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Stabilization in non-abelian Iwasawa theory

by

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1. Introduction. Let K/k be a $\mathbb{Z}_p$-extension of a number field k, let k_n be the nth layer and $A_n := A(k_n)$ the p-part of the ideal class group of k_n. Let $\Lambda := \mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$ be the Iwasawa algebra associated with K/k; all modules we shall work with will be Λ-modules. For any field E we denote by $\widetilde{L}(E)$ the maximal unramified (not necessarily abelian) pro-p-extension of E; to simplify the notation, we usually write $\widetilde{L} := \widetilde{L}(K)$ and $\widetilde{L}_n := \widetilde{L}(k_n)$ for every $n \geq 0$. In [O1] M. Ozaki studies the non-abelian extensions $\widetilde{L}/K$ and $\widetilde{L}_n/k_n$ using the *lower central series* for $\widetilde{G} := \mathrm{Gal}(\widetilde{L}/K)$ and $\widetilde{G}_n := \mathrm{Gal}(\widetilde{L}_n/k_n)$. Let

$$C_1(\widetilde{G}) := \widetilde{G} \quad \text{and} \quad C_i(\widetilde{G}) := \overline{[\widetilde{G}, C_{i-1}(\widetilde{G})]} \quad (\text{for any } i \geq 2)$$

(for details and precise definitions see Section 2), and set

$$X^{(i)} := X^{(i)}(K/k) = C_i(\widetilde{G})/C_{i+1}(\widetilde{G}) \quad (\text{for any } i \geq 1)$$

(with analogous notation $X_n^{(i)}$, depending on the series for $\widetilde{G}_n$).

Note that for $i = 1$ we obtain the classical Iwasawa module $X^{(1)} \simeq \varprojlim_n A_n$ (where the limit is with restect to the natural norm maps). Stabilization is a frequent property of Iwasawa modules like class groups (other related modules have been extensively studied in [BC]): those modules associated with the fields k_n tend to remain the same (i.e., *stabilize*) from the very first step in which they stop growing. For example we have the following results (see [Fu] and [B3]).

THEOREM 1.1. *Assume all ramified primes in K/k are totally ramified in K/k_{n_0}.*

(i) *If $|A_n| = |A_{n+1}|$ for some $n \geq n_0$, then $|A_m| = |A_n| = |X^{(1)}|$ for all $m \geq n$.*

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(ii) *If* $\mathrm{rk}_p(A_n) = \mathrm{rk}_p(A_{n+1})$ *for some* $n \geq n_0$, *then* $\mathrm{rk}_p(A_m) = \mathrm{rk}_p(A_n) = \mathrm{rk}_p(X^{(1)})$ *for all* $m \geq n$ (*where* rk_p *denotes the* p*-rank of a module*).

Since capitulation of ideals is related to the generalized Greenberg Conjecture (see, e.g., [B1], [B2] and [LN]), in [BC] we studied the capitulation kernels and provided an example of delayed stabilization. For any $m \geq n$, let $i_{n,m} : A_n \to A_m$ be the map induced by the natural inclusions, and define $H_{n,m} := \mathrm{Ker}(i_{n,m})$ and $H_n := \bigcup_{m \geq n} H_{n,m}$. The H_n have natural stabilization properties similar to the ones of class groups (see [BC, Theorem 3.7]), but we noticed a possible different behaviour for the $H_{n,m}$:

Theorem 1.2 ([BC, Theorem 3.11]). *Let* $n \geq n_0$. *Then there exist constants* r (*independent of* n) *and* $h(n)$ *such that, if* $n < r$, *then*

$$\begin{aligned} 1 = |H_{n,n}| \leq |H_{n,n+1}| \leq \cdots \leq |H_{n,r}| < |H_{n,r+1}| < \cdots \\ < |H_{n,h(n)}| = |H_{n,h(n)+1}| = \cdots = |H_n|. \end{aligned}$$

Our goal is to study stabilization for the orders (and the p-ranks) of the $X_n^{(i)}$ to see which of the above statements could be generalized to the non-abelian setting. Theorem 1.1 expresses the stabilization of the $X_n^{(1)}$, but delayed stabilization seems to be more common in the non-abelian case, as we shall show in a crucial example in Section 2.1. The stabilization of the $X_n^{(i)}$ is (perhaps not surprisingly) strictly related to the behaviour of the modules $X_n^{(j)}$ for $1 \leq j \leq i$; indeed, our main results are the following (see Theorems 2.13 and 3.1).

Theorem 1.3. *Let* $i \geq 1$ *and* $n \geq n_0$.

(i) *If* $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ *for all* $1 \leq j \leq i$, *then* $X_n^{(j)} \simeq X_m^{(j)} \simeq X^{(j)}$ *for any* $1 \leq j \leq i$ *and for all* $m \geq n$.

