Improved upper bounds for the star discrepancy of digital nets in dimension 3

by

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1. Introduction. The concept of digital nets provides at the moment the most efficient method to generate point sets with small star discrepancy $D^*_N$. The star discrepancy of a set of points $x_0, \ldots, x_{N-1}$ in $[0,1)^d$ is defined by

$$D^*_N = \sup_B \left| \frac{A_N(B)}{N} - \lambda(B) \right|,$$

where the supremum is taken over all subintervals $B$ of $[0,1)^d$ of the form $B = \prod_{i=1}^d [0,b_i)$, $0 < b_i \leq 1$, $A_N(B)$ denotes the number of $i$ with $x_i \in B$ and $\lambda$ is the Lebesgue measure.

A digital $(0,s,3)$-net in base 2 is a set of $N = 2^s$ points $x_0, \ldots, x_{N-1}$ in $[0,1)^3$ which is generated as follows: Choose three $s \times s$-matrices $C_1$, $C_2$ and $C_3$ over $\mathbb{Z}_2$ with the following property: For all integers $d_1, d_2, d_3 \geq 0$ with $d_1 + d_2 + d_3 = s$, the system of the first $d_1$ rows of $C_1$ together with the first $d_2$ rows of $C_2$ and the first $d_3$ rows of $C_3$ is linearly independent over $\mathbb{Z}_2$. Then to construct $x_n := (x_n^{(1)}, x_n^{(2)}, x_n^{(3)})$ for $0 \leq n \leq 2^s - 1$, represent $n$ in base 2:

$$n = n_0 + n_1 2 + \ldots + n_{s-1} 2^{s-1}$$

with $n_j \in \{0,1\}$. Now multiply $C_i$ with the vector of digits:

$$C_i(n_0, \ldots, n_{s-1})^T := (y_1^{(i)}, \ldots, y_s^{(i)})^T \in \mathbb{Z}_2^s$$

and set

$$x_n^{(i)} := \sum_{j=1}^s \frac{y_j^{(i)}}{2^s}.$$

Further let us recall the definition of digital $(0,2)$-sequences in base 2: A digital $(0,2)$-sequence in base 2 is a sequence $x_0, x_1, \ldots$ in $[0,1)^2$ which is

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generated as follows: Choose two \( N \times N \)-matrices \( C_1 \) and \( C_2 \) over \( \mathbb{Z}_2 \) such that for every integer \( s \geq 1 \) the upper left \( s \times s \)-matrices \( C_1(s) \) and \( C_2(s) \) generate a digital \((0, s, 2)\)-net in base 2 (a digital \((0, s, 2)\)-net in base 2 is defined analogously as a digital \((0, s, 3)\)-net in base 2—see Section 3). Then to construct \( x_n := (x_n^{(1)}, x_n^{(2)}) \) for \( n \geq 0 \), represent \( n \) in base 2:

\[
n = n_0 + n_1 2 + n_2 2^2 + \ldots
\]

with \( n_j \in \{0, 1\} \). Now multiply \( C_i \) with the vector of digits:

\[
C_i(n_0, n_1, n_2, \ldots)^T =: (y_i^{(1)}, y_i^{(2)}, \ldots)^T
\]

and set

\[
x_n^{(i)} := \sum_{j=1}^{\infty} \frac{y_j^{(i)}}{2^j} .
\]

It was shown by H. Niederreiter in [6] that for the star discrepancy of any digital \((0, s, 3)\)-net in base 2 we have

\[
ND_N^* \leq \frac{s^2}{4} + \frac{s}{2} + \frac{9}{4}
\]

and hence

\[
\limsup_{N \to \infty} \frac{ND_N^*}{(\log N)^2} \leq \frac{1}{4(\log 2)^2} = 0.5203 \ldots ,
\]

where the maximum is taken over all digital \((0, s, 3)\)-nets in base 2 with \( N = 2^s \) elements.

Again in [6] Niederreiter proved that for the star discrepancy of the first \( N \) elements of a digital \((0, 2)\)-sequence in base 2 we have

\[
ND_N^* \leq \frac{1}{8(\log 2)^2} (\log N)^2 + \frac{11}{8\log 2} \log N + \frac{9}{4}
\]

and hence

\[
\limsup_{N \to \infty} \frac{ND_N^*}{(\log N)^2} \leq \frac{1}{8(\log 2)^2} = 0.26017 \ldots ,
\]

where the maximum is taken over all digital \((0, 2)\)-sequences in base 2. From this result he concluded for every integer \( s \geq 1 \) the existence of a digital \((0, s, 3)\)-net in base 2 such that

\[
ND_N^* \leq s^2/8 + O(s),
\]

where \( N = 2^s \).

In [1] H. Faure constructed a digital \((0, 2)\)-sequence in base 2 such that

\[
\limsup_{N \to \infty} \frac{ND_N^*}{(\log N)^2} \geq \frac{1}{24(\log 2)^2} = 0.0867 \ldots
\]

In this paper we study the star discrepancy of digital \((0, s, 3)\)-nets in base 2 and of digital \((0, 2)\)-sequences in base 2. With the help of Walsh
series analysis we improve the general bound for the star discrepancy of digital \((0, s, 3)\)-nets in base 2 given by Niederreiter (Theorem 1). Further we give an improved upper bound for the star discrepancy of digital \((0, 2)\)-sequences in base 2 (Theorem 3) from which we conclude—in the same way as Niederreiter did in [6]—the existence of digital \((0, s, 3)\)-nets in base 2 with an essentially smaller bound for the star discrepancy than the general bound given in Theorem 1 (Theorem 2).

2. The results. We have the following general upper bound for the star discrepancy of digital \((0, s, 3)\)-nets in base 2. This bound improves the discrepancy bound given in [6].

**Theorem 1.** For all digital \((0, s, 3)\)-nets in base 2 we have

\[
ND^*_N \leq s^2/6 + O(s),
\]

where \(N = 2^s\).

The proof will be given in Section 4. From Theorem 1 we immediately get the following corollary:

**Corollary 1.** We have

\[
\limsup_{N \to \infty} \max_{x} \frac{ND^*_N}{(\log N)^2} \leq \frac{1}{6(\log 2)^2} = 0.34689 \ldots,
\]

where the maximum is taken over all digital \((0, s, 3)\)-nets in base 2.

Actually we can prove the existence of digital \((0, s, 3)\)-nets in base 2 with an essentially smaller constant at the leading term in the discrepancy bound as given in Theorem 1. We have

**Theorem 2.** For every \(s \geq 1\) there exists a digital \((0, s, 3)\)-net in base 2 such that

\[
ND^*_N \leq s^2/12 + O(s),
\]

where \(N = 2^s\).

The proof of this theorem will be given in Section 5. The digital \((0, s, 3)\)-nets in base 2 for which the discrepancy bound in Theorem 2 holds are obtained by setting \(x_n = (n/2^s, y_n), n = 0, \ldots, 2^s - 1\), where \(y_n\) is the \(n\)th element of a digital \((0, 2)\)-sequence in base 2. We shall see that the above Theorem 2 is a consequence of the following theorem:

**Theorem 3.** For the star discrepancy \(D^*_N\) of the first \(N\) elements of a digital \((0, 2)\)-sequence in base 2 we have

\[
ND^*_N \leq \frac{1}{12(\log 2)^2} \left(\log N\right)^2 + \frac{33}{36 \log 2} \log N + \frac{89}{12}.
\]

The proof of this theorem will be given in Section 5. Combining the result from Theorem 3 with the result of Faure [1] mentioned in Section 1 we obtain
Corollary 2. We have
\[ \frac{1}{24(\log 2)^2} \leq \limsup_{N \to \infty} \max_{D_N^*} \frac{N D_N^*}{(\log N)^2} \leq \frac{1}{12(\log 2)^2}, \]
where the maximum is taken over all digital \((0,2)\)-sequences in base 2.

3. Notation and auxiliary results. For \(0 \leq \alpha, \beta, \gamma \leq 1\) we consider the discrepancy function
\[ \Delta(\alpha, \beta, \gamma) := A_N([0, \alpha) \times [0, \beta) \times [0, \gamma)) - N \alpha \beta \gamma \]
for digital \((0, s, 3)\)-nets \(x_0, \ldots, x_{2^s-1}\) in base 2 (i.e. \(N = 2^s\)).

