On the irreducibility of certain polynomials with coefficients as products of terms in an arithmetic progression

by

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1. Introduction. In 1929, Schur [10] used prime ideals in algebraic number fields to prove that the Taylor polynomials for the exponential function, with some possible variations in the coefficients, are irreducible.

THEOREM 1. *Let m be a positive integer and let $a_0, \ldots, a_m$ be arbitrary integers with $|a_0| = |a_m| = 1$. Then*

$$(1.1) \qquad a_m \frac{x^m}{m!} + a_{m-1} \frac{x^{m-1}}{(m-1)!} + \cdots + a_1 x + a_0$$

is irreducible over the rationals.

Filaseta [7] used Newton polygons to obtain Schur's result, and also strengthened the result by allowing more possible values for the leading coefficient than just those of absolute value 1.

THEOREM 2. *Let m be a positive integer and let $a_0, \ldots, a_m$ be arbitrary integers with $|a_0| = 1$ and $0 < |a_m| < m$. Then the polynomial given in (1.1) is irreducible over the rationals except when*

$$a_m = \pm 5 \text{ and } m = 6 \quad \text{or} \quad a_m = \pm 7 \text{ and } m = 10.$$

Filaseta's use of Newton polygons to demonstrate the irreducibility of the polynomials in (1.1) is based on the theorem of Dumas [3] regarding the construction of the Newton polygon of the product of two polynomials. In particular, Filaseta employs the following useful lemma, from [6].

LEMMA 1. *Let k and l be integers with $k > l \geq 0$. Suppose $g(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$, $p \mid b_j$ for all $j \in \{0, 1, \ldots, n - l - 1\}$, and the rightmost edge of the Newton polygon for $g(x)$*

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with respect to p has slope $< 1/k$. Then for any integers $a_0, a_1, \ldots, a_n$ with $|a_0| = |a_n| = 1$, the polynomial $f(x) = \sum_{j=0}^{n} a_j b_j x^j$ cannot have a factor with degree in the interval $[l+1, k]$.

From this lemma, we see that using Newton polygons to eliminate the possibility of factors of a particular degree hinges on finding primes that divide certain coefficients of the polynomial. To obtain Theorem 2, Filaseta appeals to a result of Ecklund, Eggleton, Erdős and Selfridge [5] on prime divisors of binomial coefficients. We refer to [7] for details. Moreover, using the same tools, Allen and Filaseta [1], [2] proved the following result.

THEOREM 3. *Let $m > 1$ and $a_0, \ldots, a_m$ denote arbitrary integers with $|a_0| = 1$.*

(i) *Suppose $m + 1 = k' 2^u$ with k' odd and $(m+1)m = k'' 2^v 3^w$ with $\gcd(k'', 6) = 1$. Let $0 < |a_m| < \min\{k', k''\}$. Then*

$$(1.2) \qquad a_m \frac{x^m}{(m+1)!} + a_{m-1} \frac{x^{m-1}}{m!} + \cdots + a_2 \frac{x^2}{2!} + a_1 x + a_0$$

is irreducible over the rationals.

(ii) *Suppose $0 < |a_m| < 2m - 1$. Then*

$$(1.3) \qquad a_m \frac{x^{2m}}{1 \cdot 3 \cdots (2m-1)} + a_{m-1} \frac{x^{2m-2}}{1 \cdot 3 \cdots (2m-3)} + \cdots$$
$$+ a_2 \frac{x^4}{1 \cdot 3} + a_1 \frac{x^2}{1} + a_0$$

is irreducible over the rationals.

We observe that the common thread among the polynomials in (1.1), (1.2), and (1.3) is that the denominators of the coefficients are products of integers in an arithmetic progression; in the case of (1.1) and (1.2), we see an arithmetic progression with initial term 1 and common difference 1, and in (1.3) we see an arithmetic progression with initial term 1 and common difference 2. In this paper, we prove analogous results by considering denominators which are again products of integers in an arithmetic progression with initial term an odd integer a and common difference 2. Let

$$(1.4) \qquad f(x) = \frac{x^m}{a(a+2) \cdots (a + 2(m-1))} + \cdots + \frac{x^2}{a(a+2)} + \frac{x}{a} + 1,$$

$$(1.5) \qquad g(x) = a_m \frac{x^m}{a(a+2) \cdots (a + 2(m-1))}$$
$$+ a_{m-1} \frac{x^{m-1}}{a(a+2) \cdots (a + 2(m-2))} + \cdots + a_2 \frac{x^2}{a(a+2)} + a_1 \frac{x}{a} + a_0.$$

Letting $P(n)$ denote the greatest prime factor of the positive integer n (putting $P(1) = 1$), we prove the following results.

THEOREM 4. *Let $a \geq 1$ be an odd integer, $\max\{a, 110\} \leq k \leq m/2$ and $a_0, \ldots, a_m$ be arbitrary integers with $P(a_0 a_m) \leq 2k + a$. Then $f(x)$ and $g(x)$ do not have a factor of degree k.*

Now we restrict to $1 \leq a < 29$. We have

THEOREM 5. *Let a be an odd integer with $1 \leq a < 29$, and let $m > 1$ and $a_0, \ldots, a_m$ be arbitrary integers with $P(a_0 a_m) \leq a + 4$. Then*

  (i) *$f(x)$ has no factor of degree ≥ 2,*
  (ii) *$g(x)$ has no factor of degree ≥ 3, and*
  (iii) *$g(x)$ has no factor of degree 2 except perhaps when*

$$(a, m) \in \{(21, 4), (19, 59), (5, 121), (19, 114), (21, 113), (21, 163), (21, 554)\}.$$

Further if there exists a prime $p \geq a + 2$ dividing $a + 2(m - 1)$, then $f(x)$ has no linear factor. Also if such a prime p does not divide $a_0 a_m$, then $g(x)$ has no linear factor.

