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On square values of quadratics

by

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1. We investigate the following problem. Given a positive integer N, does there exist a quadratic function $f(x) = ax^2 + bx + c$ in $\mathbb{Z}[x]$ that is not identically square, and which takes square values for N consecutive values of x? Examples with $N = 8$ are known (Allison [1]), where the quadratic has an axis of symmetry midway between two integers; and all such examples may be described in terms of the points on an elliptic curve of rational rank 1. When the quadratic fails this symmetry condition, then examples only with $N = 7$ are known; see Allison [2], who by computer search discovers two such examples. In this note, we pursue two approaches, constructing thirteen new examples (given at (27)–(31), (49)–(56)), some with quite large coefficients, for example

$$f(x) = 20832169413896281x^2 - 98230455975155336x \\ + 174196838754626704,$$

which is square for $x = 0, \pm1, \pm2, \pm3$.

2. We reprise the methods of Allison [2], and consider first the case where the quadratic has an axis of symmetry about an integral value of x. Suppose $N \geq 5$. By translation, we may assume that $b = 0$, and $f(0) = l^2$, $f(\pm1) = k^2$, $f(\pm2) = m^2$. Then $4k^2 - 3l^2 = m^2$. If $N \geq 7$, then necessarily $f(\pm3) = n^2$, so that

$$4k^2 - 3l^2 = m^2, \quad 9k^2 - 8l^2 = n^2.$$

But this intersection of two quadrics, on taking $(1,1,1,1)$ as a zero point, represents an elliptic curve with a cubic model

$$y^2 = x(x^2 - 59x + 864).$$

This curve, of conductor 30, has rank 0, and its only rational points are the torsion points, which lead to $f(x)$ being identically square. So non-square quadratics in this case can only occur with N at most 5.

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Consider second the case that the quadratic has an axis of symmetry halfway between two integers. Suppose $N \geq 6$. By translation, there is no loss of generality in supposing that $a + b = 0$, and

$$f(0) = f(1) = l^2, \quad f(-1) = f(2) = k^2, \quad f(-2) = f(3) = m^2.$$

Then $(a, b, c) = \left(\frac{1}{2}(k^2 - l^2), -\frac{1}{2}(k^2 - l^2), l^2\right)$, and $3k^2 - 2l^2 = m^2$. Now $f(-3) = f(4) = 6k^2 - 5l^2$, so that $N \geq 8$ demands

$$3k^2 - 2l^2 = m^2, \quad 6k^2 - 5l^2 = n^2.$$

This latter is the equation of an elliptic curve, with a cubic model

$$y^2 = x(x^2 - 27x + 180);$$

this curve is of rank 1, and points (k, l) satisfying $(k^2 - l^2)(k^2 - 9l^2) \neq 0$ provide non-square quadratics that are square for values of $x = -3, \ldots, 4$. So in the case $N = 8$ there is an infinite family of non-square quadratics, corresponding to multiples of the generator of infinite order of the Mordell–Weil group. For example, the smallest two points are given by $(k, l) = (67, 73), (3089, 2231)$ leading to the respective quadratics:

$$f(x) = -420x^2 + 420x + 5329, \tag{1}$$

$$f(x) = 2282280x^2 - 2282280x + 4977361. \tag{2}$$

If we seek solutions with $N \geq 10$, then $f(-4) = f(5) = 10k^2 - 9l^2$, so that

$$3k^2 - 2l^2 = m^2, \quad 6k^2 - 5l^2 = n^2, \quad 10k^2 - 9l^2 = p^2.$$

This latter defines a curve of genus 5, so possesses only finitely many rational points. It seems plausible that these are given by precisely $(\pm k, \pm l) = (1, 1), (3, 1)$ (corresponding to the quadratic being a perfect square), but we are unable to show this. It seems likely that no non-square $f(x)$ exists with half-integer symmetry in the case $N = 10$.

3. The more interesting case is where $f(x)$ does not display either of the above symmetries. Suppose $N \geq 7$. Without loss of generality, suppose that

$$f(-3) = p^2, \quad f(-2) = q^2, \quad f(-1) = r^2,$$
$$f(0) = s^2, \quad f(1) = t^2, \quad f(2) = u^2, \quad f(3) = v^2.$$

Then

$$\begin{aligned} 9a - 3b + c &= p^2, \\ 4a - 2b + c &= q^2, \\ a - b + c &= r^2, \\ c &= s^2, \end{aligned}$$

$$a + b + c = t^2,$$
$$4a + 2b + c = u^2,$$
$$9a + 3b + c = v^2.$$

Accordingly,

$$\begin{aligned} 6r^2 - 8s^2 + 3t^2 &= p^2, \\ 3r^2 - 3s^2 + t^2 &= q^2, \\ r^2 - 3s^2 + 3t^2 &= u^2, \\ 3r^2 - 8s^2 + 6t^2 &= v^2. \end{aligned} \tag{3}$$

The condition that the corresponding quadratic $f(x)$ is not a perfect square is that

$$(r + 2s + t)(r + 2s - t)(r - 2s + t)(r - 2s - t) \neq 0. \tag{4}$$

The symmetry $\left(\begin{smallmatrix} p & q & r & s & t & u & v \\ v & u & t & s & r & q & p \end{smallmatrix}\right)$ of (3) corresponds to replacing b by $-b$, so henceforth, numerical solutions of (3) will be assumed to satisfy $p \leq v$.

Allison [2] ran a computer search and found two solutions (up to symmetry) to the equations (3). These solutions,

$$\begin{aligned}(p, q, r, s, t, u, v) = {} & (53, 173, 217, 233, 227, 197, 127), \\ & (526, 337, 160, 113, 274, 461, 652)\end{aligned}$$

deliver the quadratics

$$f(x) = -4980x^2 + 2220x + 54289, \tag{5}$$

$$f(x) = 37569x^2 + 24738x + 12769, \tag{6}$$

which have the property that they are square for the values of $x = -3, \ldots, 3$.

