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Representations of certain binary quadratic forms as Lambert series

by

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1. Introduction. Let q be a complex number with $|q| < 1$. In [2], A. Berkovich and H. Yesilyurt used Ramanujan's ${}_1\psi_1$ summation formula to write two theta series associated with binary quadratic forms as a sum of Lambert series:

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+5y^2} = 1 + \sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n} + \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}, \tag{1.1}$$

$$\sum_{x,y=-\infty}^{\infty} q^{2x^2+2xy+3y^2} = 1 + \sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n} - \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}. \tag{1.2}$$

By taking the sum and difference of (1.1) and (1.2), we obtain

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+5y^2} + \sum_{x,y=-\infty}^{\infty} q^{2x^2+2xy+3y^2} = 2 + 2\sum_{n=1}^{\infty}\left(\frac{-20}{n}\right)\frac{q^n}{1-q^n}, \tag{1.3}$$

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+5y^2} - \sum_{x,y=-\infty}^{\infty} q^{2x^2+2xy+3y^2} = 2\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{1+q^{2n}}. \tag{1.4}$$

In general, for an imaginary quadratic field, K, of discriminant d_K and class number h, Dirichlet's theorem [11, Th. 204] states that

$$h + \omega\sum_{n=1}^{\infty}\left(\frac{d_K}{n}\right)\frac{q^n}{1-q^n} = \sum_{i=1}^{h}\sum_{x,y=-\infty}^{\infty} q^{Q_i(x,y)}, \tag{1.5}$$

where ω is the number of units and $Q_i(x,y)$ are the inequivalent binary quadratic forms of the quadratic field K. When $K=\mathbb{Q}(\sqrt{-5})$, we get (1.3). The companion formula (1.4) can be obtained using the theory of genus characters. Thus, identities involving a single Lambert series as a sum of theta series are well understood. What is interesting in this case is that we

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can also isolate each theta series and rewrite it as a sum of Lambert series. This phenomenon is illustrated in the following example involving four theta series. Let $K = \mathbb{Q}(\sqrt{-30})$ and define

$$F(a,b,c) = \sum_{x,y=-\infty}^{\infty} q^{ax^2+bxy+cy^2}.$$

We have

(1.6)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} F(1,0,30) \\ F(2,0,15) \\ F(3,0,10) \\ F(5,0,6) \end{bmatrix} = \begin{bmatrix} 4+2\sum_{n=1}^{\infty}\left(\frac{-120}{n}\right)\frac{q^n}{1-q^n} \\ 2\sum_{n=1}^{\infty}\left(\frac{n}{15}\right)\frac{q^n-q^{3n}}{1+q^{4n}} \\ 2\sum_{n=1}^{\infty}\left(\frac{10}{n}\right)\frac{q^n-q^{2n}}{1-q^{3n}} \\ 2\sum_{n=1}^{\infty}\left(\frac{n}{6}\right)\frac{q^n-q^{2n}-q^{3n}+q^{4n}}{1-q^{5n}} \end{bmatrix}.$$

Observe that the matrix on the left hand side is invertible. Hence, we can rewrite each of the four theta series as a linear combination of four Lambert series.

In the next section, we shall provide a sufficient condition that allows us to express theta series as a linear combination of Lambert series. Explicit examples will be given in Section 3.

2. Genus characters and a theorem of Kronecker. Let us recall the theory of genus characters [13, pp. 59–62]. Let $K = \mathbb{Q}(\sqrt{N})$ where $N < 0$ is a square free integer. The discriminant of K is defined as

$$d_K = \begin{cases} N & \text{if } N \equiv 1 \pmod 4, \\ 4N & \text{otherwise.} \end{cases}$$

If d_K is odd then it can be written as a product of distinct odd primes, $-p_1p_2\dots p_k$. We set $P_i = \pm p_i$ so that $d_K/P_i \equiv 1 \pmod 4$, and each d_K/P_i remains an odd discriminant. If d_K is even, then either $d_K = -8p_2\dots p_k$, where $d_K/4 \equiv 2 \pmod 4$, or $d_K = -4p_2\dots p_k$, where $d_K/4 \equiv 3 \pmod 4$. In this case, we set P_1 to be ± 8 or -4 so that d_K/P_1 remains an odd discriminant.