When $(a, m) = (21, 4)$, by choosing $a_0 = a_4 = 1$, $a_1 = -15$, $a_2 = -140$, $a_3 = 35$, we see that

$$g(x) = \frac{x^4}{21 \cdot 23 \cdot 25 \cdot 27} + \frac{35x^3}{21 \cdot 23 \cdot 25} - \frac{140x^2}{21 \cdot 23} - \frac{15x}{21} + 1$$
$$= \frac{1}{326025}\,(x^2 - 90x - 315)(x^2 + 1035x - 1035).$$

We thank the referee for providing this example. As in the proofs of Theorems 2 and 3, our method also depends on the use of Dumas' theorem on Newton polygons for the irreducibility of polynomials. On the other hand, we do not use results from Ecklund, Eggleton, Erdős, and Selfridge [5]. Instead, we establish a Sylvester type result on the greatest prime factor of a product of several consecutive terms in an arithmetic progression. This result is of interest independent of its application to establish Theorems 4 and 5. We show for instance that for any $k \geq 2$,

$$P(n(n + 2) \cdots (n + 2(k - 1))) > 2k + a$$

if $n \geq 2k + a$, where n is odd and a is a positive odd integer less than 29, except for an explicitly given set of values of (n, k, a). This result depends on a result of Lehmer [8] and several computations. The above assertion is also true for any odd a provided k is large; see Lemma 5. As an application of Theorem 5 we give another criterion for the irreducibility of (1.3).

COROLLARY 6. *Let $P(a_0 a_m) \leq 5$. Suppose there exists a prime p such that*

$$p \,|\, (2m - 1) \quad and \quad p \nmid a_0 a_m.$$

Then the polynomial given in (1.3) *is irreducible over the rationals.*

Let a be an even integer equal to $2b$, say. Then $f(x)$ and $g(x)$ can be transformed into a polynomial of the form

$$a'_m \frac{x^m}{b(b+1)\cdots(b+m-1)} + \cdots + a'_1 \frac{x}{b} + a'_0$$

with $a'_m = 1, a'_0, \ldots, a'_{m-1}$ integers. The case $b = 1$ and $|a'_0| = 1$ is Schur's polynomial given in (1.1). For some results on the factors of such polynomials, we refer to [11]. We will not deal with this case in the present paper.

The remainder of this paper is organized as follows. In Section 2, we prove the result about the greatest prime factor of a product of consecutive terms in arithmetic progression. In Section 3, we use Newton polygons to exclude some factors of the polynomials in question. In particular cases, all factors of degree ≥ 2 are excluded. Theorem 4 comes out as a consequence of Lemmas 5 and 11. In Section 4 we discuss linear factors and prove Theorem 5 from Theorems 8 and 9.

2. Greatest prime factor of a product of integers in arithmetic progression. The letters n, d, k denote positive integers with $\gcd(n, d) = 1$. Set $\Delta = n(n + d) \cdots (n + (k - 1)d)$. Let $\pi(n)$ denote the number of primes $\leq n$ and $\pi_d(n)$ be the number of primes $\leq n$ that are co-prime to d. Let $\nu_p(n)$ denote the power of the prime p in n and p_i denote the ith prime. In this section, we obtain lower bounds on $P(\Delta)$, and conclude the section with Theorem 8, a particular result for $P(\Delta)$ when $d = 2$.

We state without proof our first lemma. See [9] for details.

LEMMA 2. *For* $0 \leq i < k$, *suppose* $P(n + id) \leq c_0 k$. *Let* $S = \{n, n + d, \ldots, n + (k - 1)d\}$. *For every prime* $p \leq c_0 k$ *with* $p \nmid d$, *choose* $n + i_p d \in S$ *such that* p *does not appear to a higher power in the factorization of any other element of* S. *Let* S_1 *be the subset of* S *obtained by deleting from* S *all* $n + i_p d$ *with* $p \leq c_0 k$ *and* $p \nmid d$. *Then*

$$\prod_{n+id \in S_1} (n + id) \leq (k - 1)! \prod_{p|d} p^{-\nu_p((k-1)!)}.$$

When $d = 1$, the product on the right hand side is taken as 1. In the next lemma, inequality (i) is an easy consequence of the formula of Legendre on the $\nu_p((k - 1)!)$. The estimate for $\pi(x)$ in (ii) is due to Dusart [4].

LEMMA 3.

(i) $\nu_p((k-1)!) \geq \dfrac{k - p}{p - 1} - \dfrac{\log(k - 1)}{\log p}$.

(ii) $\pi(x) \leq \dfrac{x}{\log x}\left(1 + \dfrac{1.2762}{\log x}\right)$ *for* $x > 1$.

