On certain congruences for Fourier coefficients of classical cusp forms

by

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For any even integer $k \geq 12$, let $S_k$ denote the space of cusp forms of weight $k$ on the full modular group $SL(2, \mathbb{Z})$. The Eisenstein series $E_4$ and $E_6$ are given by

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^3 \right) q^n$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \left( \sum_{d|n} d^5 \right) q^n$$

where $q = e^{2\pi iz}$, and we also define $\Delta(z) = (E_4^3 - E_6^2)/1728 \in S_{12}$ and $j(z) = E_4^3/\Delta(z)$, the modular invariant. It is well known that any cusp form $f(z) \in S_k$ can be written uniquely as

$$f(z) = \Delta(z)^m E_4(z)^{\delta} E_6(z)^{\varepsilon} g(j(z)) = \sum_{n=1}^{\infty} \gamma(n) q^n$$

for some polynomial $g(x) \in \mathbb{C}[x]$ of degree $\leq m - 1$, where $\gamma(n)$ are the Fourier coefficients, $m$ is positive integer, and $k = 12m + 4\delta + 6\varepsilon$ automatically with $\delta \in \{0, 1, 2\}$, $\varepsilon \in \{0, 1\}$.

The classical Gauss hypergeometric series are defined as follows:

$$2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n$$

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where \( (a)_n = a(a+1)(a+2) \ldots (a+n-1) \). We define the function \( F(t) \) and the coefficients \( \alpha(n) \) of \( t^n \) in \( F(t) \), for the above \( k, m, \varepsilon \) and polynomial \( g \):

\[
F(t) = t^m g \left( \frac{1}{t} \right) \sqrt{1 - 1728t^{\varepsilon-1}} 2F_1 \left( \frac{1}{12}, \frac{5}{12}, 1; 1728t \right)^{k-2} = \sum_{n=1}^{\infty} \alpha(n) t^n.
\]

We know that \( \alpha(n) \) satisfies a linear recurrence, since the function \( F(t) \) is a solution of a certain linear differential equation.

In this paper, we study the congruence properties of \( \gamma(n) \) and \( \alpha(n) \). For Fourier coefficients of cusp forms, many congruences are known, for example, for the Ramanujan \( \tau \)-function, \( \tau(n) \equiv \sum_{d|n} d^{11} \pmod{691} \) (see e.g. Swinnerton-Dyer [5]). Our congruences may be different from those. Our results are as follows:

**Theorem.** Let \( p \) be a prime. If \( g(t) \in \mathbb{Z}_p[t] \) in (1), (2), then

\[ \gamma(p) \equiv \alpha(p) \pmod{p}. \]

In particular, if a cusp form \( f(z) \) is a normalized \( (\gamma(1) = 1) \) common eigenfunction of Hecke operators, then for any positive integer \( r \),

\[ \gamma(p) \alpha(p^r) \equiv \alpha(p^{r+1}) + p^{k-1} \alpha(p^{r-1}) \pmod{p^{r+1}}. \]

**Example.** Put \( k = 12 \). We have

\[ f(z) = \Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \]

where \( \tau(n) \) is the Ramanujan \( \tau \)-function. The first few values of \( \tau(n) \) are 1, -24, 252, -1472, 4830, \ldots. Now, let

\[
\sum_{n=1}^{\infty} \alpha(n) t^n = \frac{t}{\sqrt{1 - 1728t}} 2F_1 \left( \frac{1}{12}, \frac{5}{12}, 1; 1728t \right)^{10} = t + 1464t^2 + 2197944t^3 + 3393216960t^4 + 5343171374520t^5 + \ldots
\]

Since \( \Delta(z) \) is a normalized common eigenfunction of Hecke operators (see e.g. Ogg [2]), we have

\[ \tau(p) \alpha(p^r) \equiv \alpha(p^{r+1}) + p^{11} \alpha(p^{r-1}) \pmod{p^{r+1}} \]

for prime \( p \) and integer \( r \geq 0 \). Then, if \( \tau(p) \not\equiv 0 \pmod{p} \), we can determine \( \tau(p) \) by using \( \alpha(p^r) \) from the Ramanujan conjecture: \( |\tau(p)| \leq 2p^{11/2} \).

We shall need two lemmas in order to prove our theorem.

**Lemma 1.** We have

(a) \( E_4(z) = 2F_1(1/12, 5/12, 1; 1728/j(z))^4 \),

(b) \( E_6(z) = \sqrt{1 - 1728/j(z)} 2F_1(1/12, 5/12, 1; 1728/j(z))^6 \).