Let d_1 be any product of the factors $P_1,\dots,P_k$ of d_K. Then $d_K = d_1d_2$ gives us a decomposition of d_K into a product of two coprime discriminants. For each decomposition $d_K = d_1d_2$, and any prime ideal $\mathfrak{p}$ not dividing d_K, we can define a *genus character*,

$$\chi(\mathfrak{p}) = \chi_{d_1}(\mathfrak{p}) = \left(\frac{d_1}{N(\mathfrak{p})}\right).$$

It can be shown [13, p. 60] that $\chi_{d_1} = \chi_{d_2}$. Hence we can identify $d_K = d_1d_2 = d_2d_1$, giving us a total of 2^{k-1} decompositions.

If $\mathfrak{p} \mid d_K$, then one of χ_{d_1}, χ_{d_2} is zero and the other is non-zero; we then take χ to be the non-zero value. There are 2^{k-1} different genus characters corresponding to decompositions of d_K. In fact, the genus characters form an abelian group, $\mathfrak{G}$, of order 2^{k-1} [13, p. 66].

Now, for an imaginary quadratic field K [9, p. 190], we have

$$\left(\frac{d_K}{p}\right) = \begin{cases} 0 & \text{if } p \text{ ramifies, } (p) = \mathfrak{p}^2,\ N(\mathfrak{p}) = p, \\ 1 & \text{if } p \text{ splits, } (p) = \mathfrak{p}\mathfrak{p}',\ N(\mathfrak{p}) = p, \\ -1 & \text{if } p \text{ is inert, } (p) = \mathfrak{p},\ N(\mathfrak{p}) = p^2. \end{cases}$$

If χ is a genus character (corresponding to $d_K = d_1d_2$), then the L-series equals

$$(2.1)\qquad L_K(s,\chi) = \sum_{\text{ideal } \mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p} \text{ prime}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$
$$= \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{ramified } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{split } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{inert } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}.$$

For the first product, we may assume without loss of generality that $p \mid d_2$, hence $\chi(\mathfrak{p}) = (\frac{d_1}{p})$ and $(\frac{d_2}{p}) = 0$. Thus,

$$(2.2)\qquad \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{ramified } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$
$$= \prod_{\text{ramified } p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1} \prod_{\text{ramified } p} \left(1 - \left(\frac{d_2}{p}\right)p^{-s}\right)^{-1}.$$

In the second product, since $\chi_{d_1} = \chi_{d_2}$ and p lies under $\mathfrak{p}$ and $\mathfrak{p}'$,

$$(2.3)\qquad \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{split } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$
$$= \prod_{\text{split } p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1} \prod_{\text{split } p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1}$$
$$= \prod_{\text{split } p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1} \prod_{\text{split } p} \left(1 - \left(\frac{d_2}{p}\right)p^{-s}\right)^{-1}.$$

For the last product, $\chi(\mathfrak{p}) = \left(\frac{d_1}{p^2}\right) = 1$. However, since $\left(\frac{d_K}{p}\right) = \left(\frac{d_1}{p}\right)\left(\frac{d_2}{p}\right) = -1$, we have

$$\text{(2.4)} \qquad \prod_{\substack{\mathfrak{p} \text{ over} \\ \text{inert } p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} = \prod_{\text{inert } p} (1 - p^{-2s})^{-1}$$
$$= \prod_{\text{inert } p} (1 - p^{-s})^{-1}(1 + p^{-s})^{-1}$$
$$= \prod_{\text{inert } p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1} \prod_{\text{inert } p} \left(1 - \left(\frac{d_2}{p}\right)p^{-s}\right)^{-1}.$$