Originally, we hoped to find infinitely many examples in this case for $N = 7$, by exhibiting a parametrized curve on the *surface* represented by the equations at (3), but we were unable to do so. Such a curve induces a parametrized curve on the variety, also a surface,

$$V : \begin{cases} 6r^2 - 8s^2 + 3t^2 = p^2, \\ 3r^2 - 3s^2 + t^2 = q^2, \\ r^2 - 3s^2 + 3t^2 = u^2, \end{cases} \tag{7}$$

and it is with this latter surface that we continue the investigation. Its points correspond to quadratics $f(x)$ with $N = 6$, and by finding curves on the surface V, we are able to give infinitely many such quadratics. Satisfying the fourth equation at (3) demands finding rational points on curves of genus greater than 1, and our parametrizations of (7) lead in this way to several new quadratics where $N = 7$.

The variety V is the intersection of three quadrics in $\mathbb{P}^5$, and is nonsingular. Accordingly V is a $K3$ surface. It contains the 32 straight lines shown in Table 1.

Table 1. 32 lines on V

ℓ_1 :	$p = 3r + 2s$,	$q = 2r + s$,	$t = r + 2s$,	$u = 2r + 3s$
ℓ_2 :	$-p = 3r + 2s$,	$q = 2r + s$,	$t = r + 2s$,	$u = 2r + 3s$
ℓ_3 :	$p = 3r + 2s$,	$-q = 2r + s$,	$t = r + 2s$,	$u = 2r + 3s$
ℓ_4 :	$-p = 3r + 2s$,	$-q = 2r + s$,	$t = r + 2s$,	$u = 2r + 3s$
ℓ_5 :	$p = 3r + 2s$,	$q = 2r + s$,	$-t = r + 2s$,	$u = 2r + 3s$
ℓ_6 :	$-p = 3r + 2s$,	$q = 2r + s$,	$-t = r + 2s$,	$u = 2r + 3s$
ℓ_7 :	$p = 3r + 2s$,	$-q = 2r + s$,	$-t = r + 2s$,	$u = 2r + 3s$
ℓ_8 :	$-p = 3r + 2s$,	$-q = 2r + s$,	$-t = r + 2s$,	$u = 2r + 3s$
ℓ_9 :	$p = 3r + 2s$,	$q = 2r + s$,	$t = r + 2s$,	$-u = 2r + 3s$
ℓ_{10} :	$-p = 3r + 2s$,	$q = 2r + s$,	$t = r + 2s$,	$-u = 2r + 3s$
ℓ_{11} :	$p = 3r + 2s$,	$-q = 2r + s$,	$t = r + 2s$,	$-u = 2r + 3s$
ℓ_{12} :	$-p = 3r + 2s$,	$-q = 2r + s$,	$t = r + 2s$,	$-u = 2r + 3s$
ℓ_{13} :	$p = 3r + 2s$,	$q = 2r + s$,	$-t = r + 2s$,	$-u = 2r + 3s$
ℓ_{14} :	$-p = 3r + 2s$,	$q = 2r + s$,	$-t = r + 2s$,	$-u = 2r + 3s$
ℓ_{15} :	$p = 3r + 2s$,	$-q = 2r + s$,	$-t = r + 2s$,	$-u = 2r + 3s$
ℓ_{16} :	$-p = 3r + 2s$,	$-q = 2r + s$,	$-t = r + 2s$,	$-u = 2r + 3s$
ℓ_{17} :	$p = 3r - 2s$,	$q = 2r - s$,	$t = r - 2s$,	$u = 2r - 3s$
ℓ_{18} :	$-p = 3r - 2s$,	$q = 2r - s$,	$t = r - 2s$,	$u = 2r - 3s$
ℓ_{19} :	$p = 3r - 2s$,	$-q = 2r - s$,	$t = r - 2s$,	$u = 2r - 3s$
ℓ_{20} :	$-p = 3r - 2s$,	$-q = 2r - s$,	$t = r - 2s$,	$u = 2r - 3s$
ℓ_{21} :	$p = 3r - 2s$,	$q = 2r - s$,	$-t = r - 2s$,	$u = 2r - 3s$
ℓ_{22} :	$-p = 3r - 2s$,	$q = 2r - s$,	$-t = r - 2s$,	$u = 2r - 3s$
ℓ_{23} :	$p = 3r - 2s$,	$-q = 2r - s$,	$-t = r - 2s$,	$u = 2r - 3s$
ℓ_{24} :	$-p = 3r - 2s$,	$-q = 2r - s$,	$-t = r - 2s$,	$u = 2r - 3s$
ℓ_{25} :	$p = 3r - 2s$,	$q = 2r - s$,	$t = r - 2s$,	$-u = 2r - 3s$
ℓ_{26} :	$-p = 3r - 2s$,	$q = 2r - s$,	$t = r - 2s$,	$-u = 2r - 3s$
ℓ_{27} :	$p = 3r - 2s$,	$-q = 2r - s$,	$t = r - 2s$,	$-u = 2r - 3s$
ℓ_{28} :	$-p = 3r - 2s$,	$-q = 2r - s$,	$t = r - 2s$,	$-u = 2r - 3s$
ℓ_{29} :	$p = 3r - 2s$,	$q = 2r - s$,	$-t = r - 2s$,	$-u = 2r - 3s$
ℓ_{30} :	$-p = 3r - 2s$,	$q = 2r - s$,	$-t = r - 2s$,	$-u = 2r - 3s$
ℓ_{31} :	$p = 3r - 2s$,	$-q = 2r - s$,	$-t = r - 2s$,	$-u = 2r - 3s$
ℓ_{32} :	$-p = 3r - 2s$,	$-q = 2r - s$,	$-t = r - 2s$,	$-u = 2r - 3s$

A point on any line corresponds to $f(x)$ being a perfect square. Further, V contains the eight conics

$$\begin{array}{lllll} C_1: & r=s, & q=t, & u=p, & p^2=-2s^2+3t^2,\\ C_2: & r=-s, & q=t, & u=p, & p^2=-2s^2+3t^2,\\ C_3: & r=s, & q=-t, & u=p, & p^2=-2s^2+3t^2,\\ C_4: & r=-s, & q=-t, & u=p, & p^2=-2s^2+3t^2,\\ C_5: & r=s, & q=t, & u=-p, & p^2=-2s^2+3t^2,\\ C_6: & r=-s, & q=t, & u=-p, & p^2=-2s^2+3t^2,\\ C_7: & r=s, & q=-t, & u=-p, & p^2=-2s^2+3t^2,\\ C_8: & r=-s, & q=-t, & u=-p, & p^2=-2s^2+3t^2. \end{array} \tag{8}$$

A point on a conic corresponds to $f(x)$ having an axis of symmetry halfway between two integers, which case has already been discussed.

