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On the number of rational points of Jacobians over finite fields

by

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1. Introduction

1.1. Notation. We introduce the following notation:

X	a smooth projective absolutely irreducible curve over $\mathbb{F}_q$,
g	the genus of X,
K	the function field of X,
Φ_{q^f} or B_f	the number of places of K of degree f,
h	the class number of X (the number of $\mathbb{F}_q$-points of $\mathrm{Jac}(X)$),
$Z_X(T)$	the zeta function of X which is a rational function of T,
$\omega_i\sqrt{q}$	the inverse roots of the numerator of $Z_X(T)$,
κ	the residue of $Z_X(q^{-s}) = \zeta_X(s)$ at $s = 1$,
log	the Neperian logarithm $\log_e$.

By a *curve* we always mean a smooth projective absolutely irreducible curve.

1.2. Existing lower bounds for the class number. Our goal is to provide estimates for the number of rational points on the Jacobian of a smooth projective curve that use the information on the number of points on this curve defined over $\mathbb{F}_q$ or over its extensions. The starting point for all such estimates is the interpretation of the class number as the value at 1 of the numerator of the zeta function of the curve. In order to estimate it, one uses properties of the zeta function such as its functional equation, and the Riemann Hypothesis (Weil bounds).

From the work of Weil, we know that the class number h of a smooth projective absolutely irreducible curve X of genus g defined over $\mathbb{F}_q$ is bounded

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as follows:

$$(\sqrt{q}-1)^{2g} \le h \le (\sqrt{q}+1)^{2g}.$$

Considerable effort has been devoted to sharpening these bounds. Let us cite some work in this direction. Lachaud and Martin-Deschamps [LMD] first obtained the lower bound

$$h \ge h_{\mathrm{LMD}} = q^{g-1}\frac{(q-1)^2}{(q+1)(g+1)},$$

using a formula which is a consequence of the functional equation for the zeta function:

$$h = \frac{\sum_{n=0}^{g-1} A_n + \sum_{n=0}^{g-2} q^{g-1-n}A_n}{\sum_{i=1}^{g} |1-\omega_i\sqrt{q}|^2},$$

where A_n is the number of effective divisors of degree n on X. Ever since, methods from combinatorics were used to give good bounds for the numerator and the denominator of this fraction.

In [BR], [BRT], Ballet, Rolland, and Tutdere used this approach in order to prove rather elaborate lower bounds on h. Some of these bounds turn out to be asymptotically optimal when $g \to \infty$, meaning that they converge to the lower bound from the generalized Brauer–Siegel theorem for function fields ([TVN], see also Remark 2.8). The best of their lower bounds is given by the following theorem:

THEOREM 1.1 (Ballet–Rolland–Tutdere). *Let $X/\mathbb{F}_q$ be a curve defined over $\mathbb{F}_q$ of genus $g \ge 2$ and of class number h. Let D_1, D_2 be finite sets of integers, $(\ell_r)_{r\in D_1}$, $(m_r)_{r\in D_2}$ be families of integers such that:*

(1) $D_1 \subseteq \{1,\dots,g-1\}$;
(2) $D_2 \subseteq \{1,\dots,g-2\}$;
(3) *for any* $r \in D_1$, $\Phi_{q^r} \ge 1$;
(4) *for any* $r \in D_2$, $\Phi_{q^r} \ge 1$;
(5) $l_r \ge 0$ *and* $\sum_{r\in D_1} r\ell_r \le g-1$;
(6) $m_r \ge 0$ *and* $\sum_{r\in D_2} rm_r \le g-2$.

Then $h \ge h_{\mathrm{BRT}}$ with

$$h_{\mathrm{BRT}} = \frac{(q-1)^2}{(g+1)(q+1)-\Phi_q}\Bigg(\prod_{r\in D_1}\binom{\Phi_{q^r}+\ell_r}{\ell_r} + q^g\prod_{r\in D_2}\left[\left(\frac{q^r}{q^r-1}\right)^{\phi_{q^r}} - \Phi_{q^r}\binom{\Phi_{q^r}+m_r}{m_r}\int_0^{q^{-r}}\frac{(q^{-r}-t)^{m_r}}{(1-t)^{\Phi_{q^r}+m_r+1}}\,dt\right]\Bigg).$$

From now on we denote by h_{BRT} the best possible lower bound from this theorem, that is, the one with an optimal choice of $D_1, D_2, (\ell_r)_{r\in D_1}$, and $(m_r)_{r\in D_2}$.