Since the generating matrices \(C_1, C_2\) and \(C_3\) of a \((0, s, 3)\)-net must be regular, and since multiplying \(C_1, C_2\) and \(C_3\) by a regular matrix \(A\) does not change the point set (only its order) we may always assume that
\[
C_1 = \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1
\end{pmatrix},
C_2 = \begin{pmatrix}
c_{1,1}^2 & c_{1,2}^2 & \ldots & c_{1,s}^2 \\
c_{2,1}^2 & c_{2,2}^2 & \ldots & c_{2,s}^2 \\
\ldots & \ldots & \ldots & \ldots \\
c_{s,1}^2 & c_{s,2}^2 & \ldots & c_{s,s}^2
\end{pmatrix} =: \begin{pmatrix}
c_1^2 \\
c_2^2 \\
\vdots \\
c_s^2
\end{pmatrix},
\]
\[
C_3 = \begin{pmatrix}
c_{1,1}^3 & c_{1,2}^3 & \ldots & c_{1,s}^3 \\
c_{2,1}^3 & c_{2,2}^3 & \ldots & c_{2,s}^3 \\
\ldots & \ldots & \ldots & \ldots \\
c_{s,1}^3 & c_{s,2}^3 & \ldots & c_{s,s}^3
\end{pmatrix} =: \begin{pmatrix}
c_1^3 \\
c_2^3 \\
\vdots \\
c_s^3
\end{pmatrix}.
\]

Assume that \(\alpha, \beta\) and \(\gamma\) are “s-bit”, i.e.
\[
\alpha = \frac{\alpha_1}{2} + \ldots + \frac{\alpha_s}{2^s}, \quad \beta = \frac{\beta_1}{2} + \ldots + \frac{\beta_s}{2^s}, \quad \gamma = \frac{\gamma_1}{2} + \ldots + \frac{\gamma_s}{2^s},
\]
and let \(\alpha', \beta'\) and \(\gamma'\) be arbitrary with
\[
\alpha \leq \alpha' < \alpha + \frac{1}{2^s}, \quad \beta \leq \beta' < \beta + \frac{1}{2^s}, \quad \gamma \leq \gamma' < \gamma + \frac{1}{2^s}.
\]

Then (since all coordinates of the points of a digital net are \(s\)-bit) we have
\[
\Delta(\alpha', \beta', \gamma') = \Delta(\alpha, \beta, \gamma) - 2^s(\alpha' \beta' \gamma' - \alpha \beta \gamma),
\]
and hence for the star-discrepancy \(D_N^*\) of the net we have
\[
(1) \quad \left| D_N^* - \frac{1}{N} \max_{\alpha, \beta, \gamma \text{ s-bit}} |\Delta(\alpha, \beta, \gamma)| \right| < \frac{3}{N} - \frac{3}{N^2} + \frac{1}{N^3}
\]
(note that \(N = 2^s\)).
We will call
\[ \frac{1}{N} \max_{\alpha, \beta, \gamma} |\Delta(\alpha, \beta, \gamma)| =: D_N^d \]
the \textit{discrete discrepancy} of the net. $D_N^d$ differs from $D_N^*$ at most by the almost negligible quantity $3/N$ and seems for nets to be the more natural measure for the irregularities of distribution.

We need some further notation: For any $s$-bit number $\delta = \delta_1/2 + \ldots + \delta_s/2^s$ we write
\[ \vec{\delta} := \left( \begin{array}{c} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_s \end{array} \right), \]
and for a non-negative integer $k = k_{s-1}2^{s-1} + \ldots + k_12 + k_0$ we write
\[ \vec{k} := \left( \begin{array}{c} k_0 \\ k_1 \\ \vdots \\ k_{s-1} \end{array} \right). \]

For the proof of Theorem 1 we need two auxiliary results.

**Lemma 1.** Let $z$ be of the form $z = p/2^s$, $p \in \{0, \ldots, 2^s - 1\}$ (i.e. $z$ is $s$-bit). Then for the characteristic function $\chi_{[0,z)}$ of the interval $[0, z)$ we have
\[ \chi_{[0,z)}(x) = \sum_{k=0}^{2^s-1} c_k(z) \text{wal}_k(x), \]
where $\text{wal}_k$ denotes the $k$th Walsh function in base 2 (see Remark 1),
\[ c_k(z) = \begin{cases} z & \text{if } k = 0, \\ \text{wal}_k(z) \frac{1}{2^{v(k)}} \psi(2^{v(k)}z) & \text{if } k \neq 0, \end{cases} \]
where $\psi(x)$ is periodic with period 1 and
\[ \psi(x) = \begin{cases} x & \text{if } 0 \leq x < 1/2, \\ x - 1 & \text{if } 1/2 \leq x < 1, \end{cases} \]
and $v(k) = r$ if $2^r \leq k < 2^{r+1}$ (for $k = 0$ define $v(0) := -1$).

**Remark 1.** Recall that Walsh functions in base 2 can be defined as follows: For a non-negative integer $k$ with base 2 representation $k = k_m2^m + \ldots + k_12 + k_0$ and a real $x$ with (canonical) base 2 representation $x = x_1/2 + x_2/2^2 + \ldots$ we have
\[ \text{wal}_k(x) = (-1)^{x_1k_0 + x_2k_1 + \ldots + x_mk_m} = (-1)^{(\vec{k}|\vec{x})}. \]
Proof of Lemma 1. This is a simple calculation, to be found for example in [3, Lemma 2].

Lemma 2. Let $\psi$ be as in Lemma 1. Then

$$\psi(2^{l+1}\beta) - \sum_{i=0}^{l} \psi(2^i\beta) = \{\beta\} - \beta_{l+2},$$

where $\{\beta\} = \beta - [\beta]$.

Proof. See [4, Lemma 2].

For the proof of Theorem 3 we need some further notation and auxiliary results:

The concept of shifted digital $(0,s,2)$-nets in base 2 is a slight generalization of the well known concept of digital $(0,s,2)$-nets in base 2. A shifted digital $(0,s,2)$-net in base 2 is a set of $N = 2^s$ points $x_0, \ldots, x_{N-1}$ in $[0,1)^2$ which is generated as follows: Choose two $s \times s$-matrices $C_1, C_2$ over $\mathbb{Z}_2$ with the following property: For every integer $k$, $0 \leq k \leq s$, the system of the first $k$ rows of $C_1$ together with the first $s-k$ rows of $C_2$ is linearly independent over $\mathbb{Z}_2$. Further choose two fixed vectors $\bar{k}_i = (k_1^{(i)}, \ldots, k_s^{(i)})^T \in \mathbb{Z}_2^s$, $i = 1, 2$. Then to construct $x_n := (x_n^{(1)}, x_n^{(2)})$ for $0 \leq n \leq 2^s - 1$, represent $n$ in base 2:

$$n = n_0 + n_1 2 + \ldots + n_{s-1} 2^{s-1}$$

with $n_j \in \{0, 1\}$. Now multiply $C_i$ with the vector of digits and add the vector $\bar{k}_i$, i.e.:

$$C_i(n_0, \ldots, n_{s-1})^T + \bar{k}_i =: (y_1^{(i)}, \ldots, y_s^{(i)})^T \in \mathbb{Z}_2^s$$

and set

$$x_n^{(i)} := \sum_{j=1}^{s} \frac{y_j^{(i)}}{2^j}.$$

Remark 2. In the definition of usual digital $(0,s,2)$-nets in base 2 the vectors $\bar{k}_i$, $i = 1, 2$, are omitted.