Consider the intersection of V with the family of hyperplanes

$$u-p=\lambda(r-s). \tag{9}$$

On V we have $u^2-p^2=-5(r^2-s^2)$, so that on the intersection

$$u+p=-\frac{5}{\lambda}(r+s) \tag{10}$$

giving

$$2u=\lambda(r-s)-\frac{5}{\lambda}(r+s), \qquad 2p=-\lambda(r-s)-\frac{5}{\lambda}(r+s), \tag{11}$$

with

$$E_\lambda: \quad \begin{cases} 12\lambda^2q^2=r^2(25+22\lambda^2+\lambda^4)+rs(50-2\lambda^4)\\ \qquad\qquad +s^2(25-14\lambda^2+\lambda^4),\\ 12\lambda^2t^2=r^2(25-14\lambda^2+\lambda^4)+rs(50-2\lambda^4)\\ \qquad\qquad +s^2(25+22\lambda^2+\lambda^4). \end{cases} \tag{12}$$

E_λ possesses the $\mathbb{Q}(\lambda)$ point $\mathcal{O}_\lambda(r,s,q,t)=(1-\lambda,\ -1-\lambda,\ 3-\lambda,\ 3+\lambda)$ (whose locus as λ varies is the straight line ℓ_{19}), so that E_λ is an elliptic curve over $\mathbb{Q}(\lambda)$. For finitely many values of λ the curve can acquire singularities or even split. The singular elements of the pencil occur at the following values of λ, with the corresponding Kodaira classification: $\lambda=0$ (I_2), ∞ (I_2), ±1 (I_4), ±5 (I_4), $\pm\sqrt{-2}$ (I_1), $\pm5/\sqrt{-2}$ (I_1). Indeed there are the following decompositions:

$$\begin{array}{ll} \lambda=0: & C_2, C_4,\\ \lambda=\infty: & C_5, C_7,\\ \lambda=1: & \ell_{10},\ell_{12},\ell_{14},\ell_{16}, \end{array} \tag{13}$$

$$
\begin{aligned}
&\lambda = -1: && \ell_1, \ell_3, \ell_5, \ell_7,\\
&\lambda = 5: && \ell_{18}, \ell_{20}, \ell_{22}, \ell_{24},\\
&\lambda = -5: && \ell_{25}, \ell_{27}, \ell_{29}, \ell_{31},\\
&\lambda = \pm\sqrt{-2}: && \text{nodal quartic},\\
&\lambda = \pm\frac{5}{\sqrt{-2}}: && \text{nodal quartic}.
\end{aligned}
\tag{13}
$$

[cont.]

A cubic model for E_λ is given by

$$(14)\quad \mathcal{E}_\lambda: \quad T^2 = S(S^2 + 4(25 + 55\lambda^2 + \lambda^4)S + 4(\lambda^2 - 1)^2(\lambda^2 - 25)^2),$$

and this is a base-change, given by $\mu = \lambda^2$, of the elliptic curve over $\mathbb{Q}(\mu)$

$$(15)\quad \mathcal{F}_\mu: \quad T^2 = S(S^2 + 4(25 + 55\mu + \mu^2)S + 4(\mu - 1)^2(\mu - 25)^2).$$

This latter equation defines a rational elliptic surface, and it is known from results of Shioda (see Shioda [12]) that the $\mathbb{C}(\mu)$ group of points on $\mathcal{F}_\mu$ is generated by those points in which the S-coordinate is a polynomial in μ of degree at most 2. A straightforward machine computation discovers that there are precisely twelve such points (including the point at infinity), and that $\mathcal{F}_\mu(\mathbb{C}(\mu))$ is of rank 2 and generated by

$$
\begin{aligned}
Q_1 &= (4(\mu - 1)^2,\ -12(\mu - 1)^2(\mu + 9)),\\
Q_2 &= ((\mu - 1)(\mu - 25),\ -3(\mu - 1)(\mu - 25)(\mu + 5))
\end{aligned}
\tag{16}
$$

of infinite order, and $(0, 0)$ of order 2.

However, to compute the Néron–Severi rank of V, equivalently of the $K3$ surface defined by $\mathcal{E}_\lambda$ (and to compute the associated generators of $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$), is harder.

To find a set of generators for $\mathrm{NS}(V, \mathbb{C})$ over $\mathbb{Z}$, we use ideas of Swinnerton-Dyer [13], to which article the reader is referred for full details (see also Bremner [3]–[5] for other application of these methods). The group $\mathrm{NS}(V, \mathbb{C})$ is spanned over $\mathbb{Z}$ by

(a) any non-singular fibre of the pencil $\mathcal{E}_\lambda$,
(b) the locus of the point $\mathcal{O}_\lambda$,
(c) the components of the singular fibres in the pencil $\mathcal{E}_\lambda$,
(d) the loci of the generators of the group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$.

For (b), we have seen that the locus of $\mathcal{O}_\lambda$ is the line ℓ_{19}. For (c), the components are listed at (13). This provides 16 independent elements of $\mathrm{NS}(V, \mathbb{C})$. Since the rank of $\mathrm{NS}(V, \mathbb{C})$ is at most 20, it follows that the rank of the group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ is at most 4.

Now the curve $\mathcal{F}_\mu$ has torsion group generated by $(0,0)$, and this point is divisible by 2 on $\mathcal{E}_\lambda$, with the torsion group of $\mathcal{E}_\lambda$ of order 4 generated by

$$T = [-2(\lambda^2-1)(\lambda^2-25),\ 36\lambda(\lambda^2-1)(\lambda^2-25)].$$

The points Q_1, Q_2 of $\mathcal{F}_\mu$ correspond to the following (independent) points of $\mathcal{E}_\lambda$:

$$\begin{aligned} P_1 &= (4(\lambda^2-1)^2, -12(\lambda^2-1)^2(\lambda^2+9)),\\ P_2 &= ((\lambda^2-1)(\lambda^2-25), -3(\lambda^2-1)(\lambda^2-25)(\lambda^2+5)) \end{aligned} \tag{17}$$

thereby generating a subgroup of $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ of rank 2, and contributing to a subgroup of $\mathrm{NS}(V,\mathbb{C})$ of rank 18.