(ii) *If* $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ *for all* $1 \leq j \leq i-1$, *and* $\mathrm{rk}_p(X_n^{(i)}) = \mathrm{rk}_p(X_{n+1}^{(i)})$, *then* $\mathrm{rk}_p(X_n^{(i)}) = \mathrm{rk}_p(X_m^{(i)}) = \mathrm{rk}_p(X^{(i)})$ *for all* $m \geq n$.

One immediately notices that the statement for the p-ranks is weaker since it requires hypotheses on the orders (which imply that all the λ-invariants of the $X^{(i)}$ are zero, see Remark 3.2) and not only on p-ranks. At present we are not able to remove this obstacle because a crucial step in our proof shows that the "normally generated" condition appearing in Definition 2.10 for the sets $R_n^{(i)}$ is not necessary to obtain Λ-submodules of $X^{(i)}$ if one assumes stabilization of orders (while stabilization of p-ranks seems not to be enough for it).

2. Stabilization for the order of $X_n^{(i)}$. Let H be a (topological) group. For all $a, b \in H$ we set $[a, b] = a^{-1}b^{-1}ab$. If H_1, H_2 are subgroups of H, we denote the *commutator group* of H_1 and H_2 by

$$\overline{[H_1, H_2]} := \overline{\langle [h_1, h_2] \,:\, h_1 \in H_1,\, h_2 \in H_2 \rangle}$$

(the topological closure of $[H_1, H_2]$). Moreover, we define the sequences

$$C_1(H) := H, \qquad C_i(H) := \overline{[H, C_{i-1}(H)]} \quad (\text{for any } i \geq 2)$$

and

$$D_0(H) := H, \qquad D_i(H) := \overline{[D_{i-1}(H), D_{i-1}(H)]} \quad (\text{for any } i \geq 1).$$

We call them the *lower central series* and the *derived series* of H respectively.

Let $\widetilde{G} = \mathrm{Gal}(\widetilde{L}/K)$. We get the following series for $\widetilde{G}$:

$$\widetilde{G} = C_1(\widetilde{G}) \supseteq C_2(\widetilde{G}) \supseteq C_3(\widetilde{G}) \supseteq \cdots, \tag{1}$$

$$\widetilde{G} = D_0(\widetilde{G}) \supseteq D_1(\widetilde{G}) \supseteq D_2(\widetilde{G}) \supseteq \cdots. \tag{2}$$

A well known relation is that $D_i(H) \subseteq C_{2^i}(H)$, i.e., the ith term of the derived series is contained in the 2^ith term of the lower central series. For more details see, for example, [Bo, Ch. I, §6.3, §6.4] or [Fr]. The derived series of $\widetilde{G}$ is naturally related to the class field tower of K, while the lower central series is less intuitive so we give here notations and properties for the fields it is related to.

Definition 2.1. For every $i \geq 1$, we set

(i) $L^{(i)} :=$ the subfield of $\widetilde{L}$ fixed by $C_{i+1}(\widetilde{G})$;
(ii) $G^{(i)} := \widetilde{G}/C_{i+1}(\widetilde{G}) \simeq \mathrm{Gal}(L^{(i)}/K)$;
(iii) $X^{(i)} := C_i(\widetilde{G})/C_{i+1}(\widetilde{G}) = C_i(G^{(i)}) \simeq \mathrm{Gal}(L^{(i)}/L^{(i-1)})$.

The group $X^{(i)}$ will be called the *ith Iwasawa module* of K/k.

We have analogous notation and definitions (just add an index $_n$) for the modules associated with the lower central series of $\widetilde{G}_n := \mathrm{Gal}(\widetilde{L}_n/k_n)$.

If $F^{(i)}(k_n)$ is the ith term of the class field tower of k_n (i.e., the maximal abelian unramified p-extension of $F^{(i-1)}(k_n)$), then, from the relation above, $L^{(2^i-1)}(k_n) \subseteq F^{(i)}(k_n)$.

In [O1] M. Ozaki proved several results on the structure of the $X^{(i)}$ as modules over the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(K/k)]]$, which are particularly interesting when the Iwasawa μ-invariant of K/k is trivial.

Theorem 2.2.