For the star discrepancy of shifted digital $(0,s,2)$-nets in base 2 we have the following result:

Lemma 3. For the star discrepancy $D_N^*$ of a shifted digital $(0,s,2)$-net in base 2 we have

$$ND_N^* \leq \frac{s}{3} + \frac{19}{9},$$

where $N = 2^s$. 
Proof. In [4, Theorem 5] this lemma was proved for digital \((0, s, 2)\)-nets in base 2. It easily follows from the proof that the assertion is also true for shifted digital nets. ■

Finally we need the following general result which is well known in the theory of uniform distribution modulo one:

**Lemma 4.** Let \(x_0, \ldots, x_{N-1}\) be a point set in \([0, 1]^d\) with star discrepancy \(D_\star^\ast\). Let \(x_n := (x_n^{(1)}, \ldots, x_n^{(d)})\), \(0 \leq n \leq N - 1\), and let \(\varepsilon_n^{(i)}\), \(0 \leq n \leq N - 1\), \(1 \leq i \leq d\), be non-negative reals with \(\varepsilon_n^{(i)} < 1/a\), such that \(x_n^{(i)} + \varepsilon_n^{(i)} < 1\), for all \(0 \leq n \leq N - 1\), \(1 \leq i \leq d\). Then for the star discrepancy \(\tilde{D}_\star^\ast\) of the point set \(\tilde{x}_0, \ldots, \tilde{x}_{N-1}\), with \(\tilde{x}_n^{(i)} := x_n^{(i)} + \varepsilon_n^{(i)}\) for all \(0 \leq n \leq N - 1\), \(1 \leq i \leq d\), we have

\[
|D_\star^\ast - \tilde{D}_\star^\ast| \leq d/a.
\]

4. **Proof of Theorem 1.** Due to formula (1) it suffices to show that

\[
ND_N^d \leq s^2/6 + \mathcal{O}(s)
\]

for all digital \((0, s, 3)\)-nets in base 2.

Let \(I := [0, \alpha) \times [0, \beta) \times [0, \gamma)\) with \(\alpha, \beta\) and \(\gamma\) \(s\)-bit. Then for \(y = (y^{(1)}, y^{(2)}, y^{(3)}) \in [0, 1)^3\) by Lemma 1 we have

\[
\chi_I(y) - \lambda(I) = \sum_{k,l,m=0}^{2^s-1} c_k(\alpha)c_l(\beta)c_m(\gamma)\text{wal}_k(y^{(1)})\text{wal}_l(y^{(2)})\text{wal}_m(y^{(3)})
\]

\[
= \sum_{l,m=0}^{2^s-1} c_l(\beta)c_m(\gamma)\text{wal}_l(y^{(2)})\text{wal}_m(y^{(3)})
\]

\[
+ \sum_{k,m=0}^{2^s-1} c_k(\alpha)c_m(\gamma)\text{wal}_k(y^{(1)})\text{wal}_m(y^{(3)})
\]

\[
+ \sum_{k,l=0}^{2^s-1} c_k(\alpha)c_l(\beta)\text{wal}_k(y^{(1)})\text{wal}_l(y^{(2)})
\]

\[
+ \sum_{k,l,m=1}^{2^s-1} \text{wal}_k(\alpha)\text{wal}_l(\beta)\text{wal}_m(\gamma) \frac{\psi(2^u(k)\alpha)\psi(2^u(l)\beta)\psi(2^u(m)\gamma)}{2^u(k)2^u(l)2^u(m)}
\]

\[
\times \text{wal}_k(y^{(1)})\text{wal}_l(y^{(2)})\text{wal}_m(y^{(3)}).
\]

Let now \(x_i, i = 0, \ldots, 2^s-1\), with \(x_i := (x_i^{(1)}, x_i^{(2)}, x_i^{(3)})\) be a digital \((0, s, 3)\)-
net in base 2. Then we have
\[
\Delta(\alpha, \beta, \gamma) = \alpha \sum_{l,m=0 \atop (l,m) \neq (0,0)}^{2^s-1} c_l(\beta)c_m(\gamma) \sum_{i=0}^{2^s-1} \text{wal}_l(x_{1,i}^{(2)})\text{wal}_m(x_{1,i}^{(3)})
\]
\[+ \beta \sum_{k,m=0 \atop (k,m) \neq (0,0)}^{2^s-1} c_k(\alpha)c_m(\gamma) \sum_{i=0}^{2^s-1} \text{wal}_k(x_{1,i}^{(1)})\text{wal}_m(x_{1,i}^{(3)})
\]
\[+ \gamma \sum_{k,l=0 \atop (k,l) \neq (0,0)}^{2^s-1} c_k(\alpha)c_l(\beta) \sum_{i=0}^{2^s-1} \text{wal}_k(x_{1,i}^{(1)})\text{wal}_l(x_{1,i}^{(2)})
\]
\[+ \sum_{k,l,m=1}^{2^s-1} \text{wal}_k(\alpha)\text{wal}_l(\beta)\text{wal}_m(\gamma) \frac{\psi(2^v(k)\alpha)\psi(2^v(l)\beta)\psi(2^v(m)\gamma)}{2^v(k)2^v(l)2^v(m)}
\]
\[\times \sum_{i=0}^{2^s-1} \text{wal}_k(x_{1,i}^{(1)})\text{wal}_l(x_{1,i}^{(2)})\text{wal}_m(x_{1,i}^{(3)})
\]
\[=: \alpha \Sigma_1 + \beta \Sigma_2 + \gamma \Sigma_3 + \Sigma_4.
\]
From [4, Theorem 5] together with the proof of [4, Theorem 1] it follows that
\[|\Sigma_i| \leq \frac{s}{3} + \frac{19}{9}
\]
for \(i = 1, 2, 3\), and hence it suffices to show that
\[|\Sigma_4| \leq s^2/6 + O(s)
\]
for all digital \((0, s, 3)\)-nets in base 2.

We now consider \(\sum_{i=0}^{2^s-1} \text{wal}_k(x_{1,i}^{(1)})\text{wal}_l(x_{1,i}^{(2)})\text{wal}_m(x_{1,i}^{(3)})\) with \(x_{1,i}^{(1)} := x_{i,1}^{(1)}/2 + \ldots + x_{i,s}^{(1)}/2^s\), \(x_{1,i}^{(2)} := x_{i,1}^{(2)}/2 + \ldots + x_{i,s}^{(2)}/2^s\) and \(x_{1,i}^{(3)} := x_{i,1}^{(3)}/2 + \ldots + x_{i,s}^{(3)}/2^s\). We identify \((x_{1,i}^{(1)}, x_{i}^{(2)}, x_{1,i}^{(3)})\) with
\[(x_{1,i,1}, \ldots, x_{i,s}^{(1)}, x_{i,1}, \ldots, x_{i,s}^{(2)}, x_{1,i,1}, \ldots, x_{i,s}^{(3)})^T \in (\mathbb{Z}_2)^{3s}
\]
and we define
\[(x_{i}^{(1)}, x_{i}^{(2)}, x_{1,i}^{(3)}) \oplus (\tilde{x}_{i}^{(1)}, \tilde{x}_{i}^{(2)}, \tilde{x}_{1,i}^{(3)}) := (x_{i,1}^{(1)} + \tilde{x}_{i,1}^{(1)}, \ldots, x_{i,s}^{(1)} + \tilde{x}_{i,s}^{(1)}).
\]