**Proof.** See Theorems 3 and 4 in Stiller [4].
Lemma 2. Let \( A(t) = \sum_{n=0}^{\infty} a(n)t^n \) and \( t(u) = \sum_{n=1}^{\infty} c(n)u^n \) \((c(1) = 1)\) be two formal power series with coefficients in \( \mathbb{Z}_p \) \((p\text{ prime})\), and denote by \( b(n) \) the coefficient of \( u^n \) in \( B(u) = A(t(u)) \frac{u}{t(u)} \frac{dt(u)}{du} \). If \( a(0) = 0 \), then

(a) \( a(p) \equiv b(p) \pmod{p} \),
(b) for all integers \( r > 0 \) and \( s_1(p), s_2(p) \in \mathbb{Z}_p \), conditions (i), (ii) below are equivalent:

(i) \( a(p^{r+1}) + s_1(p) a(p^r) + ps_2(p) a(p^{r-1}) \equiv 0 \pmod{p^{r+1}} \),
(ii) \( b(p^{r+1}) + s_1(p) b(p^r) + ps_2(p) b(p^{r-1}) \equiv 0 \pmod{p^{r+1}} \).

Proof. (a) is clear, since

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n} t^n = \int A(t) \frac{dt}{t} = \int B(u) \frac{du}{u} = \sum_{n=1}^{\infty} \frac{b(n)}{n} u^n.
\]

(i) implies

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n} t^n + s_1(p) \sum_{n=1}^{\infty} \frac{a(n)}{pn} t^{pn} + ps_2(p) \sum_{n=1}^{\infty} \frac{a(n)}{p^2n} t^{p^2n} \in \mathbb{Z}_p[[t]].
\]

Since \( t(u)^p - t(u^p) \), \( t(u)p^2 - t(u^2) \in p\mathbb{Z}_p[[t]] \), we have

\[
\sum_{n=1}^{\infty} \frac{a(n)}{n} u^n + s_1(p) \sum_{n=1}^{\infty} \frac{a(n)}{pn} u^{pn} + ps_2(p) \sum_{n=1}^{\infty} \frac{a(n)}{p^2n} u^{p^2n} = \int (B(u) + s_1(p)B(u^p) + ps_2(p)B(u^{p^2})) \frac{du}{u} = \sum_{n=1}^{\infty} \frac{b(n)}{n} u^n + s_1(p) \sum_{n=1}^{\infty} \frac{b(n)}{pn} u^{pn} + ps_2(p) \sum_{n=1}^{\infty} \frac{b(n)}{p^2n} u^{p^2n} \in \mathbb{Z}_p[[u]].
\]

This is equivalent to (ii). If \( c(1) = 1 \), we have \( u \in \mathbb{Z}_p[[t]] \). Then (ii) implies (i). (For the more general case of this lemma, see Appendix in Stienstra and Beukers [3] and Proposition 3 in Beukers [1].)

Proof of Theorem. Put \( E_2(z) = 1 - 24 \sum_{n=1}^{\infty} (\sum_{d|n} d)q^n \). Since \( E_4 \) and \( E_6 \) are modular forms of weight 4 and 6, respectively, and

\[
E_2 \left( \frac{az + b}{cz + d} \right) = (cz + d)^2 E_2(z) + \frac{6c(cz + d)}{\pi i}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}),
\]

it is not hard to check that

\[
\frac{d}{dq} E_4 = \frac{E_2 E_4 - E_6}{3q}, \quad \frac{d}{dq} E_6 = \frac{E_2 E_6 - E_4^2}{2q}, \quad \frac{d}{dq} \Delta = \frac{E_2 \Delta}{q}.
\]
Let \( t(q) = 1/j(z) = q - 744q^2 + \ldots \). It follows easily that
\[
\frac{q}{t(q)} \frac{dt(q)}{dq} = \frac{E_6}{E_4}.
\]
From (2) and Lemma 1, we have
\[
F(t(q)) \frac{q}{t(q)} \frac{dt(q)}{dq} = \left( \frac{\Delta}{E_4^3} \right)^m g(j) \sqrt{1 - \frac{1728}{j}^{z-1}} \ \text{$_2$F$_1$} \left( \frac{1}{12}, \frac{5}{12}, 1; \frac{1728}{j} \right)^{12m+4\delta+6\varepsilon-2} \frac{E_6}{E_4}.
\]
\[
= \Delta^m E_4^{\delta+1} E_6^{\varepsilon-1} g(j) \cdot \frac{E_6}{E_4} = f(z).
\]
If \( f(z) \) is a normalized common eigenfunction of Hecke operators, then for any positive integer \( r \),
\[
\gamma(p) \gamma(p^r) = \gamma(p^{r+1}) + p^{k-1} \gamma(p^{r-1}).
\]
From Lemma 2, our Theorem follows. ■

References


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