(i) *For all $i \geq 1$ and all $n \geq 0$ we have $X_n^{(i)} \simeq A(L_n^{(i-1)})_{G_n^{(i-1)}}$ (i.e., the $G_n^{(i-1)}$-coinvariants of $A(L_n^{(i-1)})$).*
(ii) *For all $i \geq 1$ and $m \geq n \gg 0$, the natural restriction maps $X_m^{(i)} \to X_n^{(i)}$ are surjective and $\varprojlim_n X_n^{(i)} \simeq X^{(i)}$.*
(iii) *For all $i \geq 1$, $X^{(i)}$ has a natural Λ-module structure.*

(iv) *Let* $\mu(K/k)=0$. *Then, for any* $i \geq 1$, $X^{(i)}$ *is a finitely generated* $\mathbb{Z}_p$*-module, and in particular, a finitely generated torsion* Λ*-module.*

(v) *Let* $\mu(K/k)=0$. *Then, for any* $i \geq 1$, *there exist integers* $\lambda^{(i)} := \lambda^{(i)}(K/k)$ *and* $\nu^{(i)}(K/k)$ *(independent of* n*) such that, for all* $n \gg 0$,

$$|X_n^{(i)}| = p^{\lambda^{(i)}(K/k)n+\nu^{(i)}(K/k)}.$$

Moreover, $\lambda^{(i)}$ *is the* $\mathbb{Z}_p$*-rank of* $X^{(i)}$.

Proof. See [O1, Lemmas 1 and 2, Proposition 1 and Theorem 1]. ∎

Definition 2.3. Let $\mu(K/k) = 0$. The invariant $\lambda^{(1)}$ is the familiar Iwasawa λ-invariant. For every $i \geq 1$ we will call $\lambda^{(i)}$ the *higher Iwasawa ith* λ*-invariant* for K/k. We write $\widetilde{\lambda}^{(i)} := \mathrm{rk}_p(X^{(i)}) = \dim_{\mathbb{F}_p}(X^{(i)}/pX^{(i)})$ (analogous notation $\widetilde{\lambda}_n^{(i)}$ for $\mathrm{rk}_p(X_n^{(i)})$).

The sequences $\{\widetilde{\lambda}_n^{(i)}\}_{n\in\mathbb{N}}$ are increasing (see [O1, Lemma 2]) and their behaviour only depends on $\mu(K/k)$ (in particular it is independent of i), while, as in classical Iwasawa theory, the stabilization of the orders of the $X_n^{(i)}$ yields a trivial $\lambda^{(i)}$-invariant.

Proposition 2.4.

(i) *If* $\mu(K/k) = 0$, *then* $\{\widetilde{\lambda}_n^{(i)}\}_{n\in\mathbb{N}}$ *is bounded for any* $i \geq 1$.

(ii) *If* $\mu(K/k) > 0$, *then* $\{\widetilde{\lambda}_n^{(i)}\}_{n\in\mathbb{N}}$ *diverges for any* $i \geq 1$.

Proof. (i) If $\mu(K/k) = 0$, then $X^{(i)}$ is a finitely generated $\mathbb{Z}_p$-module for any i (by Theorem 2.2). Then, for all $n \in \mathbb{N}$, we have $\widetilde{\lambda}_n^{(i)} \leq \widetilde{\lambda}^{(i)}$, which is finite.

(ii) Assume that for some $i \geq 1$ we have $\widetilde{\lambda}_n^{(i)} \leq t$ for all $n \in \mathbb{N}$. Now

$$\varprojlim_n X_n^{(i)}/pX_n^{(i)} = X^{(i)}/pX^{(i)},$$

and it is clear that $\widetilde{\lambda}^{(i)} \leq t$. This means that $X^{(i)}/\Phi(X^{(i)})$ is generated by (at most) t elements as an $\mathbb{F}_p$-vector space (where $\Phi(X^{(i)})$ is the Frattini subgroup of $X^{(i)}$). Thus, by the Burnside Basis Theorem, these t elements generate $X^{(i)}$ as a topological group, or, which is the same, as a $\mathbb{Z}_p$-module. This contradicts [O1, Proposition 2]. ∎

Before going into the details of the stabilization of the $|X_n^{(i)}|$ we give an example which shows that Theorem 1.1 is not immediately generalizable to the non-abelian setting.

2.1. Example. We first recall the following group-theoretical results (for the first one see, for example, [Ta, Section III]; we provide a short proof for the second for the convenience of the reader).

THEOREM 2.5. *Let G be a finite 2-group such that $G/D_1(G) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then $D_1(G)/D_2(G)$ is cyclic and $D_2(G) = 1$.*

REMARK 2.6. We recall the definition of nilpotent group. For a group G we set

$$[G, {}_iG] := [[\dots[[G, \underbrace{G], G], \dots], G}_{i \text{ times}}].$$

We say that G is *nilpotent of class* $i \geq 1$ if $[G, {}_iG] = 1$ and $[G, {}_{i-1}G] \neq 1$. If G is a topological group whose space is Hausdorff, the last conditions are equivalent to $C_i(G) = 1$ and $C_{i-1}(G) \neq 1$ (we shall use this equivalent definition).