Further \(\text{wal}_{k,l,m}(x_{i}^{(1)}, x_{i}^{(2)}, x_{1,i}^{(3)}) := \text{wal}_{k}(x_{i}^{(1)})\text{wal}_{l}(x_{i}^{(2)})\text{wal}_{m}(x_{1,i}^{(3)})\), hence
\[\text{wal}_{k,l,m}((x_{i}^{(1)}, x_{i}^{(2)}, x_{1,i}^{(3)}) \oplus (\tilde{x}_{i}^{(1)}, \tilde{x}_{i}^{(2)}, \tilde{x}_{1,i}^{(3)}))
\]
\[= \text{wal}_{k,l,m}(x_{i}^{(1)}, x_{i}^{(2)}, x_{1,i}^{(3)})\text{wal}_{k,l,m}(\tilde{x}_{i}^{(1)}, \tilde{x}_{i}^{(2)}, \tilde{x}_{1,i}^{(3)}),
\]
i.e. \(\text{wal}_{k,l,m}\) is a character on \(((\mathbb{Z}_2)^{3s}, \oplus)\).
The digital net $x_0, \ldots, x_{2^s-1}$ is a subgroup of $((\mathbb{Z}_2)^3, \oplus)$, hence
\[
2^s - 1 \sum_{i=0}^s \text{wal}_k(x_i^{(1)})\text{wal}_l(x_i^{(2)})\text{wal}_m(x_i^{(3)}) = \begin{cases} 2^s & \text{if } \text{wal}_{k,l,m}(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}) = 1 \\ 0 & \text{otherwise.} \end{cases}
\]
(For more details see [2] or [5].)
Now we have $\text{wal}_{k,l,m}(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}) = (-1)^{\left(\bar{k}|x_i^{(1)}\right) + \left(\bar{l}|x_i^{(2)}\right) + \left(\bar{m}|x_i^{(3)}\right)}$ for all $i = 0, \ldots, 2^s - 1$ iff
\[
\left(\bar{k}|x_i^{(1)}\right) = \left(\bar{l}|x_i^{(2)}\right) + \left(\bar{m}|x_i^{(3)}\right) \quad \text{for all } i = 0, \ldots, 2^s - 1
\]
(by the definition of the net); this means
\[
\left(\bar{k}|i\right) = \left(\bar{l}|C_2^{\bar{c}}\right) + \left(\bar{m}|C_3^{\bar{c}}\right) \quad \text{for all } i = 0, \ldots, 2^s - 1,
\]
and this is satisfied if and only if
\[
\bar{k} = C_2^{T\bar{c}} + C_3^{T\bar{m}} =: \bar{k}(l, m).
\]
Further
\[
\text{wal}_{k(l,m),l,m}(\alpha, \beta, \gamma) = \text{wal}_l(\bar{\delta})\text{wal}_m(\bar{\varepsilon})
\]
with $\bar{\delta} := C_2\bar{\alpha} + \bar{\beta}$ and $\bar{\varepsilon} := C_3\bar{\alpha} + \bar{\gamma}$ (note that $\delta_i = (\bar{c}_i^2|\bar{\alpha}) + \beta_i$ and $\varepsilon_i = (\bar{c}_i^3|\bar{\alpha}) + \gamma_i$).
Therefore we have
\[
\Sigma_4 = 2^s \sum_{l,m=1}^{2^s-1} \text{wal}_l(\bar{\delta})\text{wal}_m(\bar{\varepsilon}) \psi(2^{v(k(l,m))}\alpha)\psi(2^{v(l)}\beta)\psi(2^{v(m)}\gamma) \frac{2^v(k(l,m)) + v(l) + v(m)}{2^v(k(l,m)) + v(l) + v(m)}
\]
\[
\times \sum_{l=2^u}^{2^{u+1}-1} \sum_{m=2^v}^{2^{v+1}-1} \text{wal}_l(\bar{\delta})\text{wal}_m(\bar{\varepsilon}) \psi(2^{v(k(l,m))}\alpha) \frac{2^v(k(l,m))}{2^v(k(l,m))}.
\]
For $2^u \leq l \leq 2^{u+1} - 1$, $2^v \leq m \leq 2^{v+1} - 1$ we have
\[
\text{wal}_l(\bar{\delta})\text{wal}_m(\bar{\varepsilon}) = (-1)^{l_0\delta_1 + \ldots + l_u\delta_u + \delta_{u+1}(-1)^m_0\varepsilon_1 + \ldots + m_{v-1}\varepsilon_v + \varepsilon_{v+1}}
\]
\[
= (-1)^{l_0\delta_1 + \ldots + l_u\delta_u + m_0\varepsilon_1 + \ldots + m_{v-1}\varepsilon_v}
\]
\[
\times (-1)^{(\bar{c}^2_{u+1}|\bar{\alpha}) + (\bar{c}^3_{v+1}|\bar{\alpha}) + \beta_{u+1} + \gamma_{v+1}},
\]
by the definition of \( \delta \) and \( \varepsilon \). Hence

\[
\Sigma_4 = \sum_{u,v=0}^{s-1} \frac{\|2^u \beta\| \cdot \|2^v \gamma\|}{2^{u+v-s}} (-1)^{(\bar{c}_{u+1}^2 + \bar{c}_{v+1}^3) \alpha} \\
\times \sum_{l=0}^{2^u-1} \sum_{m=2^v} \sum_{k(l,m) \neq 0} (-1)^{l_0 \delta_1 + \ldots + l_{u-1} \delta_u + m_0 \varepsilon_1 + \ldots + m_{v-1} \varepsilon_v} \frac{\psi(2^v(k(l,m)) \alpha)}{2^v(k(l,m))} .
\]

Here \( l := l_0 + l_1 2 + \ldots + l_u 2^u \), \( m = m_0 + m_1 2 + \ldots + m_v 2^v \) and \( \| \cdot \| \) is the distance to the nearest integer function, i.e. \( \|x\| := \min(x-[x], 1-(x-[x])) \). Note that \( \psi(2^u \beta)(-1)^{\beta_{u+1}} = \|2^u \beta\| \) and \( \psi(2^v \gamma)(-1)^{\gamma_{v+1}} = \|2^v \gamma\| \).

For \( 0 \leq u, v \leq s - 1 \) we have

\[
\Sigma_5(u, v) := \sum_{l=0}^{2^u-1} \sum_{m=2^v} \sum_{k(l,m) \neq 0} (-1)^{l_0 \delta_1 + \ldots + l_{u-1} \delta_u + m_0 \varepsilon_1 + \ldots + m_{v-1} \varepsilon_v} \frac{\psi(2^v(k(l,m)) \alpha)}{2^v(k(l,m))} = \sum_{w=0}^{s-1} \psi(2^w \alpha) \sum_{l=0}^{2^u-1} \sum_{m=2^v} \sum_{v(k(l,m))=w} (-1)^{l_0 \delta_1 + \ldots + l_{u-1} \delta_u + m_0 \varepsilon_1 + \ldots + m_{v-1} \varepsilon_v}.
\]

For \( 0 \leq u, v, w \leq s - 1 \) define

\[
\Sigma_6(u, v, w) := \sum_{l=0}^{2^u-1} \sum_{m=0}^{2^v-1} \frac{(-1)^{l_0 \delta_1 + \ldots + l_{u-1} \delta_u + m_0 \varepsilon_1 + \ldots + m_{v-1} \varepsilon_v}}{v(k(l,m))=w} = \sum_{l=0}^{2^u-1} \sum_{m=0}^{2^v-1} \text{wal}_l(\delta)\text{wal}_m(\varepsilon).
\]

For \( 0 \leq l \leq 2^u - 1 \) and \( 0 \leq m \leq 2^v - 1 \), the condition \( v(k(l+2^u, m+2^v)) = w \) means that there are \( k_0, \ldots, k_{w-1} \in \mathbb{Z}_2 \) such that

\[
\bar{c}_1^2 l_0 + \ldots + \bar{c}_w^2 l_{u-1} + \bar{c}_{u+1}^2 m_0 + \ldots + \bar{c}_v^3 m_{v-1} + \bar{c}_{v+1}^3 + \bar{c}_1 k_0 + \ldots + \bar{c}_w k_{w-1} + \bar{c}_{w+1} = \bar{0},
\]

where \( \bar{c}_i \) is the ith canonical vector in \( \mathbb{Z}_2^s \) and \( \bar{0} \) is the zero vector in \( \mathbb{Z}_2^s \).

Since \( \bar{c}_1^2, \ldots, \bar{c}_{u+1}^2, \bar{c}_1, \ldots, \bar{c}_{w+1} \) by the \((0, s, 3)\)-net property are linearly independent as long as \((u + 1) + (v + 1) + (w + 1) \leq s\) we must have \( u + v + w \geq s - 2 \).
For $0 \leq l \leq 2^u - 1$ and $0 \leq m \leq 2^v - 1$, let
\[ \vec{n} := (l_0, \ldots, l_{u-1}, m_0, \ldots, m_{v-1})^T \in \mathbb{Z}_2^{u+v} \]
and define
\[ \vec{\zeta} := (\delta_1, \ldots, \delta_u, \epsilon_1, \ldots, \epsilon_v)^T \in \mathbb{Z}_2^{u+v}. \]
Further let $C^{(u,v)}$ be the $s \times (u+v)$-matrix over $\mathbb{Z}_2$ given by
\[ C^{(u,v)} := (\vec{c}_1^2, \ldots, \vec{c}_u^2, \vec{c}_1^3, \ldots, \vec{c}_v^3), \]
and define
\[ \vec{d} = d(u,v) := \vec{c}_{u+1}^2 + \vec{c}_{v+1}^3 \in \mathbb{Z}_2^s. \]
Now (with this notation) $v(k(l+2^u, m+2^v)) = w$ means
\[
(2) \quad C^{(u,v)} \vec{n} = \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \vec{d}
\]
for some $k_i \in \mathbb{Z}_2$ (therefore in the following we sometimes write $v(k(n)) = w$).