LEMMA 4. *Let $k \geq 2$, $c_0 > 1$, $c_1 > 0$, $d \geq 1$ and $k - \pi_d(c_0 k) \geq 1$. Suppose $n \geq c_1 k d$ and $P(\Delta) \leq c_0 k$. Then*

$$(2.1) \qquad (c_1 d)^{k - \pi_d(c_0 k)} \leq k^{\pi_d(c_0 k)} \prod_{p | d} p^{-\nu_p((k-1)!)}.$$

Proof. Observe that Δ is not divisible by primes dividing d and that every prime $> k$ may divide only one term of Δ. Hence there are at least $k - \pi_d(c_0 k) + \pi_d(k)$ terms which are divisible only by primes $\leq k$. By deleting a term in which a prime $p \leq k$, $p \nmid d$ appears to the maximum power, using the notation from Lemma 2, we see that $|S_1| \geq k - \pi_d(c_0 k) \geq 1$. We set $t := |S_1| - 1$. We arrange the elements of S_1 as

$$n + i_0 d < n + i_1 d < \cdots < n + i_t d.$$

Then by Lemma 2,

$$\prod_{v=0}^{t} (n + i_v d) \leq (k - 1)! \prod_{p | d} p^{-\nu_p((k-1)!)}.$$

This gives

$$n^{k - \pi_d(c_0 k)} \leq n(n + d) \cdots (n + (k - \pi_d(c_0 k) - 1)d) \leq (k - 1)! \prod_{p | d} p^{-\nu_p((k-1)!)}.$$

Since $n \geq c_1 k d$, we get

$$(c_1 d)^{k - \pi_d(c_0 k)} k^{k - \pi_d(c_0 k)} \leq k^k \prod_{p | d} p^{-\nu_p((k-1)!)},$$

which gives the assertion of the lemma. ∎

Putting together the inequalities from Lemma 3 with the result in Lemma 4 and observing that $\pi_d(c_0 k) \leq \pi(c_0 k)$, we obtain the following result.

COROLLARY 7. *Let $k \geq 2$, $c_0 > 1$, $c_1 > 0$, $d = p$ prime and $k - \pi_p(c_0 k) \geq 1$. Suppose that $n \geq c_1 k p$ and $P(\Delta) \leq c_0 k$. Let*

$$f(k, p) = \begin{cases} 0 & \text{if } p \geq k, \\ \dfrac{1}{p - 1} - \dfrac{p}{k(p - 1)} - \dfrac{\log(k - 1)}{k \log p} & \text{otherwise.} \end{cases}$$

Then

$$p \leq \exp \left[\frac{c_0 + \dfrac{1.2762 c_0}{\log c_0 k} - \left(1 - \dfrac{c_0}{\log c_0 k} - \dfrac{1.2762 c_0}{(\log c_0 k)^2}\right) \log c_1}{1 - \dfrac{c_0}{\log c_0 k} - \dfrac{1.2762 c_0}{(\log c_0 k)^2} + f(k, p)} \right].$$

For the rest of this section, we restrict our attention to arithmetic progressions with common difference $d = 2$. We denote

$$\Delta_2 = n(n+2)\cdots(n+2(k-1)) \quad \text{with } n \text{ odd.}$$

Before we state the next lemma, we note that Allen and Filaseta [1] showed that for every $n \geq 213$, there exists a prime $p \in (n, 1.05n]$. We will use this result in the next lemma.

LEMMA 5. *Let $a \geq 1$ be an odd integer, $k \geq \max\{a, 110\}$ and $n \geq 2k+a$. Then $P(\Delta_2) > 2k + a$.*

Proof. Suppose $P(\Delta_2) \leq 2k + a$. First assume that $n \geq 40(k-1)$. Note that $2 + a/k \leq 3$ since $a \leq k$. We apply Corollary 7 with $c_0 = 3$ and $c_1 = 19.5$. Note that $\pi(c_0 k) < k$ since $k \geq 110$. We find that the right hand side of the inequality in Corollary 7 is a decreasing function of k since each term involving k is a decreasing function of k. Hence if the inequality is not valid for some $k = k_0$, then it is not valid for any $k > k_0$. We check that the inequality is not valid for $k_0 = 110$. This proves the assertion of the lemma for $n \geq 40(k-1)$.

Next we assume that $n < 40(k-1)$. Note that $n \geq 213$. Then there exists a prime p in $\{n+2, \ldots, n+2(k-1)\}$ since the interval $(n, 1.05n]$ is contained in $(n, n+2(k-1)]$ as $n < 40(k-1)$. Further this prime exceeds $n \geq 2k + a$, by assumption. Thus $P(\Delta_2) > 2k + a$. ∎

Now we restrict to odd $a < 29$.

LEMMA 6. *Let $1 \leq a < 29$, a odd, $k \geq 31$ and $n \geq 2k + a$. Then $P(\Delta_2) > 2k + 29$.*

Proof. Suppose $P(\Delta_2) \leq 2k + 29$. We follow the argument as in Lemma 5. First let $n \geq 40(k-1)$. We apply Corollary 7 with $c_0 = 2 + 29/k$ and $c_1 = 19.5$. We check that the inequality in Corollary 7 is not valid for $k_0 = 100$. Thus we may assume that $k \leq 99$. Now we check that the inequality (2.1) with actual values of the π-function is invalid for all $31 \leq k \leq 99$.