PROPOSITION 2.7. *Let G be a pronilpotent group (i.e., the inverse limit of finite nilpotent groups) such that $G/C_2(G)$ is cyclic or procyclic. Then G is abelian.*

Proof. Let $H := G/C_3(G)$ and note that $C_2(H) = C_2(G)/C_3(G)$ is contained in $Z(H)$ (the centre of H). This means that $H/Z(H)$ is cyclic or procyclic, hence H is abelian. Therefore $C_2(G)/C_3(G) = C_2(H) = 1$, which yields $C_2(G) = C_3(G)$ (and $C_i(G) = C_2(G)$ for all $i \geq 2$). Since G is pronilpotent, it is clear that $\bigcap_{i=2}^{\infty} C_i(G) = 1$. Hence $C_2(G) = \bigcap_{i=2}^{\infty} C_i(G) = 1$, which means that G is abelian. ∎

Since "pro-p" implies "pronilpotent", the previous proposition shows that if there is a cyclic or procyclic quotient in the series (2), then the series stops there.

Fix $p = 2$ and consider the cyclotomic $\mathbb{Z}_2$-extension of the field $k = \mathbb{Q}(\sqrt{5 \cdot 732678913})$ as in [O2, Example 1]. Note that the prime 2 is inert in k, so ramification starts immediately, i.e., $n_0 = 0$. In the first layer we have $k_1 = k(\sqrt{2})$. Using PARI/GP we can see that $A_0 \simeq \mathbb{Z}/2\mathbb{Z}$, which yields $X_0^{(i)} = 0$ for every $i \geq 2$: indeed if the p-rank of $X_0^{(1)} = \widetilde{G}_0/C_2(\widetilde{G}_0)$ is 1, then $X_0^{(2)} = 0$ (by Proposition 2.7). Moreover,

$$k_1 \simeq \frac{\mathbb{Q}[x]}{(x^4 - 7326789134x^2 + 13420459724217960969)}$$

and $A_1 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Now let $H := \widetilde{G}_1/D_3(\widetilde{G}_1)$. Since $H/D_1(H) \simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, Theorem 2.5 implies that $D_2(H) = 1$, which yields $D_2(\widetilde{G}_1) = 1$. This means that $\widetilde{G}_1$ is nilpotent of class i_1 (for some $i_1 \geq 2$) and $X_1^{(i)} = 0$ for all $i \geq i_1$ (in other words $\widetilde{G}_1$ has derived length at most 2 and it is nilpotent of class i_1).

Going on with the computations, one finds the 4th layer k_4 with $A_4/A_4^2 \simeq (\mathbb{Z}/2\mathbb{Z})^{16}$. Now we denote by ρ the number of generators of the group of units of k_4 (since k_4 is totally real and has degree 2^5 over $\mathbb{Q}$, $\rho = 2^5$ as well). We have

$$\dim_{\mathbb{F}_2} A_4 = 16 \geq 2 + 2\sqrt{\rho+2} = 2 + 2\sqrt{2^5+2} \approx 13.662.$$

Thus by the Golod–Shafarevich inequality (see, for example, [NSW, Theorem 10.10.5], which improves a little the bound of the original article [GS]), we find that k_4 has an infinite Hilbert 2-class field tower, hence, a fortiori, the lower central series has infinite length, i.e., $X_4^{(i)} \neq 0$ for any $i \geq 1$.

Summing up we have $X_0^{(i_1)} = X_1^{(i_1)} = 0$, but certainly $X_4^{(i_1)} \neq 0$; this gives us a counterexample (for the sequence of the p-ranks as well) to a direct generalization of Theorem 1.1 to $X^{(i)}$ for $i \geq 2$.

2.2. Stabilization. We have to add other hypotheses to obtain the stabilization of the orders or of the p-ranks of the $X_m^{(i)}$.

To simplify the notation we assume here $n_0(K/k) = 0$, i.e., every ramified prime in the $\mathbb{Z}_p$-extension K/k is totally ramified (but, as usual in Iwasawa theory, everything can be proved for a general n_0). For any prime $\mathfrak{p}_i$ of k which ramifies in K/k, fix $\tilde{\mathfrak{p}}_i$ in $\widetilde{L}$ lying above $\mathfrak{p}_i$, and let $I(\tilde{\mathfrak{p}}_1), \ldots, I(\tilde{\mathfrak{p}}_s) \subseteq \mathcal{G} = \mathrm{Gal}(\widetilde{L}/k)$ be their inertia groups. Note that the natural restriction $\mathcal{G} \to \mathrm{Gal}(K/k)$ induces isomorphisms $I(\tilde{\mathfrak{p}}_j) \simeq \mathrm{Gal}(K/k)$ for every $1 \leq j \leq s$. Fix a topological generator γ of $I(\tilde{\mathfrak{p}}_1)$. For any $2 \leq j \leq s$, there exist $g_j \in \widetilde{G}$ such that γg_j is a topological generator of $I(\tilde{\mathfrak{p}}_j)$.