Now we have to consider three cases:

1. $u + v + w = s - 2$. Then the matrix $(C^{(u,v)}, \vec{e}_1, \ldots, \vec{e}_w)$ has rank $s - 2$ and therefore the system (2) has one or no solution.

2. $u + v + w = s - 1$. Then the matrix $(C^{(u,v)}, \vec{e}_1, \ldots, \vec{e}_w)$ has rank $s - 1$ and therefore the system (2) has one or no solution.

3. $u + v + w \geq s$. Then the matrix $(C^{(u,v)}, \vec{e}_1, \ldots, \vec{e}_w)$ has rank $s$ and therefore the system (2) has exactly $2^{u+v+w-s}$ solutions.

In the following we give the solutions of the system (2) in the above three cases and calculate the values of $\Sigma_6(u, v, w)$.

1. $u + v + w = s - 2$. Since $\vec{e}_1, \ldots, \vec{e}_{w+1}, \vec{c}_1^2, \ldots, \vec{c}_{u+1}^2, \vec{c}_1^3, \ldots, \vec{c}_{v+1}^3$ are linearly dependent we can find some $\lambda_1^1, \ldots, \lambda_{w+1}^1, \lambda_1^2, \ldots, \lambda_{u+1}^2, \lambda_1^3, \ldots, \lambda_{v+1}^3 \in \mathbb{Z}_2$ not all zero such that
\[
\sum_{i=1}^{w+1} \lambda_i^1 \vec{e}_i + \sum_{i=1}^{u+1} \lambda_i^2 \vec{c}_i^2 + \sum_{i=1}^{v+1} \lambda_i^3 \vec{c}_i^3 = \vec{0}.
\]
Assume that $\lambda_{w+1}^1 = 0$. Then $\vec{e}_1, \ldots, \vec{e}_w, \vec{c}_1^2, \ldots, \vec{c}_{u+1}^2, \vec{c}_1^3, \ldots, \vec{c}_{v+1}^3$ are linearly dependent. But this is a contradiction to the $(0, s, 3)$-net property and hence $\lambda_{w+1}^1 = 1$. In the same way one can show that $\lambda_{u+1}^2 = 1$ and $\lambda_{v+1}^3 = 1$ and hence the system (2) has exactly one solution.
Now let $D = D(u, v)$ be the following $(u + v) \times (u + v)$-matrix over $\mathbb{Z}_2$:

$$D := \begin{pmatrix}
  c_{1,s-(u+v)+1}^2 & \cdots & c_{u,s-(u+v)+1}^2 & c_{1,s-(u+v)+1}^3 & \cdots & c_{v,s-(u+v)+1}^3 \\
  c_{1,s}^2 & \cdots & c_{u,s}^2 & c_{1,s}^3 & \cdots & c_{v,s}^3 \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & \cdots & 1 & 0 & \cdots \\
  0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}^{-1}$$

(note that $D = D(u, v)$ exists due to the $(0, s, 3)$-net property). We have

$$\Sigma_6(u, v, w) = \sum_{n=0}^{2^u+v-1} (-1)^{(\tilde{n}, \tilde{\zeta})} = \sum_{Dn=0}^{2^u+v-1} (-1)^{(D\tilde{n}, \tilde{\zeta})}.$$ 

Now $v(k(Dn)) = w$ means that

$$C^{(u,v)}D\tilde{n} = \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \tilde{d}$$

for some $k_i \in \mathbb{Z}_2$. This is equivalent to

$$E \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_s \end{pmatrix}$$

with

$$E = \begin{pmatrix}
  c_{1,1}^2 & \cdots & c_{v,1}^3 \\
  c_{1,s-(u+v)}^2 & \cdots & c_{v,s-(u+v)}^3 \\
\end{pmatrix} \cdot D,$$

i.e. an $(s - (u + v)) \times (u + v)$-matrix. Therefore the unique solution $\tilde{n}$ is given by

$$\tilde{n} = (d_{s-(u+v)+1}, \ldots, d_s)^T \in \mathbb{Z}_2^{u+v}$$

and hence for $u + v + w = s - 2$ we have

$$\Sigma_6(u, v, w) = (-1)^{(d_{s-(u+v)+1}, \ldots, d_s)^T(D^T \tilde{\zeta})}.$$ 

2. $u + v + w = s - 1$. Let the $(u + v) \times (u + v)$-matrix $D = D(u, v)$ be as in case 1. We consider two subcases:
Improved upper bounds

(a) \( u + v \leq s - 2 \). Assume that \( D\vec{n} \) is a solution of the system (2). Then we find as in case 1 that

\[
\vec{n} = (d_{s-(u+v)+1}, \ldots, d_s)^T \in \mathbb{Z}_2^{u+v}.
\]

Let \( \vec{r} \in \mathbb{Z}_2^{u+v} \) be the last row of the \( (s-(u+v)) \times (u+v) \)-matrix

\[
E = \begin{pmatrix}
c_{1,1}^2 & \cdots & c_{v,1}^3 \\
\cdots & \cdots & \cdots \\
c_{1,s-(u+v)}^2 & \cdots & c_{v,s-(u+v)}^3
\end{pmatrix} \cdot D.
\]

Then \( \vec{r}^T \vec{n} = 1 + d_{s-(u+v)} \); but that contradicts case 1 from which we have \( \vec{r}^T \vec{n} = d_{s-(u+v)} \). Hence system (2) has no solution in this case.

(b) \( u + v = s - 1 \) (hence \( w = 0 \)). From \( (u+1) + (v+1) = s + 1 \) we deduce that \( c_1^2, \ldots, c_{u+1}^2, c_1^3, \ldots, c_{v+1}^3 \) are linearly dependent. Hence we can find some \( \lambda_1, \ldots, \lambda_{u+1}, \mu_1, \ldots, \mu_{v+1} \in \mathbb{Z}_2 \) not all zero such that

\[
\sum_{i=1}^{u+1} \lambda_i c_i^2 + \sum_{i=1}^{v+1} \mu_i c_i^3 = 0.
\]

Assume \( \lambda_{u+1} = 0 \). Then \( c_1^2, \ldots, c_u^2, c_1^3, \ldots, c_{v+1}^3 \) are linearly dependent, which contradicts the \( (0, s, 3) \)-net property. So \( \lambda_{u+1} = 1 \) and analogously \( \mu_{v+1} = 1 \). Hence there exists a vector \( \vec{n}_0 \in \mathbb{Z}_2^{u+v} \) such that

\[
C^{(u,v)} \vec{n}_0 = \vec{d}.
\]

Now consider the following linear equation system:

\[
\begin{pmatrix}
c_{1,2}^2 & \cdots & c_{u,2}^2 & c_{1,2}^3 & \cdots & c_{v,2}^3 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c_{1,s}^2 & \cdots & c_{u,s}^2 & c_{1,s}^3 & \cdots & c_{v,s}^3
\end{pmatrix} \cdot \vec{n} = \begin{pmatrix}
d_2 \\
\vdots \\
d_s
\end{pmatrix}.
\]

This system has a unique solution and this solution is \( \vec{n}_0 \). From this together with (3) it follows that the system

\[
C^{(u,v)} \vec{n} = \vec{c}_1 + \vec{d}
\]

cannot have a solution.

 Altogether for \( u + v + w = s - 1 \) we have

\[
\Sigma_6(u,v,w) = 0.
\]

3. \( u + v + w \geq s \). We know that system (2) has exactly \( 2^{u+v+w-s} \) solutions. Again we consider two subcases.