In fact, the rank of $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ is indeed exactly 2, and this can be shown by techniques developed by Jasper Scholten [10]. I am most grateful to him for undertaking the relevant computation in this example.

The idea is to consider primes p for which V has good reduction. The reduced surface $\overline{V}$ has Néron–Severi rank at least equal to that of V; so if such a prime p can be found with the Néron–Severi rank of $\overline{V}$ equal to 18, then necessarily the rank of $\mathrm{NS}(V,\mathbb{C})$ is equal to 18. Consider the ℓ-adic cohomology group $H^2(V_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)$ for some prime $\ell \neq p$. Let Fr_p be the Frobenius morphism at p, which acts linearly on the group. By a conjecture of Tate, proved for $K3$ surfaces over a finite field, the Néron–Severi rank of $\overline{V}$ equals the number of eigenvalues of Fr_p which are p times a root of unity. And the latter can be computed by counting points on the surface over various finite fields (in this example, over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$), and using the Lefschetz trace formula. For our example, the smallest primes of good reduction are 11 and 13. At $p = 11$, the characteristic polynomial of Frobenius is computed as $(x-11)^{18}(x+11)^2(x^2-6x+121)$, which tells us that the 11-adic Néron–Severi rank of V is 20, not sufficient for our purposes. However, at $p = 13$, the characteristic polynomial of Frobenius equals $(x-13)^{18}(x^4+14x^3+208x^2+2366x+28561)$, implying that the 13-adic Néron–Severi rank is indeed equal to 18.

Knowing that the rank is 2, it follows that the subgroup $\mathcal{G}$ generated by P_1 and P_2 is of finite index in the full group of points modulo torsion. The height pairing matrix of P_1, P_2 (see, for instance, Kuwata [9]) is readily calculated to equal $\left(\begin{smallmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$, of determinant $1/4$; and so from Kuwata (ibid.), it follows that if $\mathcal{G}$ is a proper subgroup of $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ modulo torsion, then 2 divides the index of $\mathcal{G}$ in $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$. Accordingly, at least one of the points P_1, P_2, P_1+P_2 (up to torsion) is divisible by 2 in $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$. But this is easily shown not to be the case, so that P_1 and P_2 are in fact generators for $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ modulo torsion. As a corollary, $\mathcal{E}_\lambda(\mathbb{C}(\lambda)) = \mathcal{E}_\lambda(\mathbb{Q}(\lambda))$.

The mapping between E_λ and $\mathcal{E}_\lambda$ sets up the following correspondences:

(18)

r	s	q	t	Point of $\mathcal{E}_\lambda$
$1-\lambda$	$-1-\lambda$	$-3+\lambda$	$3+\lambda$	$\mathbf{0}$
$1-\lambda$	$-1-\lambda$	$-3+\lambda$	$-3+\lambda$	P_1+T
$1-\lambda$	$-1-\lambda$	$3-\lambda$	$3+\lambda$	P_1-T
$1-\lambda$	$-1-\lambda$	$3-\lambda$	$-3-\lambda$	$2T$
$1+\lambda$	$-1+\lambda$	$-3-\lambda$	$3-\lambda$	P_1
$1+\lambda$	$-1+\lambda$	$-3-\lambda$	$-3+\lambda$	T
$1+\lambda$	$-1+\lambda$	$3+\lambda$	$3-\lambda$	$-T$
$1+\lambda$	$-1+\lambda$	$3+\lambda$	$-3+\lambda$	P_1+2T
$-5+\lambda$	$5+\lambda$	$5-3\lambda$	$5+3\lambda$	P_2+T
$-5+\lambda$	$5+\lambda$	$5-3\lambda$	$-5-3\lambda$	P_1-P_2
$-5+\lambda$	$5+\lambda$	$-5+3\lambda$	$5+3\lambda$	P_1-P_2+2T
$-5+\lambda$	$5+\lambda$	$-5+3\lambda$	$-5-3\lambda$	P_2-T
$5+\lambda$	$-5+\lambda$	$5+3\lambda$	$5-3\lambda$	P_1-P_2-T
$5+\lambda$	$-5+\lambda$	$5+3\lambda$	$-5+3\lambda$	P_2+2T
$5+\lambda$	$-5+\lambda$	$-5-3\lambda$	$5-3\lambda$	P_2
$5+\lambda$	$-5+\lambda$	$-5-3\lambda$	$-5+3\lambda$	P_1-P_2+T

and the locus of P_1 as λ varies is the line ℓ_{28}, that of P_2 the line ℓ_{11}, and that of T the line ℓ_{32}.

Accordingly, the following divisors span $\mathrm{NS}(V,\mathbb{C})$ over $\mathbb{Z}$:

$$(19)\qquad \ell_{19}; C_2, C_4, C_5, C_7, \ell_1, \ell_3, \ell_5, \ell_7, \ell_{10}, \ell_{12}, \ell_{14}, \ell_{16}, \ell_{18}, \ell_{20}, \ell_{22}, \ell_{24}, \ell_{25}, \ell_{27}, \ell_{29}, \ell_{31}; \ell_{28}, \ell_{11}, \ell_{32}.$$

The intersection matrix of these 24 divisors is straightforward to write down, and has rank 18. By considering appropriate linear combinations over $\mathbb{Z}$, it follows easily that the following 18 divisors, which we shall denote respectively by $\Gamma_1, \ldots, \Gamma_{18}$, generate $\mathrm{NS}(V,\mathbb{C})$ over $\mathbb{Z}$:

$$(20)\qquad \ell_1, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_9, \ell_{10}, \ell_{11}, \ell_{13}, \ell_{14}, \ell_{15}, \ell_{17}, \ell_{18}, \ell_{19}, \ell_{21}, \ell_{25}, C_1.$$

Note that $\mathrm{NS}(V,\mathbb{Q}) = \mathrm{NS}(V,\mathbb{C})$, and so to any curve Γ defined over $\mathbb{Q}$ on V there thus correspond uniquely determined integers $m_1, \ldots, m_{18}$ such that $\Gamma \sim m_1\Gamma_1 + \ldots + m_{18}\Gamma_{18}$. The genus of Γ is a quadratic form in the m_i, given by

$$p_a(\Gamma) = \frac{1}{2}(\Gamma.\Gamma) + 1$$

where $(\Gamma.\Gamma)$ is the self intersection number (see Shafarevich [11, p. 5]).