In a recent article dealing with estimates for the number of points on general abelian varieties, Aubry, Haloui, and Lauchaud [AHL] obtained certain lower bounds on class numbers that can be very sharp when the curve in question has many rational points compared to its genus. However, these bounds are all rather poor from the asymptotic point of view when $g \to \infty$. Let us recall their results concerning the Jacobian of curves.

THEOREM 1.2 (Aubry–Haloui–Lachaud). *For a smooth absolutely irreducible projective curve X defined over $\mathbb{F}_q$ of genus $g \geq 2$ and of class number h we have:*

(1) $h \geq M(q)^g \left(q+1+\dfrac{\Phi_q-(q+1)}{g} \right)^g$ *with*

$$M(q) = \frac{e \log x^{1/x-1}}{x^{1/x}-1}, \quad x = \left(\frac{\sqrt{q}+1}{\sqrt{q}-1} \right)^2.$$

(2) $h \geq \dfrac{q-1}{q^g-1} \left[\dbinom{\Phi_q+2g-2}{2g-1} + \displaystyle\sum_{r=2}^{2g-1} \Phi_{q^r} \dbinom{\Phi_q+2g-2-r}{2g-1-i} \right].$

(3) *If* $\Phi_q \geq g(\sqrt{q}-1)+1$ *then*

$$h \geq \binom{\Phi_q+g-1}{g} - q \binom{\Phi_q+g-3}{g-2}.$$

(4) $h \geq \dfrac{(q-1)^2}{(g+1)(q+1)-\Phi_q} \left[\dbinom{\Phi_q+g-2}{g-2} + \displaystyle\sum_{r=0}^{g-1} q^{g-1-1} \dbinom{\Phi_q+r-1}{r} \right].$

We denote by h_{AHL} the best possible lower bound for h given by (1)–(4) of this theorem. We remark that the estimate (3) can be very sharp when g is small and Φ_q is large. We will come back to that in §3.

1.3. The aim of this paper is to show how the Mertens theorem and the explicit Brauer–Siegel theorem lead to improvements of these bounds in many cases, most notably when g is large. This is done in §2 (Corollary 2.5). To do so we use the asymptotic theory of global fields, and more precisely the technique of explicit formulae. The third section is devoted to numerical experiments. We compare the bounds in several examples provided by recursive asymptotically good towers of function fields. Finally, in the fourth section we discuss further research directions and open problems.

2. Explicit formulae and their link to class numbers

2.1. Explicit formulae. Our starting point is the Mertens theorem [L] for curves and its relation to the generalized Brauer–Siegel theorem. Our exposition differs slightly from [L]: we take the opportunity to sharpen (and sometimes correct) the corresponding bounds.

Let us recall Serre's explicit formulae from [S].

THEOREM 2.1 (Explicit formula). *For any sequence (v_n) such that the radius of convergence ρ of the series $\sum v_n t^n$ is strictly positive, define $\psi_{m,v}(t) = \sum_{n=1}^{\infty} v_{mn} t^{mn}$, and $\psi_v(t) = \psi_{1,v}(t)$. Then for $t < q^{-1}\rho$, we have the explicit formula*

$$\sum_{f=1}^{\infty} f\Phi_{q^f}\psi_{f,v}(t) = \psi_v(t) + \psi_v(qt) - \sum_{j=1}^{2g} \psi_v(\sqrt{q}\,\omega_j t).$$

We choose $N \in \mathbb{N}$, and take $v_n = 1/n$ if $n \le N$ and 0 otherwise. Applying Theorem 2.1 with $t = q^{-1}$, we obtain the identity

$$S_0(N) = S_1(N) + S_2(N) + S_3(N),$$

where

$$S_0(N) = \sum_{n=1}^{N} n^{-1}q^{-n} \sum_{m|n} m\Phi_{q^m} = \sum_{f=1}^{N} \frac{1}{fq^f} |X(\mathbb{F}_{q^f})|,$$

$$S_1(N) = \sum_{n=1}^{N} \frac{1}{n}, \qquad S_2(N) = \sum_{n=1}^{N} \frac{1}{nq^n}, \qquad S_3(N) = -\sum_{j=1}^{2g}\sum_{n=1}^{N} \frac{1}{n}(q^{-1/2}\omega_j)^n.$$