(a) \( u + v \leq s \). Let the \( (u+v) \times (u+v) \)-matrix \( D = D(u,v) \) be like in case 1. Proceeding as in case 1 we find that the solutions of system (2) are given by \( D\vec{n} \) where

\[
\vec{n} = (n_0, \ldots, n_{u+v+w-(s+1)}, d_{w+1} + 1, d_{w+2}, \ldots, d_s)^T \in \mathbb{Z}_2^s,
\]
with arbitrary \( n_0, \ldots, n_{u+v+w-(s+1)} \in \mathbb{Z}_2 \). From this we get

\[
\Sigma_6(u, v, w) = \sum_{Dn=0}^{2^{u+v}-1} (-1)^{(\bar{D}n|\bar{\zeta})} = \sum_{n_0, \ldots, n_{u+v+w-(s+1)} \in \mathbb{Z}_2} (-1)^{(n|D^T\zeta)}
\]

\[
= (-1)^{(0, \ldots, 0, d_{w+1}+1, d_{w+2}, \ldots, d_s)^T|D^T\bar{\zeta}} \sum_{n=0}^{2^{u+v+w-s}-1} \text{wal}_n(D^T\zeta)
\]

\[
= 2^{u+v+w-s}(-1)^{(0, \ldots, 0, d_{w+1}+1, d_{w+2}, \ldots, d_s)^T|D^T\bar{\zeta}} \times \begin{cases} 1 & \text{if } (D^T\bar{\zeta}|\bar{e}_i) = 0 \\
 & \text{for all } i = 1, \ldots, u + v + w - s, \\
0 & \text{otherwise}. \end{cases}
\]

Let \((D^T\bar{\zeta}|\bar{e}_i) = 0\) for all \(i = 1, \ldots, u + v + w - s\). Then

\[
(D^T\bar{\zeta}|(0, \ldots, 0, d_{w+1}+1, d_{w+2}, \ldots, d_s)^T) = (D^T\bar{\zeta}|(d_{s-(u+v)+1}, \ldots, d_s)^T) + (D^T\bar{\zeta}|\bar{e}_{u+v+w-s+1}).
\]

Hence for \(u + v + w \geq s, u + v \leq s\) we have

\[
\Sigma_6(u, v, w) = 2^{u+v+w-s}(-1)^{(D^T\bar{\zeta}|(d_{s-(u+v)+1}, \ldots, d_s)^T)} \times (-1)^{(D^T\bar{\zeta}|\bar{e}_{u+v+w-s+1})} \kappa_1(u, v, w, s)
\]

where

\[
\kappa_1(u, v, w, s) = \begin{cases} 1 & \text{if } (D^T\bar{\zeta}|\bar{e}_i) = 0 \text{ for all } i = 1, \ldots, u + v + w - s, \\
0 & \text{otherwise}. \end{cases}
\]

(b) \(u + v > s\). Let \(F = F(u, v)\) be the following \(s \times s\)-matrix over \(\mathbb{Z}_2\):

\[
F = (\bar{e}_1^2, \ldots, \bar{e}_u^2, \bar{e}_1^3, \ldots, \bar{e}_{s-u}^3)^{-1}
\]

(note that \(F\) exists due to the \((0, s, 3)\)-net property) and let \(G = G(u, v)\) be the following \((u + v) \times (u + v)\)-matrix over \(\mathbb{Z}_2\):

\[
G = \begin{pmatrix}
F & \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix} \\
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix} & 1 & 0
\end{pmatrix}.
\]
We have

\begin{equation}
\Sigma_4(u, v, w) = \sum_{n=0}^{2^{u+v}-1} (-1)^{\bar{n} \bar{\zeta}} = \sum_{Gn=0}^{2^{u+v}-1} (-1)^{G\bar{n}\bar{\zeta}}.
\end{equation}

Now \( v(k(Gn)) = w \) means that

\[ C^{(u,v)} G\bar{n} = \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \vec{d} \]

for some \( k_i \in \mathbb{Z}_2 \). Since

\[ C^{(u,v)} G = (I, \bar{c}^{3}_{s-u+1}, \ldots, \bar{c}^{3}_v) \]

where \( I \) is the \( s \times s \) unit matrix, we get the following solutions for our equation system:

\[ \bar{n} = \begin{pmatrix} \vec{d} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \sum_{i=1}^{u+v-s} r_i \cdot \begin{pmatrix} \bar{c}^{3}_{s-u+i} \\ \vec{e}_i \end{pmatrix} \]

for arbitrary \( k_i \in \mathbb{Z}_2 \) and arbitrary \( r_i \in \mathbb{Z}_2 \) and where \( \vec{e}_i \) is the \( i \)th unit vector in \( \mathbb{Z}_2^{u+v-s} \).

Let \( H = H(u, v) \) be the \((u+v) \times (u+v)\)-matrix over \( \mathbb{Z}_2 \) given by

\[ H = \begin{pmatrix}
\begin{array}{cccc|cccc}
\bar{c}^{3}_{s-u+1,1} & \cdots & \bar{c}^{3}_{v,1} & 0 & \ldots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
\bar{c}^{3}_{s-u+1,s} & \cdots & \bar{c}^{3}_{v,s} & 0 & \ldots & 0 \\
1 & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & \ldots & 0 & \ldots & 0 \\
\end{array}
\end{pmatrix}.
\]
Then we can write $\vec{n}$ as

$$\vec{n} = \vec{e}_{w+1} + \begin{pmatrix} d^T \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} k_0 \\ \vdots \\ k_{w-1} \\ 0 \end{pmatrix} + H \cdot \begin{pmatrix} r_1 \\ \vdots \\ r_{u+v-s} \\ 0 \end{pmatrix}$$

where $\vec{e}_{w+1}$ is the $(w + 1)$th unit vector in $\mathbb{Z}^{u+v}_2$. Inserting in (5) yields

$$\Sigma_6(u, v, w) = (-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T | G^T \vec{\zeta}} (-1)^{(\vec{e}_{w+1} | G^T \vec{\zeta})} \times \left( \sum_{k_0, \ldots, k_{w-1} \in \mathbb{Z}_2} (-1)^{(k_0, \ldots, k_{w-1}, 0, \ldots, 0)^T | G^T \vec{\zeta}} \right) \times \left( \sum_{r_1, \ldots, r_{u+v-s} \in \mathbb{Z}_2} (-1)^{(H \cdot (r_1, \ldots, r_{u+v-s}, 0, \ldots, 0)^T | G^T \vec{\zeta})} \right)$$

$$= (-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T | G^T \vec{\zeta}} (-1)^{(\vec{e}_{w+1} | G^T \vec{\zeta})} \times 2^{u+v-w-1} \sum_{k=0}^{2^w-1} \text{wal}_k(G^T \vec{\zeta}) \times \sum_{r=0}^{2^{u+v-s}-1} \text{wal}_r(H^T G^T \vec{\zeta})$$

$$= (-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T | G^T \vec{\zeta}} (-1)^{(\vec{e}_{w+1} | G^T \vec{\zeta})} \times 2^{u+v+w-s} \kappa_2(u, v, w, s) \kappa_3(u, v, s),$$

where

$$\kappa_2(u, v, w, s) = \begin{cases} 1 & \text{if } (\vec{e}_i | G^T \vec{\zeta}) = 0 \text{ for all } i = 1, \ldots, w, \\ 0 & \text{else}, \end{cases}$$

$$\kappa_3(u, v, s) = \begin{cases} 1 & \text{if } (\vec{e}_i | H^T G^T \vec{\zeta}) = 0 \text{ for all } i = 1, \ldots, u + v - s, \\ 0 & \text{else}. \end{cases}$$

Now we can evaluate $\Sigma_5(u, v)$: We consider three cases.

1. $u + v > s$. Then

$$\Sigma_5(u, v) = 2^{u+v-s} (-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T | G^T \vec{\zeta}} \kappa_3(u, v, s) \times \sum_{w=0}^{s-1} \psi(2^w \alpha)(-1)^{(\vec{e}_{w+1} | G^T \vec{\zeta})} \kappa_2(u, v, w, s).$$

For $0 \leq u, v \leq s - 1$ let

$$m = m(u, v) := \max \{ 1 \leq j \leq u + v : (\vec{e}_i | G^T \vec{\zeta}) = 0, i = 1, \ldots, j \}$$