Following simple algebra, there results

$$\begin{aligned}
(21)\quad & \tfrac{1}{2}\deg(\Gamma)^2 - 4(\Gamma.\Gamma) \\
= {} & \frac{1}{2}(m_1 - m_2 + m_3 - m_4 - m_5 - m_6 + m_7 - m_8 - m_9 + m_{10} - m_{11} \\
& \qquad + m_{12} - m_{13} - m_{14} + m_{15} + m_{16} - m_{17})^2 \\
& + (m_1 - m_2 + m_3 - m_4 + m_5 + m_6 - m_7 + m_8 + m_9 + m_{10} - m_{11} \\
& \qquad - m_{12} + m_{15})^2 \\
& + (m_1 - m_2 + m_3 - m_7 + m_{10} - m_{13} + m_{14} + m_{15} + m_{17})^2 \\
& + (m_2 + m_4 - m_5 - m_6 - m_8 - m_9 + m_{11} - m_{16} + m_{18})^2 \\
& + (m_1 + m_3 - m_4 - m_{10} + 2m_{11} + m_{12} - m_{15} + m_{16})^2 \\
& + (m_1 - m_2 + m_3 + m_4 - m_{10} - m_{15} - m_{16} - m_{18})^2 \\
& + (m_2 - m_{12} - 2m_{13} + m_{14} + m_{16} + m_{17} - m_{18})^2 \\
& + (m_2 - m_4 - m_7 + 2m_{10} - m_{12} + m_{13} - 2m_{15})^2 \\
& + 2(m_2 + m_5 - m_6 + m_8 - m_9 - m_{11} + m_{12})^2 \\
& + (2m_1 - 2m_3 - m_4 - m_7 - m_{13} - m_{18})^2 \\
& + (m_{13} - m_{14} - m_{16} - m_{17} - 2m_{18})^2 \\
& + (m_4 - 2m_6 - m_7 + 2m_9 + m_{18})^2 \\
& + (m_4 - 2m_5 - m_7 + 2m_8 - m_{18})^2 \\
& + 2(m_7 - m_{12} + m_{14} - m_{17})^2 \\
& + (m_{14} - m_{16} + m_{17})^2 \\
& + 2(m_{14} - m_{17})^2 \\
& + 2m_{16}^2.
\end{aligned}$$

This is now in a form suitable for machine computation. Given the degree and self-intersection number of Γ, it is possible to tabulate the finitely many sets of integers $m_1, \ldots, m_{18}$ that are solutions of (21). In addition, since we are only interested in irreducible curves Γ, further restrictions are imposed on the m_i by insisting that Γ have non-negative intersection number with every known curve lying on the surface. In this manner, it is computed first that the only irreducible rational curves on V of degrees 1 and 2 are the known lines and conics. There are no irreducible cubics of genus 0, and, up to symmetry, precisely four irreducible quartics of genus 0. Representative divisors (in terms of the basis $\Gamma_1, \ldots, \Gamma_{18}$) of the symmetry classes are

$$(22)\qquad \begin{gathered}
\{1,0,0,1,0,1,0,0,0,0,0,0,0,0,0,0,-1,1\}, \\
\{1,0,0,1,0,1,0,0,0,0,0,0,-1,0,0,0,1\}, \\
\{1,1,0,0,0,1,0,0,0,0,0,0,-1,0,0,0,1\}, \\
\{1,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,-1,1\}.
\end{gathered}$$

Representative parametrizations for the symmetry classes are as follows:

$$p:q:r:s:t:u = 3m^4+12m^3n+20m^2n^2-8mn^3-3n^4 : \\ 3m^4+12m^3n+4m^2n^2-3n^4 : \\ 3m^4+10m^3n-2m^2n^2-6mn^3+3n^4 : \\ 3m^4+6m^3n-2m^2n^2-10mn^3+3n^4 : \\ 3m^4-4m^2n^2+12mn^3-3n^4 : \\ 3m^4-8m^3n-20m^2n^2+12mn^3-3n^4, \tag{23}$$

$$p:q:r:s:t:u = 3m^4+89m^3n-101m^2n^2+5mn^3+24n^4 : \\ 3m^4+63m^3n-63m^2n^2+7mn^3-14n^4 : \\ 9m^4+5m^3n+41m^2n^2-71mn^3+4n^4 : \\ 15m^4-11m^3n-43m^2n^2+73mn^3-6n^4 : \\ 21m^4-21m^3n-63m^2n^2+35mn^3-16n^4 : \\ 27m^4-29m^3n-59m^2n^2-25mn^3+26n^4, \tag{24}$$

$$p:q:r:s:t:u = 6m^4+26m^3n+97m^2n^2+13mn^3-40n^4 : \\ 2m^4+34m^3n+45m^2n^2+17mn^3-26n^4 : \\ 2m^4+28m^3n+25m^2n^2-25mn^3+12n^4 : \\ 6m^4+20m^3n+7m^2n^2-47mn^3+2n^4 : \\ 10m^4+26m^3n-27m^2n^2-11mn^3-16n^4 : \\ 14m^4+34m^3n-47m^2n^2+17mn^3+30n^4, \tag{25}$$

$$p:q:r:s:t:u = 4m^4+14m^3n-20m^2n^2+9mn^3-9n^4 : \\ 4m^4+2m^3n+4m^2n^2-15mn^3+9n^4 : \\ 4m^4-6m^3n+2m^2n^2+15mn^3-9n^4 : \\ 4m^4-10m^3n-2m^2n^2+9mn^3-9n^4 : \\ 4m^4-10m^3n+4m^2n^2+3mn^3+9n^4 : \\ 4m^4-6m^3n+20m^2n^2-21mn^3-9n^4. \tag{26}$$