Definition 2.8. Let g be any element of $\widetilde{G}$. For every $m \geq n \geq 0$ we define

$$\begin{aligned}\nu_{n,m}(g) = g^{\nu_{n,m}} &:= g^{\gamma^{(p^{m-n}-1)p^n} + \gamma^{(p^{m-n}-2)p^n} + \cdots + \gamma^{p^n} + 1} \\ &= \prod_{i=1}^{p^{m-n}} \gamma^{-(p^{m-n}-i)p^n} g \gamma^{(p^{m-n}-i)p^n}.\end{aligned}$$

As in the classical theory, we write simply ν_n to indicate $\nu_{0,n}$, i.e.,

$$\begin{aligned}\nu_n(g) = g^{\nu_n} &= g^{\gamma^{(p^n-1)} + \gamma^{(p^n-2)} + \cdots + \gamma + 1} \\ &= (\gamma^{-(p^n-1)} g \gamma^{p^n-1}) \cdot (\gamma^{-(p^n-2)} g \gamma^{p^n-2}) \cdot \ldots \cdot (\gamma^{-1} g \gamma) \cdot g\end{aligned}$$

(for any $g \in \widetilde{G}$).

Remark 2.9. Note that $\widetilde{G}$ is not a Λ-module (in general) because the "+" in the exponent is not commutative.

Definition 2.10. For every $n \geq 0$ and $i \geq 1$, we set

(i) $Y_n^{(i)} := \mathrm{Ker}\{\mathrm{res} : X^{(i)} \to X_n^{(i)}\}$, where res denotes the natural restriction map;

(ii) $\widetilde{R}_n := (\nu_n([\gamma, x]), \nu_n(g_j) : x \in \widetilde{G},\ 2 \le j \le s)_{\widetilde{G}}$, where $(\cdot)_{\widetilde{G}}$ stands for a normally generated closed subgroup in $\widetilde{G}$;

(iii) $R_n^{(i)} := (\nu_n([\gamma, x]), \nu_n(g_j) : x \in G^{(i)},\ 2 \le j \le s)_{G^{(i)}}$.

PROPOSITION 2.11. *For every $n \ge 0$ and $i \ge 1$:*

(i) $\widetilde{G}_n \simeq \widetilde{G}/\widetilde{R}_n$;

(ii) $G_n^{(i)} \simeq G^{(i)}/R_n^{(i)}$;

(iii) $Y_n^{(i)} = (C_i(\widetilde{G}) \cap \widetilde{R}_n)C_{i+1}(\widetilde{G})/C_{i+1}(\widetilde{G}) = X^{(i)} \cap R_n^{(i)}$.

Proof. See [O1, Lemma 3]. Note also that, if $i = 1$, then the characterization of $Y_n^{(i)}$ is consistent with the one of Y_n in the classical context (see, for example, [Wa, Ch. 13]). ∎

Now we focus our attention on the first i columns of the tower, i.e., we consider $G^{(i)} = \widetilde{G}/C_{i+1}(\widetilde{G})$ in place of $\widetilde{G}$. We continue to use only γ and g_j to indicate the cosets $\gamma C_{i+1}(\widetilde{G})$ and $g_j C_{i+1}(\widetilde{G})$ in $G^{(i)}$.

LEMMA 2.12. *For any $g \in \widetilde{G}$ and any $m \ge n \ge 0$ we have*

$$(g^{\nu_n})^{\nu_{n,m}} = g^{\nu_m}.$$

Proof. Direct computation shows that

$$\begin{aligned}(g^{\nu_n})^{\nu_{n,m}} &= (\nu_n(g))^{\gamma^{(p^{m-n}-1)p^n} + \gamma^{(p^{m-n}-2)p^n} + \cdots + \gamma^{1\cdot p^n} + 1} \\ &= \prod_{\varepsilon = p^{m-n}-1}^{0} \gamma^{-\varepsilon p^n} \cdot \nu_n(g) \cdot \gamma^{\varepsilon p^n} = \nu_m(g) = g^{\nu_m}. \ ∎\end{aligned}$$

THEOREM 2.13. *Let $i \ge 1$ and $n \ge 0$. If $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ for all $1 \le j \le i$, then $X_n^{(j)} \simeq X_m^{(j)} \simeq X^{(j)}$ and $Y_m^{(j)} = 0$ for every $1 \le j \le i$ and for all $m \ge n$.*

Proof. We use induction on i.

If $i = 1$, Theorem 1.1 yields $Y_n^{(1)} = 0$ and $X_n^{(1)} \simeq X_m^{(1)}$ for all $m \ge n$.