Combining these, we obtain Kronecker's theorem [13, p. 62],

$$\text{(2.5)} \qquad L_K(s,\chi) = \prod_{p} \left(1 - \left(\frac{d_1}{p}\right)p^{-s}\right)^{-1} \prod_{p} \left(1 - \left(\frac{d_2}{p}\right)p^{-s}\right)^{-1}$$
$$= L_{d_1}(s)L_{d_2}(s).$$

Next, applying the inverse Mellin transform to (2.5), we get

$$\text{(2.6)} \qquad \sum_{\mathfrak{a}\neq 0} \chi(\mathfrak{a})q^{N(\mathfrak{a})} = \left(\sum_{n=1}^{\infty} \left(\frac{d_1}{n}\right)q^n\right)\left(\sum_{n=1}^{\infty} \left(\frac{d_2}{n}\right)q^n\right)$$
$$= \sum_{n=1}^{\infty} \left(\frac{d_1}{n}\right) \frac{\sum_{k=1}^{|d_2|} \left(\frac{d_2}{k}\right) q^{kn}}{1 - q^{|d_2|n}}.$$

Using $\mathcal{L}_{d_1,d_2}(q)$ to denote the last sum, we can now state our result.

THEOREM 2.1. *Let K be an imaginary quadratic field, with ω units, where the ideal class group is isomorphic to the group $\mathfrak{G}$ of genus characters. Let $ax^2 + bxy + cy^2$ be a primitive binary quadratic form with discriminant d_K. If p is any unramified prime represented by this form, then*

$$F(a,b,c) = 1 + \frac{\omega}{|\mathfrak{G}|} \sum_{\chi_{d_1} \in \mathfrak{G}} \chi_{d_1}(p)\mathcal{L}_{d_1,d_2}(q)$$
$$= 1 + \frac{\omega}{|\mathfrak{G}|} \sum_{\textit{admissible } d_1} \left(\frac{d_1}{p}\right)\mathcal{L}_{d_1,d_2}(q).$$

Proof. Let $m = |\mathfrak{G}|$. Under the hypothesis of the theorem, there is an isomorphism from the form class group of primitive binary quadratic forms to the ideal class group [6, p. 113], which is hence isomorphic to $\mathfrak{G}$. Next, let $Q_i(x,y)$ be the representatives of the form class group and $F_i = \sum q^{Q_i(x,y)}$ be the associated theta series, where the sum is over all integers x and y, excluding $(x,y) = (0,0)$. Then (2.6) gives us a system of m equations of the form

$$\omega\mathcal{L}_{d_1,d_2}(q) = \left(\frac{d_1}{n_1}\right)F_1 + \left(\frac{d_1}{n_2}\right)F_2 + \cdots + \left(\frac{d_1}{n_m}\right)F_m, \tag{2.7}$$

where n_i is any integer represented by $Q_i(x,y)$, coprime to d_K. Hence we have the matrix equation

$$\omega\hat{L} = A\hat{F}. \tag{2.8}$$

Here $\hat{F} = (F_1,\ldots,F_k)$, $\hat{L}$ is the vector of the Lambert series $\mathcal{L}_{d_1,d_2}(q)$, and A is the character table for $\mathfrak{G}$. By the orthogonality relations, [1, Ch. 6], we can recover each F_j as

$$F_j = \frac{\omega}{m}\sum_{\chi_{d_1}\in\mathfrak{G}} \chi_{d_1}(n_j)\mathcal{L}_{d_1,d_2}(q). \tag{2.9}$$

The proof is complete when we add the case $(x,y) = (0,0)$ and replace n_j by a suitable prime. ∎

The condition in Theorem 2.1 is commonly described as imaginary quadratic fields having one class per genus. S. Chowla [4] proved that there are finitely many such fields. Further work by J. D. Swift [15], S. Chowla and W. E. Briggs [5], E. Grosswald [7] and P. J. Weinberger [16], showed that besides the 65 that are currently known, there exists at most one more field with one class per genus.