We transform it in order to make the desired quantities appear. For any $N \ge 1$,

$$\begin{aligned}
&\underbrace{S_0 - \sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f-1}\right)}_{\varepsilon_0(N)} + \sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f-1}\right) \\
&\qquad = S_1 + \underbrace{S_2 - \log\frac{q}{q-1}}_{\varepsilon_2(N)} + \log\frac{q}{q-1} \\
&\qquad\quad + \underbrace{S_3 - \log(\kappa\log q) + \log\frac{q}{q-1}}_{\varepsilon_3(N)} + \log(\kappa\log q) - \log\frac{q}{q-1}.
\end{aligned}$$

To get bounds for h we will not need estimates on $\varepsilon_0(N)$ and $\varepsilon_2(N)$, but they are useful for proving the Mertens theorem recalled later.

LEMMA 2.2. *We have the following bounds for $\varepsilon_i(N)$:*

$$-\frac{c_1(q)}{Nq^{N/2}} - \frac{c_2(q)g}{Nq^{3N/4}} \le \varepsilon_0(N) \le 0,$$

$$-\frac{1}{(q-1)(N+1)q^N} \le \varepsilon_2(N) \le 0, \qquad 0 \le |\varepsilon_3(N)| \le \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}},$$

with

$$c_1(q) = \frac{2q(q+1)}{(q-1)^2} \le 12 \quad \textit{and} \quad c_2(q) = \frac{2q}{q-1}\left(\frac{\sqrt{q}}{\sqrt{q}-1} + \frac{q^{3/2}}{q^{3/2}-1}\right) \le 20.$$

Proof. The following inequalities hold for $|x| > 1$ and $N > 0$:

$$\left|\log\left(\frac{x}{x-1}\right) - \sum_{n=1}^{N} \frac{1}{nx^n}\right| = \left|\sum_{n=N+1}^{\infty} \frac{1}{nx^n}\right| \le \frac{1}{(N+1)|x|^{N+1}} \sum_{n=0}^{\infty} \frac{1}{|x|^n}$$
$$\le \frac{1}{(N+1)|x|^N(|x|-1)}.$$

This implies the bounds for $\varepsilon_2(N)$.

The one for $\varepsilon_3(N)$ is derived from the classical formula [TVN, Corollary 3.1.13]

$$\log(\kappa \log q) - \log \frac{q}{q-1} = \sum_{i=1}^{2g} \log\left(1 - \frac{\omega_j}{\sqrt{q}}\right).$$

It gives

$$|\varepsilon_3(N)| = \left|-\sum_{j=1}^{2g}\sum_{n=1}^{N} \frac{1}{n}(q^{-1/2}\omega_j)^n - \log(\kappa \log q) + \log \frac{q}{q-1}\right|$$
$$= \left|\sum_{j=1}^{2g}\left(-\log\left(1 - \frac{\omega_j}{\sqrt{q}}\right) - \sum_{n=1}^{N} \frac{1}{n}\left(\frac{\omega_j}{\sqrt{q}}\right)^n\right)\right|,$$

and since $|\omega_j| = 1$, we have

$$|\varepsilon_3(N)| \le \sum_{j=1}^{2g} \frac{1}{(N+1)\sqrt{q}^N|\sqrt{q} - \omega_j|} \le \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}.$$

We finally estimate $\varepsilon_0(N)$ along the lines of [L, proof of Lemma 2]. We first transform the expression for S_0:

$$S_0(N) = \sum_{f=1}^{N} f\Phi_{q^f} \sum_{m=1}^{[N/f]} q^{-fm}(fm)^{-1} = \sum_{f=1}^{N} \Phi_{q^f} \sum_{m=1}^{[N/f]} \frac{1}{q^{fm}m}.$$