Now we assume that the statement is true for $i \ge 1$ and prove the $i+1$ case. First, note that if $m \ge n$, then $k_m \cap L_n^{(i)} = k_n$ since k_m/k_n is totally ramified and $L_n^{(i)}/k_n$ is unramified (we recall that $n_0 = 0$ here). Thus

$$\mathrm{Gal}(k_m L_n^{(i)}/k_m) \simeq \mathrm{Gal}(L_n^{(i)}/k_n) \simeq G_n^{(i)}.$$

This means that the natural restriction induces an epimorphism

$$G_m^{(i)} \simeq \mathrm{Gal}(L_m^{(i)}/k_m) \twoheadrightarrow \mathrm{Gal}(k_m L_n^{(i)}/k_m) \simeq G_n^{(i)}. \tag{3}$$

By inductive hypothesis,

$$|G_m^{(i)}| = |X_m^{(1)}| \cdot \ldots \cdot |X_m^{(i)}| = |X_n^{(1)}| \cdot \ldots \cdot |X_n^{(i)}| = |G_n^{(i)}|,$$

so (3) is an isomorphism. Note that $K \cap L_m^{(i)} = k_m$ yields $\mathrm{Gal}(L_m^{(i)}/k_m) \simeq$

$\mathrm{Gal}(KL_m^{(i)}/K)$, and from [O1, Lemma 2(2)] we have

$$\varprojlim_m G_m^{(i)} \simeq \varprojlim_m \mathrm{Gal}(L_m^{(i)}/k_m) \simeq \varprojlim_m \mathrm{Gal}(KL_m^{(i)}/K) \simeq \mathrm{Gal}\Big(\bigcup_{m\in\mathbb{N}} KL_m^{(i)}/K\Big) \simeq \mathrm{Gal}(L^{(i)}/K) \simeq G^{(i)}.$$

The isomorphism in (3) implies that

$$\mathrm{Gal}(KL_n^{(i)}/K) \simeq G_n^{(i)} \simeq \varprojlim_m G_m^{(i)} \simeq G^{(i)} \simeq \mathrm{Gal}(L^{(i)}/K),$$

and this leads to

$$KL_n^{(i)} = L^{(i)}. \tag{4}$$

Now recall that $Y_n^{(i+1)} = X^{(i+1)} \cap R_n^{(i+1)}$ and $G_n^{(i+1)} \simeq G^{(i+1)}/R_n^{(i+1)}$. So we have the isomorphisms

$$\mathrm{Gal}(KL_n^{(i+1)}/K) \simeq \mathrm{Gal}(L_n^{(i+1)}/k_n) \simeq G_n^{(i+1)} \simeq G^{(i+1)}/R_n^{(i+1)}, \tag{5}$$

and from (4),

$$\mathrm{Gal}(KL_n^{(i)}/K) = \mathrm{Gal}(L^{(i)}/K) \simeq G^{(i)} = \widetilde{G}/C_{i+1}(\widetilde{G}) \simeq G^{(i+1)}/X^{(i+1)}. \tag{6}$$

By comparing (5) and (6) we obtain $R_n^{(i+1)} \subseteq X^{(i+1)}$, which yields $R_n^{(i+1)} = Y_n^{(i+1)}$. Since $X^{(i+1)} \subseteq Z(G^{(i+1)})$ (where Z denotes the centre of the group), we can take away the "normally generated" condition in the definition of $R_n^{(i+1)}$, i.e.,

$$R_n^{(i+1)} := (\nu_n([\gamma,x]), \nu_n(g_j) : x \in G^{(i+1)},\ 2 \le j \le s)_{G^{(i+1)}} = \langle \nu_n([\gamma,x]), \nu_n(g_j) : x \in G^{(i+1)},\ 2 \le j \le s\rangle_{G^{(i+1)}},$$

where $\langle\cdot\rangle$ stands for topologically generated subgroup. We also recall that $Y_n^{(i+1)}$ is a Λ-submodule of $X^{(i+1)}$ (as the kernel of a homomorphism or because $Y_n^{(i+1)} = \mathrm{Gal}(L^{(i+1)}/L^{(i)}L_n^{(i+1)})$ is a normal subgroup of $\mathrm{Gal}(L^{(i+1)}/k)$), so we have the following equality between Λ-modules (recall also Lemma 2.12):

$$\nu_{n,n+1} Y_n^{(i+1)} = \nu_{n,n+1} R_n^{(i+1)} = R_{n+1}^{(i+1)} = Y_{n+1}^{(i+1)}.$$

Our final hypothesis $|X_n^{(i+1)}| = |X_{n+1}^{(i+1)}|$ implies that the natural restriction $X_{n+1}^{(i+1)} \twoheadrightarrow X_n^{(i+1)}$ is an isomorphism, hence $Y_{n+1}^{(i+1)} = Y_n^{(i+1)}$ as well.

