ACTA ARITHMETICA
131.2 (2008)

On the equation $aX^4 - bY^2 = 2$

by

S. Akhtari (Vancouver), A. Togbé (Westville, IN) and
P. G. Walsh (Ottawa)

1. Introduction. The problem of determining upper bounds for the number of integer points on elliptic curves has received considerable attention, and is a notoriously difficult problem. In a series of papers [11]–[15], Ljunggren proved absolute upper bounds for the number of positive integer solutions to equations of the form

$$aX^4 - bY^2 = c, \qquad c \in \{\pm 1, -2, \pm 4\}.$$

In the case $c = 1$, Ljunggren [12] proved that the equation $X^4 - bY^2 = 1$ has at most two solutions in positive integers X, Y. This result was extended by Bennett and the third author in [3], wherein it was proved that all equations of the form $a^2X^4 - bY^2 = 1$, with $a > 1$, have at most one solution in positive integers. In [5], Chen and Voutier proved that the equation $aX^4 - Y^2 = 1$ $(a > 2)$ has at most one solution in positive integers. These results have recently been extended by Akhtari [1], wherein it was proved that any equation of the form $aX^4 - bY^2 = 1$ has at most two solutions in positive integers. A key fact in the proof is that one only needs to prove the result for the subfamily of equations $(t + 1)X^4 - tY^2 = 1$.

Noticeably absent from the above list of values for c is the particular value $c = 2$. The third author [19] has recently proved, under stringent conditions on a, b, that the equation $aX^4 - bY^2 = 2$ has at most one solution in positive integers. Proving a similar result for the general equation $aX^4 - bY^2 = 2$ remains elusive. However, the methods of [1] can be employed for the particular subfamily of equations $(t + 2)X^4 - tY^2 = 2$. The purpose of the present paper is to prove an upper bound in the case that $c = 2$. In particular, we prove

2000 *Mathematics Subject Classification*: Primary 11D25; Secondary 11B39.
Key words and phrases: Diophantine equation, linear recurrence, elliptic curve, hypergeometric polynomial.

THEOREM 1.1. *For all odd positive integers $t > 40\,000$, the equation*

$$(t+2)X^4 - tY^2 = 2 \tag{1.1}$$

has at most two solutions in positive integers X, Y. For the remaining odd positive integers t, equation (1.1) *has at most three solutions in positive integers X, Y.*

Theorem 1.1 is likely not best possible. We conjecture that the only positive integer solution to equation (1.1) is $(X, Y) = (1, 1)$, and more generally, that any equation of the form $aX^4 - bY^2 = 2$, with a and b odd, has at most one solution in positive integers, and that such a solution must arise from the fundamental solution to the quadratic equation $aX^2 - bY^2 = 2$. This conjecture was verified for (1.1) in the range $1 \le t < 1200$.

The organization of the paper is as follows. In Section 2 we reduce the solution of (1.1) to the problem of determining all squares in certain linear recurrences, yielding equations of the form $X^2 = V_{2k+1}(t)$. In Section 3, the problem is further reduced to a family of Thue equations with the property that the roots of the associated family of polynomials can be given explicitly in terms of the parameter t. We finish off the first part of the paper by proving some lower bounds in Section 4, which will be needed later. In Section 5 the family of Thue equations are shown to be written in terms of resolvent forms, and the concept of a solution being associated to a fourth root of unity is introduced. In Section 6 the main lemma is proved, which shows that for each fourth root of unity, there can be at most one associated solution to the Thue equation. The proof of this fact uses the hypergeometric method, and in particular, proves that for fixed t, there is at most one solution (k, x) to the equation $X^2 = V_{4k+3}(t)$. In Section 7, we use Thue's method to solve the equation $X^2 = V_{4k+1}(t)$ completely, finishing the proof of Theorem 1.1.

2. Linear recurrences. For $t \ge 1$ and odd, let

$$\alpha = \frac{\sqrt{t+2} + \sqrt{t}}{\sqrt{2}},$$

and for $k \ge 0$, define sequences $\{V_i\}, \{U_i\}$ by

$$\alpha^{2k+1} = \frac{V_{2k+1}\sqrt{t+2} + U_{2k+1}\sqrt{t}}{\sqrt{2}}, \qquad \alpha^{2k} = V_{2k} + U_{2k}\sqrt{t(t+2)}.$$

All positive integer solutions (X, Y) to the quadratic equation

$$(t+2)X^2 - tY^2 = 2$$

are given by $(X, Y) = (V_{2k+1}, U_{2k+1})$.