Next we assume that $213 \leq n < 40(k-1)$. Since now $(n, 1.05n] \subset (n, n+2(k-1)]$, there exists a prime $\geq 2k + a$ dividing Δ_2. Hence we may assume that $n < 213$. Then we need only consider $2k + a \leq n < 213$ with n odd. For these finitely many values of n and k, we check directly that the assertion of the lemma is true. ∎

Let T be the set of all integers $M \geq 1$ with $P(M(M+2)) \leq 31$. Table 1 below shows 101 such integers put in groups according to the largest prime factor of $M(M+2)$. It follows from Lehmer's work [8] that if $n > 1$ is an

integer, then $P(n(n+2)) \geq 37$ except when $n = M$ with M given by Table 1. Thus Table 1 gives all the integers of T.

Table 1

p	Integers M with $P(M(M+2)) = p$
3	1
5	3, 25
7	5, 7, 243
11	9, 33, 75
13	11, 13, 63, 273, 845, 1573
17	15, 49, 117, 119, 187, 1375
19	17, 19, 55, 133, 169, 245, 323, 361, 625, 663, 1615, 3211, 3969
23	21, 23, 115, 207, 253, 297, 343, 1125, 1309, 2185, 2275, 2873,
	3703, 6875, 8073, 9315, 18513, 41743, 57475, 1128125, 1447873
29	27, 85, 143, 145, 375, 435, 493, 665, 2871, 8379, 9945, 12673,
	14875, 16443, 24563, 41325, 45617, 87723, 184875
31	29, 31, 91, 93, 153, 341, 403, 525, 527, 713, 897, 1083, 1519,
	1953, 2695, 3625, 4123, 5423, 7161, 19435, 22475, 86273,
	130975, 203203, 2509045, 3322053, 287080365

The next three lemmas deal with the complementary case of Lemma 6 when $k \leq 30$.

LEMMA 7. *Let* $2 \leq k \leq 30$. *Suppose no* $M \in T$ *is of the form* $n + 4j$ *for any* j *with* $0 \leq j \leq (k-2)/2$. *Then* $P(\Delta_2) \geq 2k + 29$.

Proof. We divide the integers $n, n+2, \ldots, n+2(k-1)$ into pairs

(2.2) $$(n, n+2), (n+4, n+6), \ldots.$$

Note that there are at least $[k/2]$ pairs. By hypothesis, none of these pairs coincides with $(M, M+2)$ for any $M \in T$. Then the product of integers in each pair in (2.2) has a prime factor ≥ 37. Since these integers are in a block of length at most 30, we see that each pair in (2.2) must have a distinct prime ≥ 37 dividing their product. Thus Δ_2 is divisible by at least $[k/2]$ primes ≥ 37. Hence

$$P(\Delta_2) \geq p_{[k/2]+11}.$$

We check that $p_{[k/2]+11} \geq 2k + 29$ for $2 \leq k \leq 30$, which completes the proof of the lemma. ∎

LEMMA 8. *Let* $3 \leq k \leq 30$ *and* $n \geq 2k + 29$. *Then* $P(\Delta_2) \geq 2k + 29$ *except when* (n, k) *is one of the following ten pairs:*

$$(91, 3), (115, 3), (115, 4), (117, 3), (143, 3), (243, 3), (341, 3),$$
$$(525, 3), (663, 3), (2871, 3).$$

Proof. By Lemma 7, we need only consider n such that

$$M = n + 4j \quad \text{for some } M \in T \text{ and some } j \text{ with } 0 \leq j \leq (k-2)/2.$$

Then we find that

(2.3) $$2k + 29 \leq n \leq M \leq n + 2(k-2).$$

Let p be the largest prime $< M$ and q the smallest prime $\geq \max\{2k+29, M\}$. If $p \geq n$, then (2.3) implies that p divides Δ_2. If $p < n$ and $q \leq p + 2k$, then q divides Δ_2. Thus for any $k \geq (q-p)/2$, the product Δ_2 is divisible by either p or q. So the assertion of the lemma is true provided $p \geq 2k + 29$. Thus we may assume that either

$$k < \frac{q-p}{2} \quad \text{or} \quad k > \frac{p-29}{2}.$$

Combining with (2.3), we have

(2.4)
$$\max\{2k + 29, M - 2(k-2)\} \leq n \leq M,$$
$$k < \min\left\{31, \frac{q-p}{2}\right\} \quad \text{or} \quad \frac{p-29}{2} < k \leq 30.$$

Thus for each $M \in T$, we check for the finitely many values of (n,k) in (2.4) whether $P(\Delta) \geq 2k + 29$. We illustrate the above procedure with an example. Let $M = 243$. Then $p = 241$ and $q = 251$. Hence

$$k < 5 \quad \text{and} \quad 239 \leq n \leq 243, \quad n \text{ odd}.$$

In these cases we check directly that $P(\Delta) \geq 2k + 29$, the only exception being $(n,k) = (243, 3)$. By the above procedure we find only the 10 exceptions listed in the statement of the lemma. ∎

Finally we show

LEMMA 9. *Let* $3 \leq k \leq 30$ *and* $n \geq 2k + a$ *with* $1 \leq a < 29$, *a odd. Assume that* Δ_2 *is not equal to any of the ten products in Lemma* 8. *Then*

$$P(\Delta_2) > 2k + a$$

except when $(n, k, a) \in \{(23, 3, 17), (31, 3, 25)\}$.