Thus Nakayama's Lemma yields $Y_n^{(i+1)} = R_n^{(i+1)} = 0$, and the theorem follows. ∎

Remark 2.14. (1) The hypothesis $G_n^{(i)} \simeq G_{n+1}^{(i)}$ (or $R_n^{(i)} = R_{n+1}^{(i)}$, which is the same) is equivalent to those of the above theorem, i.e., $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ for $1 \le j \le i$.

In the same way it is easy to see that the condition $\widetilde{G}_n \simeq \widetilde{G}_{n+1}$ (or $\widetilde{R}_n = \widetilde{R}_{n+1}$) is equivalent to requiring $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ for any $j \ge 1$.

(2) The equivalence $R_n^{(i+1)} \subseteq X^{(i+1)} \Leftrightarrow R_n^{(i)} = 0$ can also be proved easily by group theory, noting that

$$\begin{aligned} R_n^{(i)} &= \pi(R_n^{(i+1)}) = (R_n^{(i+1)} \cdot C_{(i+1)}(G^{(i+1)}))/C_{(i+1)}(G^{(i+1)}) \\ &\simeq R_n^{(i+1)}/(R_n^{(i+1)} \cap C_{(i+1)}(G^{(i+1)})) = R_n^{(i+1)}/(R_n^{(i+1)} \cap X^{(i+1)}) \end{aligned}$$

(where $\pi : G^{(i+1)} \to G^{(i)}$ is the natural projection).

(3) From the proof of Theorem 2.13 one can also deduce (among others) the following facts:

(i) $k_m L_n^{(j)} = L_m^{(j)}$ for all $m \geq n$ and any $1 \leq j \leq i$;
(ii) $KL_n^{(j)} = L^{(j)}$ for any $1 \leq j \leq i$;
(iii) $L^{(j)}/L_n^{(j)}$ is a $\mathbb{Z}_p$-extension of the number field $L_n^{(j)}$, and for any $1 \leq j \leq i$ the fixed field of $\mathrm{Gal}(L^{(j)}/L_n^{(j)})^{p^h}$ is $L_{n+h}^{(j)}$.

We end this section with a slightly different version of Theorem 2.13 (recall that $A(E)$ denotes the p-part of the class group of the number field E).

PROPOSITION 2.15. *Assume that $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ for all $1 \leq j \leq i-1$ and that $A(L_{n+1}^{(i-1)}) \simeq A(L_n^{(i-1)})$. Then $X_n^{(j)} \simeq X_m^{(j)} \simeq X^{(j)}$ for every $1 \leq j \leq i$ and any $m \geq n$.*

Proof. Recall that $X_n^{(i)} \simeq A(L_n^{(i-1)})_{G_n^{(i-1)}}$, i.e., $X_n^{(i)}$ is isomorphic to the maximal quotient of $A(L_n^{(i-1)})$ on which $G_n^{(i-1)} = \mathrm{Gal}(L_n^{(i-1)}/k_n)$ acts trivially (see Theorem 2.2). Since $L_n^{(i-1)} \cap k_{n+1} = k_n$, we have a surjection

$$\mathrm{Gal}(L_{n+1}^{(i-1)}/k_{n+1}) \twoheadrightarrow \mathrm{Gal}(k_{n+1}L_n^{(i-1)}/k_{n+1}) \simeq \mathrm{Gal}(L_n^{(i-1)}/k_n)$$

(i.e., $G_{n+1}^{(i-1)} \twoheadrightarrow G_n^{(i-1)}$) given by the natural restriction maps. The first hypothesis yields $G_{n+1}^{(i-1)} \simeq G_n^{(i-1)}$, so $A(L_{n+1}^{(i-1)}) \simeq A(L_n^{(i-1)})$ immediately implies $X_n^{(i)} \simeq X_{n+1}^{(i)}$ and we can apply Theorem 2.13. ∎

3. On the p-rank of $X_n^{(i)}$.

The stabilization of p-ranks turns out to be much more complicated to obtain. Up to now we can only prove it under rather stringent hypotheses similar to those of Theorem 2.13 (in particular they yield $\lambda^{(i)} = 0$ for any i). Recall that $\widetilde{\lambda}_n^{(i)} := \mathrm{rk}_p(X_n^{(i)})$.