Theorem 2.1 can easily be generalized to imaginary quadratic fields with more than one class per genus. However, in this case, it is not possible to isolate the several theta series associated to the same genus and the corresponding result is not as striking.

There are another 36 known form class groups that have one class per genus. These are said to have nonfundamental discriminants and correspond to orders [6, p. 132] rather than the ring of integers in the imaginary quadratic field. Theorem 2.1 does not apply to these 36.

For these 101 discriminants, N. A. Hall [8] has computed explicit formulas for the number of representations of an integer by binary quadratic forms, while K. S. Williams [17] has given identities analogous to (1.5).

For recent work on the problem of representations of an integer by binary quadratic forms (not necessarily having one class per genus), see Z. H. Sun and K. S. Williams [14], P. Kaplan and K. S. Williams [10] and references therein.

3. Examples and tables. There are exactly nine imaginary quadratic fields with class number $h = 1$, listed according to their discriminants:

$$d_K = -3, -4, -7, -8, -11, -19, -43, -67, -163. \tag{3.1}$$

L. C. Shen [12] has given explicit identities for these cases.

There are 18 imaginary quadratic fields with class number $h = 2$, listed according to their discriminants:

$$d_K = -15, -20, -24, -35, -40, -51, -52, -88, -91, \\ -115, -123, -148, -187, -232, -235, -267, -403, -427. \tag{3.2}$$

In the cases where $d_K = -4\ell$, $\ell = 5, 13$ or 37, we have

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+\ell y^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-4\ell}{n}\right) \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \left(\frac{\ell}{n}\right) \frac{q^n}{1+q^{2n}}, \tag{3.3}$$

$$\sum_{x,y=-\infty}^{\infty} q^{2x^2+2xy+\frac{\ell+1}{2}y^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-4\ell}{n}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{\ell}{n}\right) \frac{q^n}{1+q^{2n}}. \tag{3.4}$$

(3.3) and (3.4) were established by H. H. Chan and S. H. Chan [3].

When $d_K = -8\ell$, $\ell = 3, 5, 11$ or 29,

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+2\ell y^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-8\ell}{n}\right) \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \left(\frac{n}{\ell}\right) \frac{q^n + (\frac{-1}{\ell})q^{3n}}{1+q^{4n}},$$

$$\sum_{x,y=-\infty}^{\infty} q^{2x^2+\ell y^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-8\ell}{n}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{n}{\ell}\right) \frac{q^n + (\frac{-1}{\ell})q^{3n}}{1+q^{4n}}.$$

The case of $\ell = 3$ was also given in [2].

The remaining discriminants are all of the form $d_K = -\ell m$, according to the following table:

ℓ	m	$D = \ell m$	a	b	c
3	5	15	2	1	2
7	5	35	3	1	3
3	17	51	3	3	5
7	13	91	5	3	5
23	5	115	5	5	7
3	41	123	3	3	11
11	17	187	7	3	7
47	5	235	5	5	13
3	89	267	3	3	23
31	13	403	11	9	11
61	7	427	7	7	17

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+xy+\frac{D+1}{4}y^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \left(\frac{-\ell}{n}\right) \frac{\sum_{j=1}^{m-1} (\frac{j}{m}) q^{jn}}{1-q^{mn}},$$

$$\sum_{x,y=-\infty}^{\infty} q^{ax^2+bxy+cy^2} = 1 + \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{-\ell}{n}\right) \frac{\sum_{j=1}^{m-1} (\frac{j}{m}) q^{jn}}{1-q^{mn}}.$$

There are 24 imaginary quadratic fields with class number $h = 4$, having one class per genus. We give a table listing the discriminants together with the representatives of the form class group, as well as one example for $d_K = -84$.