Thus,

$$\varepsilon_0(N) = S_0(N) - \sum_{f=1}^{N} \Phi_{q^f} \log \frac{q^f}{q^f-1} = -\sum_{f=1}^{N} \Phi_{q^f}\left(\log \frac{q^f}{q^f-1} - \sum_{m=1}^{[N/f]} \frac{1}{q^{fm}m}\right)$$
$$= -\sum_{f=1}^{N} \Phi_{q^f} \sum_{m=[N/f]+1}^{\infty} \frac{1}{q^{fm}m}.$$

As $\frac{1}{m} \le \frac{1}{[N/f]+1}$, we get

$$0 \le -\varepsilon_0(N) \le \sum_{f=1}^{N} \frac{\Phi_{q^f}}{([N/f]+1)q^{f[N/f]}(q^f-1)}.$$

To estimate Φ_{q^f} we use $\Phi_{q^f} \le \frac{q^f+1+2gq^{f/2}}{f}$. Thus

$$0 \le -\varepsilon_0(N) \le \frac{1}{N}\sum_{f=1}^{N} \frac{q^f+1+2gq^{f/2}}{(q^f-1)q^{f[N/f]}}.$$

We split the last sum in two, using the fact that for $f > [N/2]$ we have $[N/f] = 1$, and for $f \le [N/2]$ we have $f[N/f] \ge N - f$:

$$\begin{aligned}
-\varepsilon_0(N) &\le \frac{1}{N}\sum_{f=1}^{[N/2]} \frac{q^f+1+2gq^{f/2}}{q^{N-f}(q^f-1)} + \frac{1}{N}\sum_{f=[N/2]+1}^{N} \frac{q^f+1+2gq^{f/2}}{q^f(q^f-1)} \\
&\le \frac{1}{N}\biggl(\sum_{f=1}^{[N/2]} \frac{q^f+1}{q^f-1}q^{f-N} + \sum_{f=[N/2]+1}^{N} \frac{q^f+1}{q^f-1}q^{-f}\biggr) \\
&\quad + \frac{2g}{N}\biggl(\sum_{f=1}^{[N/2]} \frac{q^f}{q^f-1}q^{f/2-N} + \sum_{f=[N/2]+1}^{N} \frac{q^f}{q^f-1}q^{-3f/2}\biggr) \\
&\le \frac{q+1}{(q-1)N}\biggl(\sum_{f=1}^{[N/2]} q^{f-N} + \sum_{f=[N/2]+1}^{N} q^{-f}\biggr) \\
&\quad + \frac{2gq}{N(q-1)}\biggl(\sum_{f=1}^{[N/2]} q^{f/2-N} + \sum_{f=[N/2]+1}^{N} q^{-3f/2}\biggr) \\
&\le \frac{(q+1)(q^{-[N/2]-1}+q^{-N+[N/2]})}{(q-1)N(1-q^{-1})} \\
&\quad + \frac{2gq}{N(q-1)}\biggl(\frac{q^{-N+[N/2]/2}}{1-q^{-1/2}} + \frac{q^{-3([N/2]+1)/2}}{1-q^{-3/2}}\biggr) \\
&\le \frac{2(q+1)q}{(q-1)^2}\cdot\frac{1}{Nq^{N/2}} + \frac{2q}{q-1}\biggl(\frac{\sqrt{q}}{\sqrt{q}-1} + \frac{q^{3/2}}{q^{3/2}-1}\biggr)\frac{g}{Nq^{-3N/4}}. \ \blacksquare
\end{aligned}$$

REMARK 2.3. The bound for $\varepsilon_0(N)$ provides a correction to [L, Lemma 2], and the bound for $\varepsilon_3(N)$ corrects Lemma 5 there. It can be easily checked that these bounds are also valid in the more general situation of varieties over finite fields treated in [L].

2.2. Bounds for the class number. Using the calculations from the previous section and applying the class number formula

$$\kappa \log q = \frac{hq^{1-g}}{q-1},$$

we get the following theorem.

THEOREM 2.4. *Let X be a smooth projective absolutely irreducible curve defined over $\mathbb{F}_q$ of class number h. Then h is given by the following formula valid for any $N \geq 1$:*

$$\log h = g \log q + \sum_{f=1}^{N} \frac{1}{fq^f} |X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N} \frac{1+q^{-n}}{n} - \varepsilon_3(N),$$

or equivalently,

$$\log h = g \log q + \sum_{r=1}^{N} \left(\Phi_{q^r} \sum_{f=1}^{\lfloor N/r \rfloor} \frac{1}{fq^{rf}} \right) - \sum_{n=1}^{N} \frac{1+q^{-n}}{n} - \varepsilon_3(N),$$

where $\varepsilon_3(N)$ satisfies $|\varepsilon_3(N)| \leq \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}}$.