THEOREM 3.1. *Suppose that $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ for all $1 \leq j \leq i$, and $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_{n+1}^{(i+1)}$. Then $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_m^{(i+1)} = \widetilde{\lambda}^{(i+1)}$ for any $m \geq n$.*

Proof. In the proof of Theorem 2.13 we have already seen that the hypothesis on the orders implies

$$R_n^{(i+1)} \subseteq X^{(i+1)} \quad \text{and} \quad Y_m^{(i+1)} = \nu_{n,m} Y_n^{(i+1)}$$

for all $m \geq n$. Moreover, from the hypothesis $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_{n+1}^{(i+1)}$, we have

$$X^{(i+1)}/(Y_n^{(i+1)} + pX^{(i+1)}) = X^{(i+1)}/(\nu_{n,n+1}Y_n^{(i+1)} + pX^{(i+1)}).$$

Then $Y_n^{(i+1)} + pX^{(i+1)} = \nu_{n,n+1}Y_n^{(i+1)} + pX^{(i+1)}$ and, moding out $pX^{(i+1)}$, we obtain

$$(Y_n^{(i+1)} + pX^{(i+1)})/pX^{(i+1)} = \nu_{n,n+1}(Y_n^{(i+1)} + pX^{(i+1)}/pX^{(i+1)}).$$

Nakayama's Lemma yields

$$Y_n^{(i+1)} + pX^{(i+1)}/pX^{(i+1)} = 0, \quad \text{i.e.,} \quad Y_n^{(i+1)} \subseteq pX^{(i+1)}.$$

Thus $Y_m^{(i+1)} \subseteq pX^{(i+1)}$ and $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_m^{(i+1)} = \widetilde{\lambda}^{(i+1)}$ for all $m \geq n$. ∎

Remark 3.2. Note that assuming $|X_n^{(1)}| = |X_{n+1}^{(1)}|$ (as in the above theorem) yields $\lambda^{(1)} = 0$, and consequently $\lambda^{(j)} = 0$ for all $j \geq 1$ (see [O1, Proposition 3]). Of course it is still possible for $\widetilde{\lambda}^{(j)}$ to be non-zero, but the applications of the theorem are limited by this rather strong hypothesis (which was natural for the stabilization of orders).

As seen in Proposition 2.15, if we replace the hypothesis $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_{n+1}^{(i+1)}$ of the previous theorem with the corresponding one on class groups, i.e., $\mathrm{rk}_p(A(L_{n+1}^{(i)})) = \mathrm{rk}_p(A(L_n^{(i)}))$, then the claim remains true.

Proposition 3.3. *Assume that* $\mathrm{rk}_p(A(L_{n+1}^{(i)})) = \mathrm{rk}_p(A(L_n^{(i)}))$ *and* $|X_n^{(j)}| = |X_{n+1}^{(j)}|$ *for all* $1 \leq j \leq i$. *Then* $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_m^{(i+1)} = \widetilde{\lambda}^{(i+1)}$ *for all* $m \geq n$.

Proof. Let

$$B_{n+1}^{(i)} := \langle\, a^{g-1} : g \in G_{n+1}^{(i)},\ a \in A(L_{n+1}^{(i)}) \rangle \subseteq A(L_{n+1}^{(i)}).$$

The hypothesis on the orders implies that $\mathrm{Gal}(L_{n+1}^{(i)}/L_n^{(i)}) \simeq \mathrm{Gal}(k_{n+1}/k_n)$ (see also Remark 2.14(3)). Thus the action of g commutes with the norm $N_{L_{n+1}^{(i)}/L_n^{(i)}}$ and one gets $N_{L_{n+1}^{(i)}/L_n^{(i)}}(B_{n+1}^{(i)}) = B_n^{(i)}$. Hence

$$N_{L_{n+1}^{(i)}/L_n^{(i)}}(pA(L_{n+1}^{(i)}) + B_{n+1}^{(i)}) = pA(L_n^{(i)}) + B_n^{(i)}.$$

By the hypothesis on ranks, the norm map $N_{L_{n+1}^{(i)}/L_n^{(i)}} : A(L_{n+1}^{(i)}) \to A(L_n^{(i)})$ induces an isomorphism

$$\overline{N} : A(L_{n+1}^{(i)})/pA(L_{n+1}^{(i)}) \xrightarrow{\sim} A(L_n^{(i)})/pA(L_n^{(i)}),$$

which yields

$$A(L_{n+1}^{(i)})/(pA(L_{n+1}^{(i)}) + B_{n+1}^{(i)}) \xrightarrow{\sim} A(L_n^{(i)})/(pA(L_n^{(i)}) + B_n^{(i)}).$$

But $A(L_n^{(i)})/(pA(L_n^{(i)}) + B_n^{(i)})$ is canonically isomorphic to $X_n^{(i+1)}/pX_n^{(i+1)}$, so $\widetilde{\lambda}_n^{(i+1)} = \widetilde{\lambda}_{n+1}^{(i+1)}$ and we can apply Theorem 3.1. ∎

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