d_k	$F(a,b,c)$			
-84	$(1,0,21)$	$(2,2,11)$	$(3,0,7)$	$(5,4,5)$
-120	$(1,0,30)$	$(2,0,15)$	$(3,0,10)$	$(5,0,6)$
-132	$(1,0,33)$	$(2,2,17)$	$(3,0,11)$	$(6,6,7)$
-168	$(1,0,42)$	$(2,0,21)$	$(3,0,14)$	$(6,0,7)$
-195	$(1,1,49)$	$(3,3,17)$	$(5,5,11)$	$(7,1,7)$
-228	$(1,0,57)$	$(2,2,29)$	$(3,0,19)$	$(6,6,11)$
-280	$(1,0,70)$	$(2,0,35)$	$(5,0,14)$	$(7,0,10)$
-312	$(1,0,78)$	$(2,0,39)$	$(3,0,26)$	$(6,0,13)$
-340	$(1,0,85)$	$(2,2,43)$	$(5,0,17)$	$(10,10,11)$
-372	$(1,0,93)$	$(2,2,47)$	$(3,0,31)$	$(6,6,17)$
-408	$(1,0,102)$	$(2,0,51)$	$(3,0,34)$	$(6,0,17)$
-435	$(1,1,109)$	$(3,3,37)$	$(5,5,23)$	$(11,7,11)$
-483	$(1,1,121)$	$(3,3,41)$	$(7,7,19)$	$(11,1,11)$
-520	$(1,0,130)$	$(2,0,65)$	$(5,0,26)$	$(10,0,13)$
-532	$(1,0,133)$	$(2,2,67)$	$(7,0,19)$	$(13,12,13)$
-555	$(1,1,139)$	$(3,3,47)$	$(5,5,29)$	$(13,11,13)$
-595	$(1,1,149)$	$(5,5,31)$	$(7,7,23)$	$(13,9,13)$
-627	$(1,1,157)$	$(3,3,53)$	$(11,11,17)$	$(13,7,13)$
-708	$(1,0,177)$	$(2,2,89)$	$(3,0,59)$	$(6,6,31)$
-715	$(1,1,179)$	$(5,5,37)$	$(11,11,19)$	$(13,13,17)$
-760	$(1,0,190)$	$(2,0,95)$	$(5,0,38)$	$(10,0,19)$
-795	$(1,1,199)$	$(3,3,67)$	$(5,5,41)$	$(15,15,17)$
-1012	$(1,0,253)$	$(2,2,127)$	$(11,0,23)$	$(17,12,17)$
-1435	$(1,1,359)$	$(5,5,73)$	$(7,7,53)$	$(19,3,19)$

For $d_K = -84$, we set

$$L_1 = \sum_{n=1}^{\infty} \left(\frac{-84}{n}\right) \frac{q^n}{1-q^n}, \quad L_2 = \sum_{n=1}^{\infty} \left(\frac{28}{n}\right) \frac{q^n - q^{2n}}{1-q^{3n}},$$

$$L_3 = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n - q^{5n}}{1+q^{6n}}, \quad L_4 = \sum_{n=1}^{\infty} \left(\frac{21}{n}\right) \frac{q^n}{1+q^{2n}}.$$

Then

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+21y^2} = 1 + \frac{1}{2}(L_1 + L_2 + L_3 + L_4),$$

$$\sum_{x,y=-\infty}^{\infty} q^{3x^2+7y^2} = 1 + \frac{1}{2}(L_1 + L_2 - L_3 - L_4),$$

$$\sum_{x,y=-\infty}^{\infty} q^{2x^2+2xy+11y^2} = 1 + \frac{1}{2}(L_1 - L_2 + L_3 - L_4),$$

$$\sum_{x,y=-\infty}^{\infty} q^{5x^2+4xy+5y^2} = 1 + \frac{1}{2}(L_1 - L_2 - L_3 + L_4).$$