(if $u + v = 0$ or if $(\vec{e}_1 | G^T \vec{\zeta}) = 1$ set $m = m(u, v) := 0$). By the definition of $m = m(u, v)$ we have $(\vec{e}_1 | G^T \vec{\zeta}) = \ldots = (\vec{e}_{m} | G^T \vec{\zeta}) = 0$ and $(\vec{e}_{m+1} | G^T \vec{\zeta}) = 1$. 
Hence $\kappa_2(u, v, w, s) = 1$ iff $w \leq m(u, v)$. So we have

$$
\Sigma_5(u, v) = 2^{u+v-s}(-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T|G^T\xi}\kappa_3(u, v, s)
$$

$$
\times \left( \sum_{w=0}^{m-1} \psi(2^w \alpha) - \psi(2^m \alpha) \right)
$$

$$
= 2^{u+v-s}(-1)^{(d_1, \ldots, d_s, 0, \ldots, 0)^T|G^T\xi}\kappa_3(u, v, s)(\alpha_{m+1} - \alpha),
$$

where we used Lemma 2. Hence

$$
|\Sigma_5(u, v)| \leq 2^{u+v-s}.
$$

2. $u + v \leq s - 2$. We have

$$
\Sigma_5(u, v) = \sum_{w=s-2-(u+v)}^{s-1} \frac{\psi(2^w \alpha)}{2^w} \Sigma_6(u, v, w)
$$

$$
= \frac{\psi(2^{s-2-(u+v)} \alpha)}{2^{s-2-(u+v)}} \Sigma_6(u, v, s - 2 - (u + v))
$$

$$
+ \sum_{w=s-(u+v)}^{s-1} \frac{\psi(2^w \alpha)}{2^w} \Sigma_6(u, v, w)
$$

$$
= 2^{u+v-s}(-1)^{(d_1-\frac{u+v}{s}+1, \ldots, d_s)^T|D^T\bar{\xi})}
$$

$$
\times \left[ 4\psi(2^{s-2-(u+v)} \alpha) + \sum_{w=s-(u+v)}^{s-1} \psi(2^w \alpha) \right]
$$

$$
\times (-1)^{(e_{u+v+w-s+1}|D^T\bar{\xi})}\kappa_1(u, v, w, s). \right].
$$

For $0 \leq u, v \leq s - 1$ let

$$
p = p(u, v) := \max\{1 \leq j \leq u + v : (e_i|D^T\bar{\xi}) = 0, \ i = 1, \ldots, j\}
$$

(if $u + v = 0$ or if $(e_1|D^T\bar{\xi}) = 1$ set $p = p(u, v) := 0$). By the definition of $p = p(u, v)$ we have

$$(e_1|G^T\bar{\xi}) = \ldots = (e_p|G^T\bar{\xi}) = 0 \quad \text{and} \quad (e_{p+1}|G^T\bar{\xi}) = 1.$$

Hence $\kappa_1(u, v, w, s) = 1$ iff $u + v + w - s \leq p(u, v)$. So we have

$$
\Sigma_5(u, v) = 2^{u+v-s}(-1)^{(d_1-\frac{u+v}{s}+1, \ldots, d_s)^T|D^T\bar{\xi})}
$$

$$
\times \left[ 4\psi(2^{s-2-(u+v)} \alpha) - \psi(2^{s-(u+v)+p} \alpha) + \sum_{w=s-(u+v)}^{s-(u+v)+p-1} \psi(2^w \alpha) \right].
$$
Now with Lemma 2 we get
\[
\sum_{w=s-(u+v)}^{s-(u+v)+p-1} \psi(2^w \alpha) - \psi(2^{s-(u+v)+p} \alpha) = \sum_{w=0}^{s-(u+v)+p-1} \psi(2^w \alpha) - \psi(2^{s-(u+v)+p} \alpha) - \sum_{w=0}^{s-(u+v)-1} \psi(2^w \alpha) = \alpha_{s-(u+v)+p+1} - \alpha_{s-(u+v)+1} - \psi(2^{s-(u+v)} \alpha).
\]
Moreover we have
\[
4\psi(2^{s-(u+v)-2} \alpha) = 4 \left( \frac{\alpha_{s-(u+v)}}{2^2} + \ldots + \frac{\alpha_s}{2^{u+v+2}} - \frac{\alpha_{s-(u+v)-1}}{2} \right) = \alpha_{s-(u+v)} + \{2^{s-(u+v)} \alpha\} - 2\alpha_{s-(u+v)-1}.
\]
Hence
\[
4\psi(2^{s-2-(u+v)}) - \psi(2^{s-(u+v)+p} \alpha) + \sum_{w=s-(u+v)}^{s-(u+v)+p-1} \psi(2^w \alpha) = \alpha_{s-(u+v)} + \{2^{s-(u+v)} \alpha\} - 2\alpha_{s-(u+v)-1} + \alpha_{s-(u+v)+p+1} - \alpha_{s-(u+v)+1} - \psi(2^{s-(u+v)} \alpha) = \alpha_{s-(u+v)+p+1} + \alpha_{s-(u+v)} - 2\alpha_{s-(u+v)-1}.
\]
Therefore we have
\[
\Sigma_5(u, v) = 2^{u+v-s}(-1)^{(d_{s-(u+v)+1}, \ldots, d_s)^T[D^T \bar{z}]} \times [\alpha_{s-(u+v)+p+1} + \alpha_{s-(u+v)} - 2\alpha_{s-(u+v)-1}].
\]
Hence
\[
|\Sigma_5(u, v)| \leq 2 \cdot 2^{u+v-s}.
\]
3. \(s - 1 \leq u + v \leq s\). Then we get, as in case 2,
\[
\Sigma_5(u, v) = 2^{u+v-s}(-1)^{(d_{s-(u+v)+1}, \ldots, d_s)^T[D^T \bar{z}]} \times [\alpha_{s-(u+v)+p+1} - \alpha_{s-(u+v)+1} - \psi(2^{s-(u+v)} \alpha)].
\]
From
\[
\psi(2^{s-(u+v)} \alpha) = \frac{\alpha_{s-(u+v)+2}}{2^2} + \ldots + \frac{\alpha_s}{2^{u+v}} - \frac{\alpha_{s-(u+v)+1}}{2} = \frac{1}{2}(\{2^{s-(u+v)+1} \alpha\} - \alpha_{s-(u+v)+1})
\]
we get
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\[ |\alpha_{s-(u+v)+p+1} - \alpha_{s-(u+v)+1} - \psi(2^{s-(u+v)}\alpha)| \]

\[ = \left| \alpha_{s-(u+v)+p+1} - \frac{1}{2} (\alpha_{s-(u+v)+1} + \{2^{s-(u+v)+1}\alpha\}) \right| \leq 1 \]

and hence

\[ |\Sigma_5(u,v)| \leq 2^{u+v-s} \]

Summing up we have

\[ |\Sigma_5(u,v)| \leq \begin{cases} 
2 \cdot 2^{u+v-s} & \text{for } u + v \leq s - 2, \\
2^{u+v-s} & \text{for } u + v \geq s - 1.
\end{cases} \]

Therefore

\[ |\Sigma_4| \leq \sum_{u,v=0}^{s-1} \frac{\|2^u\beta\| \cdot \|2^v\gamma\|}{2^{u+v-s}} |\Sigma_5(u,v)| \]

\[ \leq \sum_{u,v=0}^{s-1} \|2^u\beta\| \cdot \|2^v\gamma\| + \sum_{u,v=0}^{s-1} \|2^u\beta\| \cdot \|2^v\gamma\|. \]

From [4, Theorem 3] we get

\[ \sum_{u,v=0}^{s-1} \|2^u\beta\| \cdot \|2^v\gamma\| \leq \left( \frac{s}{3} + \frac{1}{9} - \frac{(-1)^s}{9 \cdot 2^s} \right)^2. \]

Further, by [4, Theorem 2], we have

\[ \sum_{u,v=0}^{s-1} \|2^u\beta\| \cdot \|2^v\gamma\| = \sum_{u=0}^{s-1} \|2^u\beta\| \sum_{v=0}^{s-u-2} \|2^v\gamma\| \]

\[ \leq \sum_{u=0}^{s-1} \|2^u\beta\| \left( \frac{s - u - 1}{3} + \frac{1}{9} - \frac{(-1)^{s-u-1}}{9 \cdot 2^{s-u-1}} \right) \]

and

\[ \sum_{u=0}^{s-1} \|2^u\beta\| \frac{s - u - 1}{3} \leq \frac{1}{6} + \frac{1}{3} \sum_{k=1}^{s-2} \sum_{u=0}^{s-k-1} \|2^u\beta\| \]

\[ \leq \frac{1}{6} + \frac{1}{3} \sum_{k=1}^{s-2} \left( \frac{s - k}{3} + \frac{1}{9} - \frac{(-1)^{s-k}}{9 \cdot 2^{s-k}} \right) \]

\[ = \frac{s^2}{18} - \frac{s}{54} - \frac{4}{162} + \frac{(-1)^s}{162 \cdot 2^{s-2}}. \]
Together we have

\[
|\Sigma_4| \leq \left( \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s} \right)^2 + \frac{s^2}{18} - \frac{s}{54} - \frac{4}{162} + \frac{(-1)^s}{162 \cdot 2^{s-2}} \\
+ \frac{2}{9} \left( \frac{s}{3} + \frac{1}{9} - (-1)^s \frac{1}{9 \cdot 2^s} \right) \\
= \frac{s^2}{6} + s \cdot \left( \frac{7}{54} - \frac{2(-1)^s}{27 \cdot 2^s} \right) + \frac{1}{81} - \frac{2(-1)^s}{81 \cdot 2^s} + \frac{1}{81 \cdot 2^{2s}}
\]

and the result follows.