Thus, a positive integer solution (X, Y), with $X > 1$, to equation (1.1) is equivalent to the existence of positive integers (t, k) satisfying

$$X^2 = V_{2k+1}(t).$$

Our strategy to prove Theorem 1.1 is to first prove that for fixed t, the equations

$$X^2 = V_{4k+1}(t) \tag{2.1}$$

and

$$X^2 = V_{4k+3}(t) \tag{2.2}$$

are solvable for at most one integer $k > 0$. This results in an upper bound of three solutions for equation (1.1). We then show that for t large enough, equation (2.1) has no solution with $k > 0$.

3. Reduction to Thue equations. We show here that a solution to equation (1.1) gives rise to a solution to a Thue equation.

It is easily proved by induction that the following relation holds for $\{V_n\}$:

$$V_{4k+1} = V_{2k+1}^2 + 2tU_{2k}^2. \tag{3.1}$$

Therefore, if $V_{4k+1} = X^2$, then

$$X^2 = V_{2k+1}^2 + 2tU_{2k}^2,$$

and hence, $(X - V_{2k+1})(X + V_{2k+1}) = 2tU_{2k}^2$. It follows that there are positive integers r, s, A, B, with $rs = 2t$ and $U_{2k} = 2AB$, for which

$$X - V_{2k+1} = 2rA^2, \qquad X + V_{2k+1} = 2sB^2.$$

Consequently, $V_{2k+1} = sB^2 - rA^2$, and from the easily seen identity $V_{2k+1} = V_{2k} + tU_{2k}$, one deduces that $V_{2k} = sB^2 - rA^2 - 2tAB$. Substituting these expressions for V_{2k} and U_{2k} into the equation $V_{2k}^2 - t(t+2)U_{2k}^2 = 1$ results in the equation

$$s^2B^4 - 4tsAB^3 - 12tA^2B^2 + 4rtBA^3 + r^2A^4 = 1.$$

Multiplying this equation through by s^2 and setting

$$x = -sB, \qquad y = A \tag{3.2}$$

shows that x and y satisfy the Thue equation

$$x^4 + 4tx^3y - 12tx^2y^2 - 8t^2xy^3 + 4t^2y^4 = s^2. \tag{3.3}$$

Similar to (3.1), one has the relation

$$V_{4k+3} = V_{2k+1}^2 + 2tU_{2k+2}^2,$$

and so if $V_{4k+3} = X^2$, then

$$X^2 = V_{2k+1}^2 + 2tU_{2k+2}^2.$$

Therefore, $(X - V_{2k+1})(X + V_{2k+1}) = 2tU_{2k+2}^2$, and so there are positive integers r, s, A, B, with $rs = 2t$ and $U_{2k+2} = 2AB$, for which

$$X - V_{2k+1} = 2rA^2, \qquad X + V_{2k+1} = 2sB^2.$$

Hence $V_{2k+1} = sB^2 - rA^2$, and from the identity $V_{2k+2} = V_{2k+1} + tU_{2k+2}$, one has $V_{2k+2} = sB^2 - rA^2 + 2tAB$. Substituting these expressions into the equation $V_{2k+2}^2 - t(t+2)U_{2k+2}^2 = 1$ gives

$$s^2B^4 + 4tsAB^3 - 12tA^2B^2 - 4rtBA^3 + r^2A^4 = 1.$$

Multiplying through by s^2 and letting

(3.4) $$x = sB, \qquad y = A$$

shows that x and y satisfy equation (3.3).

Asymptotically, the roots of the polynomial

(3.5) $$p_t(x) = x^4 + 4tx^3 - 12tx^2 - 8t^2x + 4t^2$$

are given as follows. We adopt the L-notation defined in [8, pp. 1151–1152] that we recall here. Let c be a real number, assume $f(x)$, $g(x)$, and $h(x)$ are real functions and $h(x) > 0$ for $x > c$. We will write