For imaginary quadratic fields with class number $h = 8$, the following 13 have one class per genus:

d_k	$F(a,b,c)$			
-420	$(1,0,105)$	$(2,2,53)$	$(3,0,35)$	$(5,0,21)$
	$(6,6,19)$	$(7,0,15)$	$(10,10,13)$	$(11,8,11)$
-660	$(1,0,165)$	$(2,2,83)$	$(3,0,55)$	$(5,0,33)$
	$(6,6,29)$	$(10,10,19)$	$(11,0,15)$	$(13,4,13)$
-840	$(1,0,210)$	$(2,0,105)$	$(3,0,70)$	$(5,0,42)$
	$(6,0,35)$	$(7,0,30)$	$(10,0,21)$	$(14,0,15)$
-1092	$(1,0,273)$	$(2,2,137)$	$(3,0,91)$	$(6,6,47)$
	$(7,0,39)$	$(13,0,21)$	$(14,14,23)$	$(17,8,17)$
-1155	$(1,1,289)$	$(3,3,97)$	$(5,5,59)$	$(7,7,43)$
	$(11,11,29)$	$(15,15,23)$	$(17,1,17)$	$(19,17,19)$
-1320	$(1,0,330)$	$(2,0,165)$	$(3,0,110)$	$(5,0,66)$
	$(6,0,55)$	$(10,0,33)$	$(11,0,30)$	$(15,0,22)$
-1380	$(1,0,345)$	$(2,2,173)$	$(3,0,115)$	$(5,0,69)$
	$(6,6,59)$	$(10,10,37)$	$(15,0,23)$	$(19,8,19)$
-1428	$(1,0,357)$	$(2,2,179)$	$(3,0,119)$	$(6,6,61)$
	$(7,0,51)$	$(14,14,29)$	$(17,0,21)$	$(19,4,19)$
-1540	$(1,0,385)$	$(2,2,193)$	$(5,0,77)$	$(7,0,55)$
	$(10,10,41)$	$(11,0,35)$	$(14,14,31)$	$(22,22,23)$
-1848	$(1,0,462)$	$(2,0,231)$	$(3,0,154)$	$(6,0,77)$
	$(7,0,66)$	$(11,0,42)$	$(14,0,33)$	$(21,0,22)$
-1995	$(1,1,499)$	$(3,3,167)$	$(5,5,101)$	$(7,7,73)$
	$(15,15,37)$	$(19,19,31)$	$(21,21,29)$	$(23,11,23)$
-3003	$(1,1,751)$	$(3,3,251)$	$(7,7,109)$	$(11,11,71)$
	$(13,13,61)$	$(21,21,41)$	$(29,19,29)$	$(31,29,31)$
-3315	$(1,1,829)$	$(3,3,277)$	$(5,5,167)$	$(13,13,67)$
	$(15,15,59)$	$(17,17,53)$	$(29,7,29)$	$(31,23,31)$

We illustrate Theorem 2.1 by writing $F(1,0,105)$ in terms of eight Lambert series:

$$\sum_{x,y=-\infty}^{\infty} q^{x^2+105y^2} = 1 + \frac{1}{4}\Bigg(\sum_{n=1}^{\infty} \left(\frac{-420}{n}\right) \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \left(\frac{105}{n}\right) \frac{q^n}{1+q^{2n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{140}{n}\right) \frac{q^n - q^{2n}}{1-q^{3n}} + \sum_{n=1}^{\infty} \left(\frac{-84}{n}\right) \frac{\sum_{k=1}^{5} \left(\frac{5}{k}\right) q^{kn}}{1-q^{5n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{60}{n}\right) \frac{\sum_{k=1}^{7} \left(\frac{-7}{k}\right) q^{kn}}{1-q^{7n}} + \sum_{n=1}^{\infty} \left(\frac{-35}{n}\right) \frac{\sum_{k=1}^{12} \left(\frac{12}{k}\right) q^{kn}}{1-q^{12n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{28}{n}\right) \frac{\sum_{k=1}^{15} \left(\frac{-15}{k}\right) q^{kn}}{1-q^{15n}} + \sum_{n=1}^{\infty} \left(\frac{-20}{n}\right) \frac{\sum_{k=1}^{21} \left(\frac{21}{k}\right) q^{kn}}{1-q^{21n}} \Bigg).$$