The condition that (23) gives a point on (3) becomes

$$9m^8-108m^7n-72m^6n^2+876m^5n^3+478m^4n^4-932m^3n^5 \\ +376m^2n^6-60mn^7+9n^8=\square,$$

a curve of genus 3 with only finitely many rational points. Computer search with $|m|+n \le 500$ produces solutions with $(m,n)=(0,1),(1,0),(-1,1),(1,1),(-2,1),(1,2),(-3,1),(-1,3),(1,3),(3,1),(7,9),(55,19)$. The latter two points correspond to non-square polynomials $f(x)$, the first recovering the quadratic $f(x)$ listed at (6), the second giving

$$
(27)\quad
\begin{aligned}
&(m,n) = (55,19),\\
&(p,q,r,s,t,u,v) = (10477119, 8670314, 6875821, 5106636,\\
&\qquad\qquad 3402469, -1942706, 1633911),\\
&f(x) = 3349123623505x^2 - 17850059564040x + 26077731236496.
\end{aligned}
$$

In similar vein, (24) gives a point on (3) provided

$$
\begin{aligned}
1089m^8 &- 2382m^7n - 1589m^6n^2 - 2996m^5n^3 + 13587m^4n^4\\
&+ 9394m^3n^5 - 11207m^2n^6 - 1416mn^7 + 1296n^8 = \square,
\end{aligned}
$$

with known points at $(m,n) = (0,1), (1,0), (-1,1), (1,1), (2,1), (-1,3), (1,3), (-3,2), (1,7), (4,5), (7,3), (-2,9), (-13,1), (-5,17), (-89,90)$. New non-square $f(x)$ arise from the following points:

$$
(28)\quad
\begin{aligned}
&(m,n) = (-5,17),\\
&(p,q,r,s,t,u,v)\\
&\qquad = (40196, -80351, 98726, -107179, -108064, 101579, 86074),\\
&f(x) = -775012455x^2 + 965502510x + 11487338041;
\end{aligned}
$$

$$
(29)\quad
\begin{aligned}
&(m,n) = (-89,90),\\
&(p,q,r,s,t,u,v) = (-6735041, -5812061, 4881537,\\
&\qquad\qquad -3938125, -2969567, 1938531, 572321),\\
&f(x) = 815037309304x^2 - 7505537657440x + 15508828515625.
\end{aligned}
$$

The curve (25) gives a point on (3) provided

$$
\begin{aligned}
324m^8 &+ 1536m^7n - 404m^6n^2 - 3572m^5n^3 + 11297m^4n^4\\
&+ 1462m^3n^5 - 8311m^2n^6 + 1816mn^7 + 1936n^8 = \square,
\end{aligned}
$$

with known points at $(m,n) = (0,1), (1,0), (-1,1), (1,1), (-2,1), (1,2), (3,2), (-2,5), (-5,3), (1,7), (-8,1), (-9,2), (-7,8), (-13,4), (16,17)$, delivering the following new non-square $f(x)$:

$$
(30)\quad
\begin{aligned}
&(m,n) = (-9,2),\\
&(p,q,r,s,t,u,v) = (3131, -2351, -1761, 1589, 1949, 2631, 3449),\\
&f(x) = 924940x^2 + 348740x + 2524921.
\end{aligned}
$$

The curve (26) gives a point on (3) provided

$$
\begin{aligned}
16m^8 &+ 16m^7n + 276m^6n^2 - 944m^5n^3 + 1408m^4n^4 - 1584m^3n^5\\
&+ 117m^2n^6 + 810mn^7 + 81n^8 = \square,
\end{aligned}
$$

with known points at $(m,n) = (0,1), (1,0), (-1,1), (1,1), (-1,2), (1,2), (2,1), (-3,1), (3,1), (-3,2), (3,2), (3,4), (7,2), (6,5), (11,14)$, delivering the following new non-square $f(x)$:

$$(31)\quad \begin{aligned} &(m,n) = (11,14),\\ &(p,q,r,s,t,u,v)\\ &\qquad = (-28621, 10460, 12651, -31162, 50423, -69816, 89255),\\ &f(x) = 380193121x^2 + 1191215564x + 971070244. \end{aligned}$$

In like manner, we find that up to symmetry, there are 10 quintics of genus 0. One such is provided by the divisor $\ell_1 + \ell_3 + \ell_5 - \ell_{10} + \ell_{21} + \ell_{22} + \ell_{30}$ with parametrization

$$(32)\quad \begin{aligned} p:q:r:s:t:u = {} & 285a^5 + 55a^4b - 110a^3b^2 + 6a^2b^3 + 5ab^4 - b^5 :\\ & 165a^5 + 57a^4b + 18a^3b^2 - 22a^2b^3 - 3ab^4 + b^5 :\\ & 15a^5 + 209a^4b - 2a^3b^2 - 30a^2b^3 - ab^4 + b^5 :\\ & 15a^5 - 209a^4b - 2a^3b^2 + 30a^2b^3 - ab^4 - b^5 :\\ & 165a^5 - 57a^4b + 18a^3b^2 + 22a^2b^3 - 3ab^4 - b^5 :\\ & 285a^5 - 55a^4b - 110a^3b^2 - 6a^2b^3 + 5ab^4 + b^5. \end{aligned}$$

For the corresponding v to be rational, we seek points on the curve of genus 4

$$\begin{aligned} &162225a^{10} - 43890a^9b - 162971a^8b^2 + 12152a^7b^3 + 43786a^6b^4\\ &\qquad + 1876a^5b^5 - 3670a^4b^6 - 392a^3b^7 + 85a^2b^8 + 14ab^9 + b^{10} = \square. \end{aligned}$$

The only points found occur at $(a,b) = (0,1), (-1,1), (1,1), (1,2), (-1,3), (1,3), (-1,5), (1,5)$ with only $(a,b) = (1,2)$ delivering a non-square $f(x)$, but one already known, that at (1).