$$f(x) = g(x) + L_c(h(x))$$

if

$$g(x) - h(x) \le f(x) \le g(x) + h(x) \quad \text{ for } x > c.$$

Therefore we obtain

$$\begin{aligned}
\beta^{(1)} &= \sqrt{2t} + 1 + \frac{1}{2\sqrt{2t}} - \frac{1}{2t} - \frac{9}{16t\sqrt{2t}} + L_6\left(\frac{0.59}{t^2}\right),\\
\beta^{(2)} &= -\sqrt{2t} + 1 - \frac{1}{2\sqrt{2t}} - \frac{1}{2t} + \frac{9}{16t\sqrt{2t}} + L_{792}\left(\frac{0.48}{t^2}\right),\\
\beta^{(3)} &= \frac{1}{2} - \frac{5}{16t} + \frac{23}{64t^2} + L_{105}\left(\frac{0.49}{t^3}\right),\\
\beta^{(4)} &= -4t - \frac{5}{2} + \frac{21}{16t} - \frac{84}{64t^2} + L_5\left(\frac{1.349}{t^3}\right).
\end{aligned}$$

Carefully analyzing the construction of the Thue equation (3.3), it is not difficult to verify that if $X^2 = V_{4k+1}$ with $k > 0$, then the corresponding positive integer solution (x, y), given by (3.2), to equation (3.3) has the property that the closest root to x/y is $\beta^{(4)}$, whereas if $X^2 = V_{4k+3}$, then the corresponding positive integer solution (x, y), given by (3.4), to equation (3.3) has the property that the closest root to x/y is $\beta^{(1)}$. We make this comment more concrete in the following.

LEMMA 3.1. *If* $X^2 = V_{4k+1}$ *is solvable with* X *an integer and* $k > 0$, *and if* (x, y), *given by* (3.2), *is the corresponding solution to the Thue equation*

(3.3), *then*

$$|x/y - \beta^{(4)}| < \frac{1}{16t|y|^4}.$$

If $X^2 = V_{4k+3}$ *is solvable with* X *an integer and* $k > 0$, *and if* (x, y), *given by* (3.4), *is the corresponding solution to the Thue equation* (3.3), *then*

$$|x/y - \beta^{(1)}| < \frac{1}{4t|y|^4}.$$

Proof. We will prove the result for the equation $X^2 = V_{4k+1}$, as the proof for the second case is essentially identical. We will use the fact that V_{2k}/U_{2k} is a close approximation to $\sqrt{t(t+2)}$. The definition of (x, y) gives

$$\frac{x}{y} = \frac{-sB}{A} = \frac{-2sB^2}{2AB} = -\frac{X + V_{2k+1}}{U_{2k}} = -\frac{\sqrt{V_{4k+1}} + V_{2k+1}}{U_{2k}}.$$

Using the relations $V_{4k+1} = V_{2k+1}^2 + 2tU_{2k}^2$ and $V_{2k+1} = V_{2k} + tU_{2k}$, we find that

$$x/y = -(\sqrt{(V_{2k}/U_{2k})^2 + 2t(V_{2k}/U_{2k}) + (t^2 + t)} + (V_{2k}/U_{2k}) + t).$$

Using the fact that V_{2k}/U_{2k} is a close approximation to $\sqrt{t(t+2)}$, we see that the above expression is closely approximated by $-4t - 5/2$, which shows that the closest root to x/y is $\beta^{(4)}$. As

$$|p_t(x, y)| = |y|^4 \prod_{i=1}^{4} |x/y - \beta^{(i)}| \leq 4t^2,$$

we see that

$$|x/y - \beta^{(4)}| \leq \frac{4t^2}{|y|^4 \prod_{i \neq 4} |x/y - \beta^{(i)}|}.$$

Using the crude estimate $4t$ for each factor $|x/y - \beta^{(i)}|$, we see that

$$|x/y - \beta^{(4)}| \leq \frac{1}{16t|y|^4}. \blacksquare$$

4. Lower bounds for k, t and $|y|$. In order to prove Theorem 1.1, we first need to verify that equation (1.1) has only the positive integer solution $(X, Y) = (1, 1)$ for all t up to a certain bound. Two independent computations, using PARI and MAGMA, were run in order to verify that equation (1.1) has no positive integer solutions other than $(X, Y) = (1, 1)$ for all $1 \leq t \leq 1200$. This was achieved by computing all integer solutions to each Thue equation of the form $p_t(x, y) = s^2$, where $p_t(x, y)$ is given in (3.5), and s is a divisor of $2t$.

We will also need a lower bound for k. The polynomial $V_{4k+1}(t)$ is monic and of even degree. Therefore, Runge's method can be applied directly to equation (2.1). Fortunately, Runge's method has been implemented in

MAGMA by Beukers and Tengely (see [4]). This rather short MAGMA computation verified that no positive integer solutions (X, t) to equation (2.1) exist for each $1 \le k \le 24$.