Finally for imaginary quadratic fields with class number $h = 16$, -5460 is the only fundamental discriminant with one class per genus.

d_k	$F(a,b,c)$			
-5460	$(1,0,1365)$	$(2,2,683)$	$(3,0,455)$	$(5,0,273)$
	$(6,6,229)$	$(7,0,195)$	$(10,10,139)$	$(13,0,105)$
	$(14,14,101)$	$(15,0,91)$	$(21,0,65)$	$(26,26,59)$
	$(30,30,53)$	$(35,0,39)$	$(37,4,37)$	$(42,42,43)$

We again illustrate Theorem 2.1 by writing $F(37,4,37)$ in terms of sixteen Lambert series:

$$\sum_{x,y=-\infty}^{\infty} q^{37x^2+4xy+37y^2}$$
$$= 1 + \frac{1}{8}\Bigg(\sum_{n=1}^{\infty} \left(\frac{-5460}{n}\right) \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \left(\frac{1365}{n}\right) \frac{q^n}{1+q^{2n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{1820}{n}\right) \frac{q^n - q^{2n}}{1-q^{3n}} - \sum_{n=1}^{\infty} \left(\frac{-1092}{n}\right) \frac{\sum_{k=1}^{5} \left(\frac{5}{k}\right) q^{kn}}{1-q^{5n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{780}{n}\right) \frac{\sum_{k=1}^{7} \left(\frac{-7}{k}\right) q^{kn}}{1-q^{7n}} - \sum_{n=1}^{\infty} \left(\frac{-420}{n}\right) \frac{\sum_{k=1}^{13} \left(\frac{13}{k}\right) q^{kn}}{1-q^{13n}}$$
$$+ \sum_{n=1}^{\infty} \left(\frac{-455}{n}\right) \frac{\sum_{k=1}^{12} \left(\frac{12}{k}\right) q^{kn}}{1-q^{12n}} - \sum_{n=1}^{\infty} \left(\frac{364}{n}\right) \frac{\sum_{k=1}^{15} \left(\frac{-15}{k}\right) q^{kn}}{1-q^{15n}}$$
$$- \sum_{n=1}^{\infty} \left(\frac{273}{n}\right) \frac{\sum_{k=1}^{20} \left(\frac{-20}{k}\right) q^{kn}}{1-q^{20n}} + \sum_{n=1}^{\infty} \left(\frac{-260}{n}\right) \frac{\sum_{k=1}^{21} \left(\frac{21}{k}\right) q^{kn}}{1-q^{21n}}$$

$$+\sum_{n=1}^{\infty}\left(\frac{-195}{n}\right)\frac{\sum_{k=1}^{28}\left(\frac{28}{k}\right)q^{kn}}{1-q^{28n}}-\sum_{n=1}^{\infty}\left(\frac{156}{n}\right)\frac{\sum_{k=1}^{35}\left(\frac{-35}{k}\right)q^{kn}}{1-q^{35n}}$$
$$-\sum_{n=1}^{\infty}\left(\frac{140}{n}\right)\frac{\sum_{k=1}^{39}\left(\frac{-39}{k}\right)q^{kn}}{1-q^{39n}}-\sum_{n=1}^{\infty}\left(\frac{105}{n}\right)\frac{\sum_{k=1}^{52}\left(\frac{-52}{k}\right)q^{kn}}{1-q^{52n}}$$
$$-\sum_{n=1}^{\infty}\left(\frac{-91}{n}\right)\frac{\sum_{k=1}^{60}\left(\frac{60}{k}\right)q^{kn}}{1-q^{60n}}+\sum_{n=1}^{\infty}\left(\frac{-84}{n}\right)\frac{\sum_{k=1}^{65}\left(\frac{65}{k}\right)q^{kn}}{1-q^{65n}}\Bigg).$$

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