4. As an alternative approach to the problem of discovering points on (3), we recall that the intersection of two quadrics in $\mathbb{P}^4$ is a Del Pezzo surface, and birationally equivalent to the plane. We represent the intersection of two quadrics at (3) upon the plane as follows. In the equations (3) put

$$r + t = R,\quad r - t = T,\quad p - v = P,\quad p + v = V,$$
$$q + u = Q,\quad q - u = U,\quad s = S$$

and there results

$$(33)\quad V^2 = -P^2 + 9R^2 - 32S^2 + 9T^2,$$
$$(34)\quad U^2 = -Q^2 + 4R^2 - 12S^2 + 4T^2,$$
$$(35)\quad 2RT = QU,$$
$$(36)\quad 3RT = PV.$$

The first and fourth quadrics intersect where

$$3R = \beta P,\quad V = \beta T,\quad 32S^2 = P^2(\beta^2 - 1) - T^2(\beta^2 - 9).$$

The latter has point at $(P,T,S)=(2,2,1)$, and on putting $P:T=2S+A:2S+B$ the following parametrization results:

$$(37)\quad \begin{aligned} P:T:S = {} & 2(9A^2(\beta^2-1)-2AB(\beta^2-9)+B^2(\beta^2-9)): \\ & -2(9A^2(\beta^2-1)-18AB(\beta^2-1)+B^2(\beta^2-9)): \\ & -\beta(9A^2(\beta^2-1)-B^2(\beta^2-9)). \end{aligned}$$

This gives the representation on the plane

$$(38)\quad \begin{aligned} P:R:S:T:V = {} & 6(9A^2(\beta^2-1)-2AB(\beta^2-9)+B^2(\beta^2-9)): \\ & 2\beta(9A^2(\beta^2-1)-2AB(\beta^2-9)+B^2(\beta^2-9)): \\ & \beta(9A^2(\beta^2-1)-B^2(\beta^2-9)): \\ & -2(9A^2(\beta^2-1)-18AB(\beta^2-1)+B^2(\beta^2-9)): \\ & -2\beta(9A^2(\beta^2-1)-18AB(\beta^2-1)+B^2(\beta^2-9)) \end{aligned}$$

from which, with $G=A-B$,

$$(39)\quad \begin{aligned} p:r:s:t:v = {} & 8\beta^2(\beta+3)B^2+48\beta^2BG-9(\beta-3)(\beta^2-1)G^2: \\ & 8\beta^2(\beta+1)B^2+16\beta^3BG+9(\beta-1)(\beta^2-1)G^2: \\ & \beta(8\beta^2B^2+18(\beta^2-1)BG+9(\beta^2-1)G^2): \\ & 8\beta^2(\beta-1)B^2+16\beta^3BG+9(\beta+1)(\beta^2-1)G^2: \\ & 8\beta^2(\beta-3)B^2-48\beta^2BG-9(\beta+3)(\beta^2-1)G^2. \end{aligned}$$

Then the remaining equations for q, u demand:

$$(40)\quad \begin{aligned} \square = {} & 64\beta^4(\beta+2)^2B^4 \\ & +32\beta^4(5\beta^2+16\beta+27)B^3G \\ & +4\beta^2(49\beta^4+306\beta^2-99)B^2G^2 \\ & +36\beta^2(\beta^2-1)(5\beta^2-16\beta+27)BG^3 \\ & +81(\beta^2-1)^2(\beta-2)^2G^4, \end{aligned}$$

$$(41)\quad \begin{aligned} \square = {} & 64\beta^4(\beta-2)^2B^4 \\ & +32\beta^4(5\beta^2-16\beta+27)B^3G \\ & +4\beta^2(49\beta^4+306\beta^2-99)B^2G^2 \\ & +36\beta^2(\beta^2-1)(5\beta^2+16\beta+27)BG^3 \\ & +81(\beta^2-1)^2(\beta+2)^2G^4. \end{aligned}$$

The first equation (40) above may be considered a quartic cover over $\mathbb{Q}(\beta)$ of an elliptic curve, and using standard techniques (see for example Cassels [6, Chapter 8]), we can obtain a cubic model for this curve, together with the relevant mapping. Computation gives the cubic model

$$(42)\quad T^2 = S(S^2+2(11\beta^4+18\beta^2+99)S-135(\beta^2-1)^2(\beta^2-9)^2),$$

with mapping from (42) to (40) given by

$$B/G = \frac{-135\beta(\beta^2-1)^2(\beta^2-9) - \beta(5\beta^2+16\beta+27)S + (\beta+2)T}{8\beta(15\beta^2(\beta^2-1)(\beta^2-9) + (\beta+2)^2 S)}. \tag{43}$$

Note that the symmetry $\beta \mapsto -\beta$ taking (41) to (40) implies that (41) is also a quartic cover of the same elliptic curve (42), with corresponding mapping obtained from (43).

Accordingly, to find $B : G$ simultaneously satisfying (41) and (40), we seek two points (S_1, T_1), (S_2, T_2) on the curve (42) with corresponding values of B, G equal, that is,

$$\begin{aligned} &\frac{(-135\beta(\beta^2-1)^2(\beta^2-9) - \beta(5\beta^2+16\beta+27)S_1 + (\beta+2)T_1)}{8\beta(15\beta^2(\beta^2-1)(\beta^2-9) + (\beta+2)^2 S_1)} \\ &\quad = \frac{(-135\beta(\beta^2-1)^2(\beta^2-9) - \beta(5\beta^2-16\beta+27)S_2 + (\beta-2)T_2)}{8\beta(15\beta^2(\beta^2-1)(\beta^2-9) + (\beta-2)^2 S_2)}. \end{aligned} \tag{44}$$

To deal with the case where the denominator of the left hand side of (44) is zero, we observe the following identity on the curve (42):

$$\begin{aligned} &\frac{(-135\beta(\beta^2-1)^2(\beta^2-9) - \beta(5\beta^2+16\beta+27)S + (\beta+2)T)}{8\beta(15\beta^2(\beta^2-1)(\beta^2-9) + (\beta+2)^2 S)} \\ &\quad = \frac{(45(\beta^2-1)^2 - S)(27(\beta^2-1)(\beta^2-9) + S)}{-8\beta(135\beta(\beta^2-1)^2(\beta^2-9) + \beta(5\beta^2+16\beta+27)S + (\beta+2)T)}. \end{aligned} \tag{45}$$

If the denominator of the left hand side of (44) is zero, then

$$S_1 = -15\beta^2(\beta^2-1)(\beta^2-9)/(\beta+2)^2,$$

with corresponding T_1 given by

$$T_1 = \pm 60\beta(\beta^2-1)(\beta^2-9)(\beta^4+5\beta^3-9\beta-9)/(\beta+2)^3.$$

With the lower sign, the left hand side of (44) takes the value $1 : 0$; and with the upper sign, using (45), it takes the value

$$\frac{-3(\beta^2-1)(\beta^2+9\beta+9)(\beta^4+3\beta^3-3\beta-3)}{4\beta^2(\beta+2)^2(\beta^4+5\beta^3-9\beta-9)}. \tag{46}$$

