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# On certain congruences for Fourier coefficients of classical cusp forms

by

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For any even integer $k \geq 12$, let $S_k$ denote the space of cusp forms of weight $k$ on the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The Eisenstein series $E_4$ and $E_6$ are given by

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^3\Big) q^n$$
$$= 1 + 240q + 2160q^2 + 6720q^3 + 17520q^4 + 30240q^5 + \ldots,$$
$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \Big(\sum_{d|n} d^5\Big) q^n$$
$$= 1 - 504q - 16632q^2 - 122976q^3 - 532728q^4 - \ldots$$

where $q = e^{2\pi i z}$, and we also define $\Delta(z) = (E_4^3 - E_6^2)/1728 \in S_{12}$ and $j(z) = E_4^3/\Delta(z)$, the modular invariant. It is well known that any cusp form $f(z) \in S_k$ can be written uniquely as

$$f(z) = \Delta(z)^m E_4(z)^{\delta} E_6(z)^{\varepsilon} g(j(z)) = \sum_{n=1}^{\infty} \gamma(n) q^n \tag{1}$$

for some polynomial $g(x) \in \mathbb{C}[x]$ of degree $\leq m-1$, where $\gamma(n)$ are the Fourier coefficients, $m$ is positive integer, and $k = 12m + 4\delta + 6\varepsilon$ automatically with $\delta \in \{0,1,2\}$, $\varepsilon \in \{0,1\}$.

The classical Gauss hypergeometric series are defined as follows:

$${}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}\, x^n$$

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where $(a)_n = a(a+1)(a+2)\dots(a+n-1)$. We define the function $F(t)$ and the coefficients $\alpha(n)$ of $t^n$ in $F(t)$, for the above $k, m, \varepsilon$ and polynomial $g$:

$$F(t) = t^m g\left(\frac{1}{t}\right)\sqrt{1-1728t}^{\,\varepsilon-1}\,{}_2F_1\left(\frac{1}{12}, \frac{5}{12}, 1; 1728t\right)^{k-2} = \sum_{n=1}^{\infty} \alpha(n)t^n. \tag{2}$$

We know that $\alpha(n)$ satisfies a linear recurrence, since the function $F(t)$ is a solution of a certain linear differential equation.

In this paper, we study the congruence properties of $\gamma(n)$ and $\alpha(n)$. For Fourier coefficients of cusp forms, many congruences are known, for example, for the Ramanujan $\tau$-function, $\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}$ (see e.g. Swinnerton-Dyer [5]). Our congruences may be different from those. Our results are as follows:

THEOREM. *Let $p$ be a prime. If $g(t) \in \mathbb{Z}_p[t]$ in* (1), (2), *then*

$$\gamma(p) \equiv \alpha(p) \pmod{p}.$$

*In particular*, *if a cusp form $f(z)$ is a normalized* ($\gamma(1) = 1$) *common eigenfunction of Hecke operators*, *then for any positive integer $r$*,

$$\gamma(p)\alpha(p^r) \equiv \alpha(p^{r+1}) + p^{k-1}\alpha(p^{r-1}) \pmod{p^{r+1}}.$$

EXAMPLE. Put $k = 12$. We have

$$f(z) = \Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

where $\tau(n)$ is the Ramanujan $\tau$-function. The first few values of $\tau(n)$ are $1, -24, 252, -1472, 4830, \dots$ Now, let

$$\sum_{n=1}^{\infty} \alpha(n)t^n = \frac{t}{\sqrt{1-1728t}}\,{}_2F_1\left(\frac{1}{12}, \frac{5}{12}, 1; 1728t\right)^{10}$$
$$= t + 1464t^2 + 2197944t^3 + 3393216960t^4 + 5343171374520t^5 + \dots$$

Since $\Delta(z)$ is a normalized common eigenfunction of Hecke operators (see e.g. Ogg [2]), we have

$$\tau(p)\alpha(p^r) \equiv \alpha(p^{r+1}) + p^{11}\alpha(p^{r-1}) \pmod{p^{r+1}}$$

for prime $p$ and integer $r \ge 0$. Then, if $\tau(p) \not\equiv 0 \pmod{p}$, we can determine $\tau(p)$ by using $\alpha(p^r)$ from the Ramanujan conjecture: $|\tau(p)| \le 2p^{11/2}$.

We shall need two lemmas in order to prove our theorem.

LEMMA 1. *We have*

(a) $E_4(z) = {}_2F_1(1/12, 5/12, 1; 1728/j(z))^4$,

(b) $E_6(z) = \sqrt{1-1728/j(z)}\,{}_2F_1(1/12, 5/12, 1; 1728/j(z))^6$.