COROLLARY 2.5 (Bounds for the class number). *The number of rational points h on the Jacobian of X satisfies $h_{\min}(N) \leq h \leq h_{\max}(N)$, where*

$$h_{\min}(N) = q^g \exp\left(\sum_{f=1}^{N} \frac{1}{fq^f} |X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N} \frac{1+q^{-n}}{n} - \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}} \right),$$

$$h_{\max}(N) = q^g \exp\left(\sum_{f=1}^{N} \frac{1}{fq^f} |X(\mathbb{F}_{q^f})| - \sum_{n=1}^{N} \frac{1+q^{-n}}{n} + \frac{2g}{(\sqrt{q}-1)(N+1)q^{N/2}} \right).$$

REMARK 2.6. The knowledge of a given (small) number of Φ_{q^f}'s allows us, nevertheless, to apply Corollary 2.5 for any N. For example, in the case of lower bounds, one can bound from below the unknown Φ_{q^f} by 0, or by the quantities arising from the Weil bounds, depending on which one is better. We thus get a family of bounds parametrized by N, and we can choose the best one.

2.3. Mertens theorem and class numbers. Putting together estimates from Section 2.1, we find once again:

THEOREM 2.7 (Mertens theorem [L]). *Let X be a smooth projective absolutely irreducible curve of genus g defined over $\mathbb{F}_q$. Then*

$$\sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f-1} \right) = \log(\kappa \log q) - \varepsilon_0(N) + \varepsilon_2(N) + \varepsilon_3(N) - \sum_{n=1}^{N} \frac{1}{n}.$$

For any $N \geq 1$, we can deduce from this a weaker form of our bound, which might be easier to compare to Ballet–Rolland–Tutdere's bound:

$$\log h = g \log q + \left[\sum_{f=1}^{N} \Phi_{q^f} \log\left(\frac{q^f}{q^f - 1} \right) \right] - \sum_{n=1}^{N} \frac{1 + q^{-n}}{n} + \varepsilon_0(N) - \varepsilon_3(N).$$

REMARK 2.8. Theorem 2.7 implies that our bounds on h are asymptotically optimal. More precisely, recall that a family of curves $\{X_i\}$ over $\mathbb{F}_q$ of genus $g_i \to \infty$ is *asymptotically exact* if the limits

$$\phi_{q^r} = \lim_{i \to \infty} \frac{\Phi_{q^r}(X_i)}{g_i}$$

exist for all r. For asymptotically exact families of curves the generalized Brauer–Siegel theorem [TVN] states that

$$\lim_{i \to \infty} \frac{\log h(X_i)}{g_i} = \log q + \sum_{r=1}^{\infty} \phi_{q^r} \log\left(\frac{q^r}{q^r - 1} \right).$$

We see that when $g_i \to \infty$ and then $N \to \infty$, the bounds $h_{\min}(N)$ and $h_{\max}(N)$ from Corollary 2.5 divided by g_i converge to the right hand side of the above equality.

3. Numerical computations. In this section, we compare the lower bound $h_{\min}(N)$ given by Theorem 2.4 with h_{BRT} and h_{AHL} in the situation of recursive towers. We denote by h_{LZ} the bound from Theorem 2.4 for the optimal choice of N. Such a number N is found by computer-aided calculations where the missing information on the number of points on a curve X over $\mathbb{F}_{q^r}$ is obtained either from the inequality $X(\mathbb{F}_{q^r}) \geq X(\mathbb{F}_{q^d})$ when $d \mid r$, or from Serre's bound $X(\mathbb{F}_{q^r}) \geq q^r + 1 - g\lfloor 2q^{r/2} \rfloor$, depending on which one is more precise. We follow closely [BRT, Section 5].