Proof. By Lemmas 7 and 8, we need to check the assertion only when

$$2k + a \leq n < 2k + 29, \quad 1 \leq a < 29, \quad a \text{ odd}, \quad 3 \leq k \leq 30,$$

which is done by direct computation. ∎

For each odd $a \in [1, 27]$, let $T(a)$ be the set of $M \in T$ for which $P(M(M + 2)) \leq a + 4 \leq M$. For example, when $a = 1$, $T(a) = \{25\}$; when $a = 3$, $T(a) = \{7, 25, 243\}$. Let k be given and $n \geq 2k + a$ with

$1 \leq a < 29$, a odd. We denote by a^* the smallest a such that

$$P(\Delta_2) \leq 2k + a.$$

With the above notation, we combine Lemmas 6 and 9 to obtain the following theorem.

THEOREM 8. *Let* $k \geq 2$ *and* $n \geq 2k + a$ *with* $1 \leq a < 29$, *a odd. Then*

(2.5) $$P(\Delta_2) > 2k + a$$

except for the following values of n, k *and* a:

$$k = 2, 1 \leq a < 29 \text{ with } n \in T(a),$$
$$k = 3, (n, a) = (23, 17), (31, 25),$$
(2.6) $$k = 3, (n, a^*) = (91, 25), (115, 17), (117, 11), (143, 23), (243, 13),$$
$$(341, 25), (525, 25), (663, 23), (2871, 23),$$
$$k = 4, (n, a^*) = (115, 15).$$

Proof. Let $k \geq 3$. Suppose $(n, k, a) \in \{(23, 3, 17), (31, 3, 25)\}$. Then $P(\Delta_2) \leq 2k + a$ and these exceptions are listed in (2.6). Now assume that $(n, k, a) \notin \{(23, 3, 17), (31, 3, 25)\}$. Then by Lemmas 6 and 9 we find that $P(\Delta_2) > 2k + a$ except possibly when (n, k) equals any of the ten pairs in Lemma 8. Let us take $(n, k) = (91, 3)$. Then $P(\Delta_2) = 31 > 2k + a$ except when $a = 25, 27$. Thus $a^* = 25$. Similarly a^* for other pairs in Lemma 8 are found and listed in (2.6).

Now we take $k = 2$. Then

$$P(n(n + 2)) \geq 37 > 2k + a$$

for all n except those $n = M$ listed in T. For any given odd a, $1 \leq a < 29$, by our notation $T(a)$ denotes the values of $n \geq 2k + a$ for which (2.5) does not hold. Hence $T(a)$ gives the set of exceptional values of n. This proves the theorem. ∎

3. Newton polygons. As mentioned in the Introduction, a result of Dumas [3], from 1906, led Filaseta [6] to Lemma 1. Filaseta also remarks in [6] that this lemma may be strengthened by only requiring that p not divide $a_0 a_m$ in place of the condition that $|a_0| = |a_m| = 1$; we make use of this stronger version of the lemma here. In [11], Shorey and Tijdeman gave a refinement of Lemma 1 using the notion of Newton function. Let f be any polynomial of degree n in $\mathbb{Z}[x]$. The *Newton function* $Nf_p(x)$ with respect to a prime p is a real valued function on the interval $[0, n]$ which has the Newton polygon of f with respect to p as its graph. We shall give below a slightly modified version of their lemma.

LEMMA 10. *Let k and l be integers with $k > l \geq 0$. Suppose $u(x) = \sum_{j=0}^{n} b_j x^j \in \mathbb{Z}[x]$ and p is a prime such that $p \nmid b_n$ and $p \mid b_j$ for all $j \in \{0, 1, \ldots, n-l-1\}$. Let $a_0, a_1, \ldots, a_n$ be integers with $p \nmid a_0 a_n$. Put $v(x) = \sum_{j=0}^{n} a_j b_j x^j$. Then for any factor $h(x)$ of $v(x)$ having degree $k > l$, we have*

$$Nu_p(y) \leq Nh_p(y), \qquad Nu_p(n) - Nu_p(n-k+y) \geq Nh_p(k) - Nh_p(y)$$

for any $y \in [0, k]$ where $Nh_p(k)$ is a positive integer.

Proof. Suppose

$$v(x) = h(x)w(x)$$

with $\deg h(x) = k > l$. If the leftmost edge of the Newton polygon of $v(x)$ is of slope 0, then its x-length is $\leq l$. Since $h(x)$ is of degree $k > l$, the Newton polygon of $h(x)$ has at least one edge of non-zero slope. Thus $Nh_p(k)$ is a positive integer.

From the hypothesis it is clear that

(3.1) $$\begin{cases} Nu_p(0) = Nv_p(0) = 0, \quad Nu_p(n) = Nv_p(n), \\ Nu_p(x) \leq Nv_p(x) \quad \text{for } x \in (0, n). \end{cases}$$

Further, by Dumas' theorem, we have

(3.2) $$\begin{cases} Nv_p(y) \leq Nh_p(y) \quad \text{for } y \in [0, k], \\ Nv_p(n) - Nv_p(n-k+y) \leq Nu_p(n) - Nu_p(n-k+y) \quad \text{for } y \in [0, k]. \end{cases}$$