*Proof.* See Theorems 3 and 4 in Stiller [4]. ∎

LEMMA 2. *Let* $A(t)=\sum_{n=0}^{\infty}a(n)t^n$ *and* $t(u)=\sum_{n=1}^{\infty}c(n)u^n$ $(c(1)=1)$ *be two formal power series with coefficients in* $\mathbb{Z}_p$ $(p$ *prime*$)$, *and denote by* $b(n)$ *the coefficient of* $u^n$ *in* $B(u)=A(t(u))\frac{u}{t(u)}\frac{dt(u)}{du}$. *If* $a(0)=0$, *then*

(a) $a(p)\equiv b(p)\pmod p$,

(b) *for all integers* $r>0$ *and* $s_1(p),s_2(p)\in\mathbb{Z}_p$, *conditions* (i), (ii) *below are equivalent*:

(i) $a(p^{r+1})+s_1(p)a(p^r)+ps_2(p)a(p^{r-1})\equiv 0\pmod{p^{r+1}}$,

(ii) $b(p^{r+1})+s_1(p)b(p^r)+ps_2(p)b(p^{r-1})\equiv 0\pmod{p^{r+1}}$.

*Proof.* (a) is clear, since

$$\sum_{n=1}^{\infty}\frac{a(n)}{n}t^n=\int A(t)\frac{dt}{t}=\int B(u)\frac{du}{u}=\sum_{n=1}^{\infty}\frac{b(n)}{n}u^n.$$

(i) implies

$$\sum_{n=1}^{\infty}\frac{a(n)}{n}t^n+s_1(p)\sum_{n=1}^{\infty}\frac{a(n)}{pn}t^{pn}+ps_2(p)\sum_{n=1}^{\infty}\frac{a(n)}{p^2n}t^{p^2n}\in\mathbb{Z}_p[[t]].$$

Since $t(u)^{pn}-t(u^p)^n,\ t(u)^{p^2n}-t(u^{p^2})^n\in p\mathbb{Z}_p[[t]]$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{a(n)}{n}t(u)^n+s_1(p)\sum_{n=1}^{\infty}\frac{a(n)}{pn}t(u^p)^n+ps_2(p)\sum_{n=1}^{\infty}\frac{a(n)}{p^2n}t(u^{p^2})^n\\
&\quad=\int A(t(u))\frac{dt(u)}{t(u)}+s_1(p)\int A(t(u^p))\frac{dt(u^p)}{t(u^p)}+ps_2(p)\int A(t(u^{p^2}))\frac{dt(u^{p^2})}{t(u^{p^2})}\\
&\quad=\int (B(u)+s_1(p)B(u^p)+ps_2(p)B(u^{p^2}))\frac{du}{u}\\
&\quad=\sum_{n=1}^{\infty}\frac{b(n)}{n}u^n+s_1(p)\sum_{n=1}^{\infty}\frac{b(n)}{pn}u^{pn}+ps_2(p)\sum_{n=1}^{\infty}\frac{b(n)}{p^2n}u^{p^2n}\in\mathbb{Z}_p[[u]].
\end{aligned}$$

This is equivalent to (ii). If $c(1)=1$, we have $u\in\mathbb{Z}_p[[t]]$. Then (ii) implies (i). (For the more general case of this lemma, see Appendix in Stienstra and Beukers [3] and Proposition 3 in Beukers [1].) ∎

*Proof of Theorem.* Put $E_2(z)=1-24\sum_{n=1}^{\infty}(\sum_{d|n}d)q^n$. Since $E_4$ and $E_6$ are modular forms of weight 4 and 6, respectively, and

$$E_2\left(\frac{az+b}{cz+d}\right)=(cz+d)^2E_2(z)+\frac{6c(cz+d)}{\pi i},\qquad \begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}(2,\mathbb{Z}),$$

it is not hard to check that

$$\frac{d}{dq}E_4=\frac{E_2E_4-E_6}{3q},\qquad \frac{d}{dq}E_6=\frac{E_2E_6-E_4^2}{2q},\qquad \frac{d}{dq}\Delta=\frac{E_2\Delta}{q}.$$

Let $t(q) = 1/j(z) = q - 744q^2 + \dots$ It follows easily that

$$\frac{q}{t(q)}\,\frac{dt(q)}{dq} = \frac{E_6}{E_4}.$$

From (2) and Lemma 1, we have

$$\begin{aligned} &F(t(q))\frac{q}{t(q)}\,\frac{dt(q)}{dq} \\ &\quad = \left(\frac{\Delta}{E_4^3}\right)^m g(j)\sqrt{1-\frac{1728}{j}}^{\varepsilon-1}{}_2F_1\left(\frac{1}{12},\frac{5}{12},1;\frac{1728}{j}\right)^{12m+4\delta+6\varepsilon-2}\cdot\frac{E_6}{E_4} \\ &\quad = \Delta^m E_4^{\delta+1}E_6^{\varepsilon-1}g(j)\cdot\frac{E_6}{E_4} = f(z). \end{aligned}$$

If $f(z)$ is a normalized common eigenfunction of Hecke operators, then for any positive integer $r$,

$$\gamma(p)\gamma(p^r) = \gamma(p^{r+1}) + p^{k-1}\gamma(p^{r-1}).$$

From Lemma 2, our Theorem follows. ■

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