We now describe how to obtain the lower bound $k > 6$ in the case of solving $X^2 = V_{4k+3}$. Firstly, it is trivial to see that $V_{4k+3} \equiv 3 \pmod 4$ for k even. Indeed, first note that from the definition of V_k,

$$V_{2k+1} = (t+1)V_{2k-1} + tU_{2k+1},$$

and as $(t+1+\sqrt{t(t+2)})^{-1} = t+1-\sqrt{t(t+2)}$, we also have the relation

$$V_{2k-3} = (t+1)V_{2k-1} - tU_{2k+1}.$$

Combining these two equations gives the second order linear recurrence

$$V_{2k+1} = (2t+2)V_{2k-1} - V_{2k-3}.$$

Since t is odd, 4 divides $2t+2$, and so for each k,

$$V_{2k+1} \equiv -V_{2k-3} \pmod 4.$$

As $V_1 = 1$ and $V_3 = 2t+1 \equiv 3 \pmod 4$, it follows that

$$V_{2k+1} \equiv 1 \pmod 4$$

for $2k+1 \equiv 1, 7 \pmod 8$, and

$$V_{2k+1} \equiv 3 \pmod 4$$

for $2k+1 \equiv 3, 5 \pmod 8$.

If $k = 1$, then the equation is simply $X^2 = 8t^3 + 20t^2 + 12t + 1$, which is easily shown to have no solutions in positive integers X, t using MAGMA. For the case $k = 3$, the equation $V_{15} = X^2$ implies that $V_5 = 4t^2 + 6t + 1$ is either a square or three times a square by elementary divisibility properties of terms in the sequence $\{V_n\}$ (see [9] for details). Evidently, neither of these possibilities is possible. Finally, if $k = 5$, we use the fact that for each $i \ge 0$, the Jacobi symbol (V_{16i+7}/V_{16i+5}) equals -1, which is easily proved by induction, however we provide the details for the convenience of the reader. The proof uses the above linear recurrence equation for $\{V_{2k+1}\}$, the above congruences $\pmod 4$ for V_{2k+1}, along with basic manipulation of Jacobi symbols. First,

$$\begin{aligned}(V_7/V_5) &= (((2t+2)V_5 - V_3)/V_3) = (-V_3/V_5) = -(V_3/V_5) = (V_5/V_3) \\ &= (-V_1/V_3) = (-1/V_3) = -1.\end{aligned}$$

Thus, the result holds for $i = 1$. We next show that

$$(V_{8j+7}/V_{8j+5}) = -(V_{8j-1}/V_{8j-3})$$

for all $j \ge 0$, which upon putting $j = 2i$ and $j = 2i-1$ gives the desired result. We calculate

$$\begin{aligned}(V_{8j+7}/V_{8j+5}) &= (-V_{8j+3}/V_{8j+5}) = -(V_{8j+3}/V_{8j+5}) = (V_{8j+5}/V_{8j+3})\\ &= (-V_{8j+1}/V_{8j+3}) = -(V_{8j+1}/V_{8j+3})\\ &= -(V_{8j+3}/V_{8j+1}) = -(V_{8j-1}/V_{8j+1})\\ &= -(V_{8j+1}/V_{8j-1}) = -(V_{8j-3}/V_{8j-1}) = -(V_{8j-1}/V_{8j-3}).\end{aligned}$$

We use the above lower bounds for k to obtain lower bounds for $|y|$, where (x, y) is a solution to the Thue equation (3.3) arising from a solution to equation (1.1). In the case of a solution to (1.1) with $X^2 = V_{4k+3}$, we see from the above construction that

$$2ry^2 = \sqrt{V_{4k+3}} - V_{2k+1} = \sqrt{V_{2k+1}^2 + 2tU_{2k+2}^2} - V_{2k+1}.$$

It follows that

$$2ry^2 = \frac{2tU_{2k+2}^2}{\sqrt{V_{2k+1}^2 + 2tU_{2k+2}^2} + V_{2k+1}}.$$

Dividing the numerator and denominator of this equation by $\sqrt{t}\,U_{2k+2}$ gives

$$2ry^2 = \frac{2\sqrt{t}\,U_{2k+2}}{\sqrt{(V_{2k+1}^2/tU_{2k+2}^2) + 2} + (V_{2k+1}/\sqrt{t}\,U_{2k+2})}.$$