Recall that a *tower* of function fields over $\mathbb{F}_q$ is an infinite sequence $\{F_k/\mathbb{F}_q\}_{k \in \mathbb{N}}$ of function fields such that for all k the ground field $\mathbb{F}_q$ is algebraically closed in F_k, $F_k \subset F_{k+1}$, and the genus satisfies $g(F_k) \to \infty$. A *recursive tower* is a tower $\{F_k\}$ of function fields over $\mathbb{F}_q$ such that $F_0 = \mathbb{F}_q(x_0)$ is a rational function field and $F_{k+1} = F_k(x_{k+1})$ where x_{k+1} satisfy the equation $f(x_k, x_{k+1}) = 0$ for a given polynomial $f(X, Y)$ in $\mathbb{F}_q[X, Y]$.

3.1. The first tower of Garcia–Stichtenoth. Assume that q^r is a square, and consider the tower $\{H_k\} = \mathcal{H}/\mathbb{F}_{q^r}$ defined recursively by the polynomial

$$f(X, Y) = Y^{q^{r/2}} X^{q^{r/2-1}} + Y - X^{q^{r/2}} \in \mathbb{F}_q[X, Y].$$

We also consider the recursive tower $\{F_k\} = \mathcal{F}/\mathbb{F}_q$ of function fields defined by the same polynomial starting with the rational function field $\mathbb{F}_q(x_0)$. The base change of F_k to $\mathbb{F}_{q^r}$ gives H_k.

We compare the numerical estimates from [BRT, Section 5.1] with what we obtain using our bound h_{LZ}. We take $q = 2$, $r = 2$ and consider the fields H_2, H_3, and H_4. Note an error in [BRT, Section 5.1] where for $k = 3$ the genus is erroneously taken to be equal to 14 instead of 13 (this was pointed out by Julia Pieltant). Recall that $B_1(H_k)$ denotes the number of $\mathbb{F}_4$-points of the curve corresponding to H_k.

Step k	$g(H_k)$	$B_1(H_k)$	h_{BRT}	h_{AHL}	h_{LZ}	N
2	5	16	7434	12240	9230	10
3	13	30	16 911 279 581	16 271 525 520	26 274 427 880	33
4	33	56	1.43×10^{25}	0.075×10^{25}	4.149×10^{25}	83

Here is a similar comparison for $q = 2$ and the tower $\mathcal{F}$ with $B_1(F_k)$ and $B_2(F_k)$ denoting respectively the number of $\mathbb{F}_2$- and $\mathbb{F}_4$-rational points of the curve corresponding to F_k:

Step k	$g(F_k)$	$B_1(F_k)$	$B_2(F_k)$	h_{BRT}	h_{LZ}	N
2	5	2	7	7	30	12
3	13	2	14	10453	42898	26
4	33	2	27	343 733 443 618	1 543 267 494 985	74

We notice that our bound is better than the other ones except for the case of $H_2/\mathbb{F}_4$ where we cannot beat h_{AHL}. The situation changes, however, if we use some additional information on the places of $H_2/\mathbb{F}_4$. Namely, one can calculate that $B_2(H_2) = 0$ and $B_3(H_2) = 24$. These values give the bound $h_{\mathrm{LZ}} = 13430$ reached for $N = 11$. Using MAGMA we calculated that the exact value of the class number is 16200.

3.2. The tower of Bassa–Garcia–Stichtenoth. Consider the tower $\{H_k\} = \mathcal{H}/\mathbb{F}_{q^3}$ defined recursively by the polynomial

$$f(X,Y) = (Y^q - Y)^{q-1} + 1 + \frac{X^{q(q-1)}}{(X^{q-1} - 1)^{q-1}} \in \mathbb{F}_q[X,Y],$$

and let $\{F_k\} = \mathcal{F}/\mathbb{F}_q$ be the same recursive tower over $\mathbb{F}_q$. We have the following numerical estimates for the class numbers when $q = 2$, that is, over $\mathbb{F}_8$ for H_k and over $\mathbb{F}_2$ for F_k. The value of h_{BRT} bound is taken from [BRT, Section 5.1].