Let us translate parallel to xy-axes the Newton polygon of h defined on the interval $[0, k]$ so that the point $(k, Nh_p(k))$ coincides with $(n, Nv_p(n))$. Then the origin is shifted to $(n-k, Nv_p(n) - Nh_p(k))$ and any $(x, Nh_p(x))$ for $x \in [0, k]$ goes to $(n-k+x, Nh_p(x) + Nv_p(n) - Nh_p(k))$. Thus the shifted Newton polygon of h goes from $(n-k, Nv_p(n) - Nh_p(k))$ to $(n, Nv_p(n))$ and it lies on or above the Newton polygon of v in the interval $[n-k, n]$, by Dumas' theorem. Hence for any $x \in [0, k]$, we have

$$Nh_p(x) + Nv_p(n) - Nh_p(k) \geq Nv_p(n-k+x)$$

or

$$Nv_p(n) - Nv_p(n-k+x) \geq Nh_p(k) - Nh_p(x).$$

Thus from (3.1) and (3.2) we get the assertion of the lemma. ∎

Note that Lemma 1 follows from the above lemma, since, when the last edge of the Newton polygon of g has slope $< 1/k$, then $Ng_p(n) - Ng_p(n-k) < 1$, by taking $u = g$ in the above lemma, from which we get $Nh_p(k) = 0$, a contradiction.

LEMMA 11. *Let a, m and k be positive integers with a odd and $k \leq m/2$. Assume that there exists a prime $p > 2k + a$ dividing*

$$(a + 2(m-k)) \cdots (a + 2(m-1)).$$

Then the polynomial

$$F(x) = x^m + (a + 2(m-1))x^{m-1} + \cdots + (a + 2(m-1))(a + 2(m-2))\cdots a$$

has no factor of degree k. Also the polynomial

$$G(x) = a_m x^m + a_{m-1}(a + 2(m-1))x^{m-1} + \cdots + a_0(a + 2(m-1))(a + 2(m-2))\cdots a$$

has no factor of degree k, where $a_0, \ldots, a_m$ are integers with $p \nmid a_0 a_m$.

Proof. Write $F(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_1 x + c_0$ and $G(x) = a_m x^m + a_{m-1}c_{m-1}x^{m-1} + \cdots + a_1 c_1 x + a_0 c_0$, where

$$(3.3) \qquad c_j = (a+2j)\cdots(a+2(m-1)) \quad \text{for } 0 \le j < m \quad \text{and} \quad c_m = 1.$$

By assumption, there exists a prime $p > 2k + a$ dividing $c_0, c_1, \ldots, c_{m-k}$. From Lemma 1 we see that it suffices to show that the slope of the rightmost edge of the Newton polygon of $F(x)$ with respect to the prime p is $< 1/k$. The slope of the rightmost edge is

$$(3.4) \qquad L = \max_{1 \le j \le m} \left\{ \frac{\nu_p(c_0) - \nu_p(c_j)}{j} \right\}.$$

Therefore L equals

$$\nu_p(a(a+2)\cdots(a+2(j-1)))/j \quad \text{for some } 1 \le j \le m.$$

Let $j \le k$. Since $p > 2k + a$, we see that

$$\nu_p(a(a+2)\cdots(a+2(j-1)))/j = 0 < 1/k.$$

Let $j > k$. Since $p \ge 2k + a + 2$, we have

$$\nu_p(a(a+2)\cdots(a+2(j-1))) \le \nu_p((a+2j)!) < \frac{a+2j}{p-1} \le \frac{a+2j}{a+1+2k} < \frac{j}{k}.$$

Thus, $L < 1/k$, as desired, completing the proof. ∎

Proof of Theorem 4. The assertion is a direct consequence of Lemmas 5 and 11. ∎

We combine Theorem 8 and Lemma 11 to obtain the following theorem.

THEOREM 9. *Let $a \ge 1$ be an odd integer less than 29, and let m be any integer ≥ 2. Assume that $2m \ne n + 2k - a$ for any (n, k, a) given by (2.6). Then the polynomial $F(x)$ in Lemma 11 has no factor of degree ≥ 2. Further the polynomial $G(x)$ also has no factor of degree ≥ 2 provided a_0 and a_m are composed of primes $\le a + 4$.*

4. Linear factors of $F(x)$ and $G(x)$. In this section we deal with the linear factors of $F(x)$ and $G(x)$. Again using Lemma 11, we show

LEMMA 12. *Let $a \geq 1$ be an odd integer and let $m \geq 2$. Assume that there exists a prime p such that*

$$p \nmid a, \quad p \mid (a + 2(m - 1)), \quad p^{1 - 1/(p-1)} \geq (a + 2)^{1/2}.$$

Then the polynomial $F(x)$ has no linear factor. Also the polynomial $G(x)$ has no linear factor if such a prime p does not divide $a_0 a_m$.

