) Conjecture, there exists $K > 0$ such that $x^m < Kr^{1+\varepsilon}$ where

$$r = \mathrm{rad}\left(x^m \frac{x^m-1}{x-1}(x-1)\right) = \mathrm{rad}(x^m \cdot az \cdot (x-1)) \leq x^2 R z^{1/\alpha}.$$

Hence

$$x^{m-2(1+\varepsilon)} < KR^{1+\varepsilon} z^{(1+\varepsilon)/\alpha} \leq KR^{1+\varepsilon}(az)^{(1+\varepsilon)/\alpha}.$$

From $2x^{m-1} > (x^m-1)/(x-1) \leq az$ it follows that

$$\frac{(az)^{(m-2(1+\varepsilon))/(m-1)}}{2} < \left(\frac{az}{2}\right)^{(m-2(1+\varepsilon))/(m+1)} < KR^{1+\varepsilon}(az)^{(1+\varepsilon)/\alpha}.$$

So

$$(az)^{(m-2(1+\varepsilon))/(m-1)} < K'(az)^{(1+\varepsilon)/\alpha} \quad \text{where } K' = 2KR^{1+\varepsilon}.$$

We show that $[m-2(1+\varepsilon)]\alpha > (m-1)(1+\varepsilon)$. It suffices to show that

$$(3\alpha-2)(\alpha-1-\varepsilon) > (2\alpha-1)(\alpha-1)(1+\varepsilon)$$

or equivalently

$$(3\alpha-2)\alpha > (2\alpha^2-1)(1+\varepsilon).$$

But this is true, since

$$\frac{3\alpha^2-2\alpha}{2\alpha^2-1} - 1 > \frac{(\alpha-1)^2}{2\alpha^2-1} > \varepsilon.$$

This shows that $az = (x^m-1)/(x-1)$ remains bounded, showing that the set S_α is finite. ∎

In particular, $S_2 = \{(x,m) \mid x \geq 2,\ m \geq 4,\ (x^m-1)/(x-1) = az$, with $\mathrm{rad}(a) \mid R,\ z \in N_2\}$ is a finite set.

As a corollary, we have:

(4.6) Corollary. *Let R, α be as above.*

(1) *For each $x \geq 2$ the set $\{m \geq 2 \mid (x^m - 1)/(x - 1) = az$ with $\operatorname{rad}(a) \mid R$ and $z \in N_\alpha\}$ is finite.*

(2) *For each $m \geq 4$ the set $\{x \geq 2 \mid (x^m - 1)/(x - 1) = az$ with $\operatorname{rad}(a) \mid R$ and $z \in N_\alpha\}$ is finite.*

The following very special case is found in Shorey's paper [8]:

(4.7) Corollary. *The set $S = \{(x, m) \mid x \geq 2, m \geq 3, (x^m - 1)/(x - 1)$ is a power$\}$ is finite.*

Proof. By the preceding result, the set $\{(x, m) \in S \mid m \geq 4\}$ is finite. We show that the set of integers x such that $(x^3 - 1)/(x - 1) = x^2 + x + 1 = a^k$, where $a \geq 2$, $k \geq 2$, is also finite.

Since the roots of $X^2 + X + 1 = 0$ are simple, by the theorem of Schinzel and Tijdeman [7], there are only finitely many integers $x^2 + x + 1$ of the form a^k with $a \geq 2$ and $k \geq 3$. Finally, if $x^2 + x + 1 = a^2$ with $a \geq 2$, then $x = (-1 \pm \sqrt{1 - 4(1 - a^2)})/2$. Since x is an integer we then have $4a^2 - 3 = b^2$. So $a = 1$, $x = -1$, which has been excluded. This concludes the proof. ∎

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Department of Mathematics and Statistics
Queen's University
Kingston, ON, K7L 3N6 Canada
E-mail: mathstat@mast.queensu.ca

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