Step k	$g(H_k)$	$B_1(H_k)$	h_{BRT}	h_{LZ}	N
2	5	24	125 537	126 832	9
3	13	48	2.556×10^{13}	4.039×10^{13}	29
4	29	96	2.010×10^{30}	5.778×10^{30}	11

Step k	$g(F_k)$	$B_3(F_k)$	h_{BRT}	h_{LZ}	N
2	5	8	3	3	5
3	13	16	771	1623	19
4	29	32	212 127 395	751 622 136	61

3.3. Composite towers. The next example is the composite tower $\{E_k/\mathbb{F}_{q^2}\}$ constructed in [HST]. It is obtained as a composite of the tower of Garcia and Stichtenoth from Section 3.1 with a certain explicitly given function field. The details can be found in [BRT, Proposition 5.11]. The following table combines the estimates for $q^2 = 4$:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	$B_3(E_k)$	h_{BRT}	h_{LZ}	N
2	55	1	12	12	3.657×10^{31}	23.55×10^{31}	14
3	132	1	24	24	9.198×10^{77}	121.02×10^{77}	15

For two other composite towers $\{E_k/\mathbb{F}_2\}$ and $\{E'_k/\mathbb{F}_8\}$ this time based on the tower from Section 3.2 (see [BRT, Proposition 5.17] for a detailed description), we get the following numerical data:

Step k	$g(E_k)$	$B_3(E_k)$	$B_6(E_k)$	h_{BRT}	h_{LZ}	N
2	17	16	8	10 254	27563	30
3	49	32	16	1.718×10^{14}	9.173×10^{14}	94

Step k	$g(E'_k)$	$B_1(E'_k)$	$B_2(E'_k)$	h_{BRT}	h_{LZ}	N
2	17	48	24	1.002×10^{17}	2.304×10^{17}	35
3	49	96	48	2.426×10^{48}	13.08×10^{48}	10

One more composite tower $E_k/\mathbb{F}_4$ introduced in [W] (see also [BRT, Proposition 5.18]) gives us the following table:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	$B_3(E_k)$	h_{BRT}	h_{LZ}	N
2	30	1	9	9	4.625×10^{16}	18.329×10^{16}	52
3	89	1	27	27	2.236×10^{52}	21.39×10^{52}	16

For the composite tower $E_k/\mathbb{F}_9$ from [BRT, Proposition 5.20] we obtain:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	h_{BRT}	h_{LZ}	N
2	15	36	4	8.563×10^{14}	18.76×10^{14}	30
3	46	72	8	7.470×10^{45}	41.64×10^{45}	10

Finally, for yet another composite tower $E_k/\mathbb{F}_4$ from [BRT, Proposition 5.22] we get:

Step k	$g(E_k)$	$B_1(E_k)$	$B_2(E_k)$	h_{BRT}	h_{LZ}	N
2	25	36	9	1.415×10^{18}	3.835×10^{18}	56
3	124	108	27	3.501×10^{86}	36.23×10^{86}	16

In all these examples with one exception we manage to improve on the previously known bounds.

4. Open questions. Several natural questions arise in connection with the bounds obtained in this paper.

QUESTION 4.1. *Is it possible to compare the bounds h_{BRT}, h_{AHL}, and h_{LZ}?*

We would like to have a more or less explicit description of the cases when each of the bounds is the best one. In the above examples our bound h_{LZ} always turned out to be better than h_{BRT}. However, we were not able to establish this fact in general. Comparing the bounds h_{LZ} and h_{BRT} does not seem to be easy, in particular due to the fact that the number N corresponding to the optimal $h_{\min}(N)$ can vary significantly and does not correspond at all to the number of known Φ_{q^r}'s.

QUESTION 4.2. *Can one improve (or even optimize) the bound h_{LZ} using different test functions in the explicit formulae?*

Oesterlé managed to get the best possible bounds for $|X(\mathbb{F}_{q^r})|$ available from explicit formulae using the linear programming approach (see [S]). This technique, however, does not seem to be applicable directly in our case due to the non-linearity of the problem in question. The optimization seeming difficult, it would be interesting at least to find examples where a different choice of test functions in the explicit formulae leads to better bounds than h_{LZ}.

QUESTION 4.3. *What are the analogues of the above bounds in the number field case?*

This question seems to be more directly accessible than the previous ones, since there are both the Mertens theorem and an explicit version of the Brauer–Siegel theorem available in the number field case [L], [LZ]. Nevertheless, analytic components of the proofs will certainly be more substantial, and the application of the Generalized Riemann Hypothesis might be necessary in certain cases.

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