Proof. Since p divides $a + 2(m - 1)$, we see that $p \mid c_j$ for $0 \leq j < m$ where c_j is given by (3.3). Further we require that $L < 1$ where L is given by (3.4). Thus we need

$$(4.1) \qquad \nu_p(a(a + 2) \cdots (a + 2(j - 1))) < j \quad \text{for } 1 \leq j < m.$$

Note that $\nu_p(a) = 0$. Hence we may assume that $j > 1$. Also we may suppose that $p \leq a + 2(j - 1)$, otherwise (4.1) holds since the left hand side of the inequality is 0. Suppose $p^\alpha \leq a + 2(j - 1) < p^{\alpha + 1}$. Then by taking blocks of $p, p^2, \ldots, p^\alpha$ successive terms we see that

$$\nu_p(a(a + 2) \cdots (a + 2(j - 1))) \leq \left(\left[\frac{j}{p} \right] + 1 \right) + \cdots + \left(\left[\frac{j}{p^\alpha} \right] + 1 \right)$$

$$< \frac{j}{p - 1} + \alpha \leq \frac{j}{p - 1} + \frac{\log(a + 2(j - 1))}{\log p}.$$

Thus (4.1) is valid if

$$p^{j(1 - 1/(p-1))} \geq a + 2(j - 1),$$

which is true for $j \geq 2$ by the assumption on p. Now the result follows by Lemma 1. ∎

5. Proof of Theorem 5. Suppose $g(x)$ has a factor of degree k with $2 \leq k \leq m/2$. By Theorem 9 we need only consider the exceptional values given in (2.6) with $n = M = a + 2(m - k) \geq a + 2k$. To exclude these cases, we use Lemma 1. This requires finding a suitable prime so that

$$(5.1) \qquad p \mid c_0, \ldots, c_{m-k} \quad \text{and} \quad L < 1/k$$

where L is given by (3.4). For this, we use the following procedure.

Let p be a prime such that

$$(5.2) \qquad p \mid c_{m-k}$$

and $p \mid (a + 2(m - \delta))$ with $\delta \geq 1$ chosen as small as possible.

Suppose $a + 2h$ is the least integer in $\{a, a + 2, \ldots, a + 2(m - 1)\}$ divisible by p. Let $\nu_p(a + 2(h + ip)) = \theta_i$ for $0 \leq i \leq r = (m - \delta - h)/p$. Let

$$L' = \max\left\{ \frac{\theta_0}{h + 1}, \frac{\theta_0 + \theta_1}{h + p + 1}, \ldots, \frac{\theta_0 + \cdots + \theta_r}{h + rp + 1} \right\}.$$

Then we see that

$$\nu_p(c_0) = \cdots = \nu_p(c_h),$$
$$\nu_p(c_{h+1}) = \cdots = \nu_p(c_{h+p}) = \nu_p(c_0) - \theta_0,$$
$$\vdots$$
$$\nu_p(c_{h+(r-1)p+1}) = \cdots = \nu_p(c_{h+rp}) = \nu_p(c_0) - \theta_0 - \cdots - \theta_{r-1},$$
$$\nu_p(c_{h+rp+1}) = \cdots = \nu_p(c_{m-1}) = \nu_p(c_0) - \theta_0 - \cdots - \theta_r.$$

Since $\theta_r > 0$, we have $L' > 0$. Thus by definition, $L = L'$.

Assume that

$$(5.3) \qquad \frac{\theta_0}{h+1} < \frac{1}{k} \quad \text{and} \quad \theta_i < \frac{p}{k} \quad \text{for } 1 \le i \le r.$$

Then

$$\frac{\theta_0 + \theta_1}{h+p+1} = \frac{\theta_0}{h+1}\frac{h+1}{h+p+1} + \frac{\theta_1}{h+p+1}$$
$$< \frac{h+1}{k(h+p+1)} + \frac{p}{k(h+p+1)} = \frac{1}{k}$$

and by induction, we see that

$$\frac{\theta_0 + \cdots + \theta_s}{h+sp+1} < \frac{1}{k} \quad \text{for } 1 < s \le r.$$

Thus $L < 1/k$, which is required in (5.1). Thus we need only satisfy (5.3). Since

$$p^{\theta_i} \le a + 2(h + ip) \le a + 2(m-1) = M + 2(k-1),$$

condition (5.3) is satisfied if

$$(5.4) \qquad \frac{\theta_0}{h+1} < \frac{1}{k} \quad \text{and} \quad M + 2(k-1) < p^{p/k}.$$

Thus we need only choose a prime p satisfying (5.2) and (5.4). In Tables 2 and 3, we give a choice of p for most of the values of $(M; a)$ listed in (2.6). Note that the choice of p is not unique. When $k = 4$, we have $(M; a) = (115; 15\text{--}27)$. We exclude this case by taking $p = 13$ so that $a + 2h = 39$ giving $6 \le h \le 12$ and $\theta_0 = 1$, which satisfy (5.2) and (5.4).

Table 2 $(k = 3)$

p	$(M; a)$
7	$(31; 25)$
11	$(117; 13)$, $(143; 23\text{--}27)$, $(341; 25\text{--}27)$, $(2871; 23\text{--}27)$
13	$(91; 25\text{--}27)$, $(115; 17\text{--}27)$, $(117; 15\text{--}27)$, $(243; 15\text{--}27)$, $(663; 23\text{--}27)$
17	$(117; 11)$, $(525; 25\text{--}27)$
19	$(243; 13)$
23	$(23; 17)$