Now $B : G = 1 : 0$ certainly provides a point on both (40) and (41), but leads to the trivial form $f(x) = (x-\beta)^2$. If B/G is given by (46), then there is a corresponding point on (40), and the condition that there is a corresponding point on (41) becomes

$$\begin{aligned} \square = {} & 4\beta^{26} + 108\beta^{25} + 1329\beta^{24} + 10312\beta^{23} + 58422\beta^{22} + 252210\beta^{21} \\ & + 820285\beta^{20} + 1949724\beta^{19} + 3108555\beta^{18} + 1905066\beta^{17} \\ & - 6380310\beta^{16} - 25572924\beta^{15} - 49193973\beta^{14} - 52414560\beta^{13} \\ & - 5467221\beta^{12} + 87813396\beta^{11} + 169671834\beta^{10} + 172959138\beta^{9} \end{aligned} \tag{47}$$

$$+ 94201299\beta^8 + 883548\beta^7 - 42120891\beta^6 - 31882086\beta^5 - 7017354\beta^4 + 5091336\beta^3 + 4730481\beta^2 + 1653372\beta + 236196.$$

This hyperelliptic curve of genus 12 can be searched for solutions (we find only $\beta = \pm 1, -2, \pm 3$, leading to no new forms f). So in what follows, we shall assume that the denominator of the left hand side of (44) does not vanish. By replacing β by $-\beta$, we can also assume that the right hand side of (44) does not vanish.

Then on clearing denominators, (44) takes the form

$$60\beta(\beta^2-1)(\beta^2-9)((\beta^4+5\beta^3-9\beta-9)S_1-(\beta^4-5\beta^3+9\beta-9)S_2) + 8\beta^2(\beta^2+11)S_1S_2 + 15\beta^2(\beta^2-1)(\beta^2-9)((\beta+2)T_1-(\beta-2)T_2) + (\beta^2-4)((\beta-2)S_2T_1-(\beta+2)S_1T_2) = 0. \tag{48}$$

A search procedure can be instituted, where first, for given β (and it is only necessary to consider $\beta > 0$), as many independent points as possible are found on (42). Both Connell's Apecs [7] and Cremona's mwrank [8] programs were invaluable for this purpose, and oftentimes with luck we were able to compute the exact value of the rank, and even on occasion an actual basis for the group of rational points. (It perhaps should be noted that the $\mathbb{Q}(\beta)$-rank of (42) is 3, and over the range we considered, given by numerator(β) + denominator$(\beta) \leq 100$, the maximum rank found was 7, on nine occasions). Second, letting the point (S_1, T_1) run through small linear combinations of the known points (in practice, we allowed the coefficients of the linear combinations to be at most 2 in absolute value) the condition (48) is now a linear relation connecting S_2, T_2 on the curve (42). The coefficient of T_2 is $-(\beta-2)(15\beta^2(\beta^2-1)(\beta^2-9)+(\beta+2)^2S_1)$, which by our assumption on the non-vanishing of the denominator at (44) is non-zero. Hence T_2 is determined in terms of S_2, in turn determining (S_2, T_2), which may or may not be rational of course.

In this way, the following additional solutions to (3) were discovered, in the range numerator(β) + denominator$(\beta) \leq 100$. The given ranks of (42) are unconditionally correct.

$$\begin{aligned} &\beta = 19/4 \quad (\text{rank} = 5), \\ &(p,q,r,s,t,u,v) = (-5207, 3025, 337, -41, 2969, 5153, 7295), \\ &f(x) = 4462584x^2 + 4350696x + 1681; \end{aligned} \tag{49}$$

$$\begin{aligned} &\beta = 14/9 \quad (\text{rank} = 6), \\ &(p,q,r,s,t,u,v) = (-2256701, 1444880, -1051139, \\ &\qquad\qquad 1464358, -2281673, 3207796, -4170865), \\ &f(x) = 1011118085961x^2 + 2050569240804x + 2144344352164; \end{aligned} \tag{50}$$

$$\begin{aligned}
&\beta = 41/9 \quad (\text{rank} = 6),\\
&(p,q,r,s,t,u,v) = (-4630, 2713, 1544, -2551, 4442, 6485, 8572),\\
&f(x) = 4550049x^2 + 8673714x + 6507601;
\end{aligned} \tag{51}$$

$$\begin{aligned}
&\beta = 53/4 \quad (\text{rank} = 5),\\
&(p,q,r,s,t,u,v) = (-490912179, 325332727, 178638667,\\
&\qquad\qquad 138585489, 260238115, 421266649, 590280507),\\
&f(x) = 30611887161775936x^2 + 17906051575608168x\\
&\qquad\qquad + 19205937761369121;
\end{aligned} \tag{52}$$

$$\begin{aligned}
&\beta = 7/64 \quad (\text{rank} = 7),\\
&(p,q,r,s,t,u,v)\\
&\ = (1343701, 929731, 530581, -238499, -436459, 826051, 1237931),\\
&f(x) = 179124555120x^2 - 45509869440x + 56881773001;
\end{aligned} \tag{53}$$

$$\begin{aligned}
&\beta = 71/8 \quad (\text{rank} = 6),\\
&(p,q,r,s,t,u,v)\\
&\qquad = (-53331, 36943, 25885, 27567, 40429, 57391, 75747),\\
&f(x) = 392329144x^2 + 482235408x + 759939489;
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\beta = 67/28 \quad (\text{rank} = 7),\\
&(p,q,r,s,t,u,v) = (-810171421, 673785150, -541534361,\\
&\qquad\qquad 417368948, -311124657, 247112534, -258833915),\\
&f(x) = 20832169413896281x^2 - 98230455975155336x\\
&\qquad\qquad + 174196838754626704;
\end{aligned} \tag{55}$$

$$\begin{aligned}
&\beta = 61/38 \quad (\text{rank} = 5),\\
&(p,q,r,s,t,u,v)\\
&\qquad = (-89255, 69816, -50423, 31162, -12651, 10460, -28621),\\
&f(x) = 380193121x^2 - 1191215564x + 971070244.
\end{aligned} \tag{56}$$

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