5. Proof of Theorems 2 and 3

Proof of Theorem 3. We use the technique of Niederreiter introduced in [6, Proof of Lemma 4.1] (or see [7, Proof of Lemma 4.11]) and an idea of G. Larcher.

Let \( N = b_0 + b_1 2 + \ldots + b_r 2^r \), with \( b_r = 1 \) and \( b_k \in \{0, 1\} \), \( 0 \leq k < r \), be the base 2 representation of \( N \) and let the integer \( p \) be maximal such that \( 2^p \) is a divisor of \( N \).

Let the digital \((0,2)\)-sequence in base 2 be generated by the \( \mathbb{N} \times \mathbb{N} \)-matrices \( C_1 \) and \( C_2 \). Divide the sequence \( x_0, \ldots, x_N-1 \) into subsequences \( \omega_{m,b} \) for \( b = 0, \ldots, b_m - 1 \) and \( m = 0, \ldots, r \), where \( \omega_{m,b} \) is the subsequence \( x_n \) with \( \sum_{k=m+1}^r b_k 2^k + b 2^m \leq n < \sum_{k=m+1}^r b_k 2^k + (b+1) 2^m \). For fixed \( m \) divide the matrices \( C_i, i = 1, 2 \), into the following parts:

\[
C_i = \begin{pmatrix} C_i(m) & D_i(m) \\ \hline E_i(m) \end{pmatrix},
\]

where \( C_i(m) \) is the upper left \( m \times m \)-submatrix of \( C_i \). If

\[
n = \sum_{k=m+1}^r b_k 2^k + b 2^m + \sum_{k=0}^{m-1} a_k 2^k,
\]

then

\[
\tilde{n} = (a_0, a_1, \ldots, a_{m-1}, b, b_{m+1}, \ldots, b_r, 0, 0, \ldots)^T
\]

and
\[ C_i \vec{n} = \begin{pmatrix} C_i(m) \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-2} \\ a_{m-1} \end{pmatrix} \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} D_i(m) \cdot \begin{pmatrix} b \\ b_{m+1} \\ \vdots \\ b_r \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ \vdots \end{pmatrix} + \begin{pmatrix} E_i(m) \vec{n} \end{pmatrix}. \]

Hence \( \omega_{m,b} \) is a modulo \( \mathbb{Z}_2 \) shifted digital \((0, m, 2)\)-net in base 2 generated by \( C_1(m) \) and \( C_2(m) \), which finally is translated by a vector with positive coordinates less than \( 2^{-m} \). Let \( \tilde{\omega}_{m,b} \) be the shifted digital net without the final translation. Let \( D_{m,b}^* \) (resp. \( \tilde{D}_{m,b}^* \)) be the star discrepancy of \( \omega_{m,b} \) (resp. \( \tilde{\omega}_{m,b} \)). Then by Lemma 4 we have

\[ |D_{m,b}^* - \tilde{D}_{m,b}^*| \leq \frac{2}{2^m}. \]

Therefore we get, by Lemma 3,

\[ ND_N^* \leq \sum_{m=0}^{r} \sum_{b=0}^{b_{m-1}} 2^m D_{m,b}^* \leq \sum_{m=0}^{r} \sum_{b=0}^{b_{m-1}} 2^m \left( \frac{2}{2^m} + \tilde{D}_{m,b}^* \right) \]

\[ \leq 2 \sum_{m=0}^{r} b_m + \sum_{m=0}^{r} b_m \left( \frac{m}{3} + \frac{19}{9} \right) \]

\[ = 2 \sum_{m=p}^{r} b_m + \sum_{m=p}^{r} b_m \left( \frac{m}{3} + \frac{19}{9} \right), \]

where \( p \) is the maximal integer such that \( 2^p \) is a divisor of \( N \).

Now apply the same method to the set consisting of the \( x_n \) with \( N \leq n \leq 2^{r+1} - 1 \). This set consists of \( 2^{r+1} - N \) points. Let

\[ 2^{r+1} - N = \sum_{m=0}^{r} c_m 2^m, \]

with \( c_m \in \{0, 1\} \). Again we can split up this set into a union of subsequences.

Let \( \omega_{m,c} \) for \( c = 0, \ldots, c_{m-1} \) and \( m = 0, \ldots, r \) be the subsequence \( x_n \) with \( 2^{r+1} - \sum_{k=m+1}^{r} c_k 2^k - c_{2^m} \leq n < 2^{r+1} - \sum_{k=m+1}^{r} c_k 2^k - c_{2^m} + 2^m \). As above one can see that \( \omega_{m,c} \) is a modulo \( \mathbb{Z}_2 \) shifted digital \((0, m, 2)\)-net in base 2 generated by \( C_1(m) \) and \( C_2(m) \), which finally is translated by a vector with positive coordinates less than \( 2^{-m} \). As above, for the star discrepancy of our
set we get
\[(2^r + 1 - N)D_{2^r+1-N}^r \leq 2 \sum_{m=0}^{r} c_m + \sum_{m=0}^{r} c_m \left( \frac{m}{3} + \frac{19}{9} \right).\]

The first \(2^r + 1\) points of the \((0, 2)\)-sequence build a digital \((0, r + 1, 2)\)-net. Our initial set is the difference between this \((0, r + 1, 2)\)-net and the set of \(x_n\) with \(N \leq n \leq 2^r + 1 - 1\). Hence
\[ND_N^r \leq 2 \sum_{m=0}^{r} c_m + \sum_{m=0}^{r} c_m \left( \frac{m}{3} + \frac{19}{9} \right) + \left( \frac{r + 1}{3} + \frac{19}{9} \right).\]

Now
\[2^{r+1} = 2^{r+1} - N + N = \sum_{m=0}^{r} (c_m + b_m)2^m.\]

Hence we have \(c_0 = \ldots = c_{p-1} = 0\), \(b_p + c_p = 2\) and \(b_m + c_m = 1\) for \(m = p + 1, \ldots, r\). Therefore
\[ND_N^r \leq 2 \left( 2 - b_p + \sum_{m=p+1}^{r} (1 - b_m) \right) + (2 - b_p) \left( \frac{p}{3} + \frac{19}{9} \right)\]
\[+ \sum_{m=p+1}^{r} (1 - b_m) \left( \frac{m}{3} + \frac{19}{9} \right) + \left( \frac{r + 1}{3} + \frac{19}{9} \right).\]

Hence
\[ND_N^r \leq \min \left\{ 2 \sum_{m=p}^{r} b_m + \sum_{m=p}^{r} b_m \left( \frac{m}{3} + \frac{19}{9} \right), \right.\]
\[\left. \quad 2 \left( 1 + \sum_{m=p}^{r} (1 - b_m) \right) + \left( \frac{p}{3} + \frac{19}{9} \right)\right.\]
\[\left. \quad + \sum_{m=p}^{r} (1 - b_m) \left( \frac{m}{3} + \frac{19}{9} \right) + \left( \frac{r + 1}{3} + \frac{19}{9} \right) \right\}.\]

Now, since \(\min(A, B) \leq (A + B)/2\), the result follows. 

Finally we give the proof of Theorem 2, which is an easy consequence of Theorem 3.

**Proof of Theorem 2.** Let \(x_0, x_1, \ldots\) be a digital \((0, 2)\)-sequence in base 2 (such a sequence exists by [6, Corollary 6.19]) and let \(s \geq 1\) be an integer. Then the set of
\[y_n := \left( \frac{n}{2^s}, x_n \right), \quad n = 0, \ldots, 2^s - 1,\]
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is a digital \((0, s, 3)\)-net in base 2. For the star discrepancy of this net, by [6, Lemma 8.9] and by Theorem 3 we have

\[ ND_N^* \leq \frac{1}{12(\log 2)^2} (\log N)^2 + \mathcal{O}(\log N) , \]

where \( N = 2^s \), and the result follows.

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