Table 3 $(k = 2)$

p	$(M; a)$
5	$(5; 1)$, $(13; 9)$, $(25; 1, 7, 9, 11, 17)$
7	$(7; 3)$, $(19; 15)$, $(33; 9\text{–}17)$, $(49; 13\text{–}17)$, $(63; 9\text{–}17)$, $(75; 9\text{–}17)$, $(117; 13\text{–}17)$, $(133; 15\text{–}17, 23\text{–}27)$, $(145; 25\text{–}27)$, $(243; 3, 9\text{–}17, 23\text{–}27)$, $(273; 9\text{–}17)$, $(343; 23\text{–}27)$, $(845; 9\text{–}17)$, $(1125; 23\text{–}27)$
11	$(31; 27)$, $(33; 7, 19\text{–}27)$, $(55; 15\text{–}27)$, $(75; 7, 19\text{–}27)$, $(119; 13\text{–}27)$, $(143; 25\text{–}27)$, $(187; 13\text{–}27)$, $(207; 19\text{–}27)$, $(253; 19\text{–}27)$, $(273; 19\text{–}27)$, $(297; 19\text{–}27)$, $(341; 27)$, $(361; 15\text{–}27)$, $(493; 25\text{–}27)$, $(625; 15\text{–}27)$, $(713; 27)$, $(845; 19\text{–}27)$, $(1309; 19\text{–}27)$, $(1375; 13\text{–}27)$, $(1573; 13\text{–}27)$, $(1615; 15\text{–}27)$, $(2275; 19\text{–}27)$, $(2695; 27)$, $(2871; 25\text{–}27)$, $(3969; 15\text{–}27)$, $(4123; 27)$, $(5423; 27)$, $(6875; 19\text{–}27)$, $(7161; 27)$, $(9315; 19\text{–}27)$, $(16443; 25\text{–}27)$, $(18513; 19\text{–}27)$, $(19435; 27)$, $(24563; 25\text{–}27)$, $(41325; 25\text{–}27)$, $(41743; 19\text{–}27)$, $(45617; 25\text{–}27)$, $(57475; 19\text{–}27)$, $(86273; 27)$, $(87723; 25\text{–}27)$, $(130975; 27)$, $(184875; 25\text{–}27)$, $(203203; 27)$
13	$(63; 19\text{–}27)$, $(91; 27)$, $(115; 19\text{–}27)$, $(117; 19\text{–}27)$, $(169; 15\text{–}27)$, $(245; 15\text{–}27)$, $(323; 15\text{–}27)$, $(375; 25\text{–}27)$, $(403; 27)$, $(663; 15\text{–}27)$, $(897; 27)$, $(1519; 27)$, $(1573; 9)$, $(2873; 19\text{–}27)$, $(3211; 15\text{–}27)$, $(3625; 27)$, $(3703; 19\text{–}27)$, $(8073; 19\text{–}27)$, $(9945; 25\text{–}27)$, $(12673; 27)$, $(22475; 27)$, $(1128125; 19\text{–}27)$, $(1447873; 19\text{–}27)$
17	$(49; 19\text{–}27)$, $(85; 25\text{–}27)$, $(153; 27)$, $(525; 27)$, $(527; 27)$, $(1953; 27)$, $(8379; 25\text{–}27)$, $(14875; 25\text{–}27)$, $(2509045; 27)$, $(3322053; 27)$
19	$(93; 27)$, $(133; 21)$, $(435; 25\text{–}27)$, $(665; 25\text{–}27)$, $(1083; 27)$, $(2185; 21\text{–}27)$, $(287080365; 27)$
23	$(23; 19)$, $(343; 19)$, $(1125; 19)$, $(2185; 19)$

For the choices of p given in Tables 2 and 3, conditions (5.2) and (5.4) are satisfied and thus all these values are excluded. When $k = 2$ and $(M; a) = (243; 7), (1573; 11)$, we take $p = 5, 7$, respectively, and compute L' to get $L < 1/k$. Hence these cases are also excluded. Thus all values of $(M; a)$ given in (2.6) are excluded except when

$$(5.5) \qquad (M; a) \in \{(25; 3, 5, 13, 15, 19, 21), (133; 19), (243; 5, 19, 21),$$
$$(343; 21), (1125; 21)\}.$$

Next we illustrate the application of Lemma 10 with an example. Let $(M; a) = (25; 3)$. Then $m = 13$ and $u(x) = x^{13} + 27x^{12} + \cdots + 3 \cdot 5 \cdots 27$. The vertices of the Newton polygon of $u(x)$ with respect to the prime $p = 3$ are

$$(0, 0) - (9, 5) - (12, 7) - (13, 8).$$

By Lemma 10, any quadratic factor $h(x)$ satisfies $Nh_3(2) \geq Nu_3(2) = 10/9$

implying $Nh_3(2) \geq 2$. Also we have

$$Nu_3(13) - Nu_3(11) = 8 - 19/3 = 5/3 \geq Nh_3(2),$$

which is a contradiction. Thus the case $(M; a) = (25; 3)$ is excluded. Below we give the values of $(M; a, p)$ together with the vertices of the corresponding Newton polygon which are excluded by Lemma 10.

$$
\begin{aligned}
(25; 5, 3) : & \quad (0,0) - (9,5) - (12,7), \\
(25; 13, 3) : & \quad (0,0) - (8,5), \\
(25; 15, 3) : & \quad (0,0) - (6,4) - (7,5), \\
(25; 19, 3) : & \quad (0,0) - (5,4).
\end{aligned}
$$

As already noted, in the case $(M; a) = (25; 21)$ there are reducible polynomials. Thus we are left with six undecided cases in (5.5).

In these cases including $(M; a) = (25; 21)$ we check directly with MATHEMATICA that the resulting polynomials $f(x)$ do not factor. Thus $f(x)$ has no factors of degree ≥ 2. This completes the proof of the theorem. ∎

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