By the fact that $V_{2k+1} < U_{2k+2}$, it follows that

$$2ry^2 > \sqrt{t}\,U_{2k+2},$$

and from the lower bound

$$U_{2k+2} > (2t)^k > (2t)^6,$$

it follows that

$$|y| > 2\sqrt{2}\,t^{9/4}.$$

In the case of a solution to $X^2 = V_{4k+1}$ with $k > 0$, we obtain a much larger lower bound since the equation $X^2 = V_{4k+1}$ was solved using Runge's method for $1 \le k \le 24$. In this case, an analysis similar to the one given above shows that $|y| > 2^{10}t^{11}$.

5. Associated fourth roots of unity. Let

$$p_t(x, y) = x^4 + 4tx^3y - 12tx^2y^2 - 8t^2xy^3 + 4t^2y^4$$

and t be a positive integer. Our goal is to find, for fixed t, an upper bound upon the number of coprime nonzero integer solutions to the inequality

$$0 < p_t(x, y) \le 4t^2. \tag{5.1}$$

To proceed, let $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ be linear functions of (x,y) so that

$$\xi^4 = 4(-\sqrt{-t/2}+1)(x+\sqrt{-2t}\,y)^4, \qquad \eta^4 = 4(-\sqrt{-t/2}-1)(x-\sqrt{-2t}\,y)^4.$$

We refer to (ξ,η) as a pair of *resolvent forms*. Note that $\xi^4 = -\overline{\eta}^4$ and that

$$p_t(x,y) = \frac{1}{8}\,(\xi^4 - \eta^4),$$

and if (ξ,η) is a pair of resolvent forms then there are precisely three others with distinct ratios, say $(-\xi,\eta)$, $(i\xi,\eta)$ and $(-i\xi,\eta)$. Let ω be a fourth root of unity, (ξ,η) a fixed pair of resolvent forms and set

$$z = 1 - \left(\frac{\eta(x,y)}{\xi(x,y)}\right)^4.$$

We say that the integer pair (x,y) is *related* to ω if

$$\left|\omega - \frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{\pi}{12}\,|z|.$$

It turns out that each nontrivial solution (x,y) to (3.3) is related to a fourth root of unity:

LEMMA 5.1. *Suppose that (x,y) is a positive integral solution to* (5.1), *with*

$$\left|\omega_j - \frac{\eta(x,y)}{\xi(x,y)}\right| = \min_{0\le k\le 3}\left|e^{k\pi i/2} - \frac{\eta(x,y)}{\xi(x,y)}\right|.$$

Then

$$\left|\omega_j - \frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{\pi}{12}\,|z(x,y)|. \tag{5.2}$$

Proof. We begin by noting that

$$|z| = \left|\frac{\xi^4-\eta^4}{\xi^4}\right| = \frac{8p_t(x,y)}{|\xi^4|},$$

and from $xy \ne 0$,

$$|\xi^4(x,y)| \ge 8\sqrt{2}\,(\sqrt{t})^5,$$

whence

$$|z| \le \frac{4t^2}{\sqrt{2}\,(\sqrt{t})^5} < 1.$$

Since $\eta = -\overline{\xi}$, it follows that

$$|\eta/\xi| = 1, \qquad |1-z| = 1.$$

Now let $4\theta = \arg(\eta(x,y)^4/\xi(x,y)^4)$. We have

$$\sqrt{2-2\cos(4\theta)} = |z| < 1,$$

and so $|\theta| < \pi/12$. Since

$$\left|\omega_j - \frac{\eta(x,y)}{\xi(x,y)}\right| \le |\theta|,$$

it follows that

$$\left|\omega_j - \frac{\eta(x,y)}{\xi(x,y)}\right| \le \frac{1}{4}\,\frac{|4\theta|}{\sqrt{2-2\cos(4\theta)}}\left|1 - \frac{\eta(x,y)^4}{\xi(x,y)^4}\right|.$$

From the fact that $|4\theta|/\sqrt{2-2\cos(4\theta)} < \pi/3$ whenever $0 < |\theta| < \pi/12$, we obtain the desired inequality. ∎

We now put $\omega_i = \eta(\beta^{(i)},1)/\xi(\beta^{(i)},1)$ for $1 \le i \le 4$. The ω_i are the distinct fourth roots of unity. The following lemma represents a key step towards the proof of Theorem 1.1.

LEMMA 5.2. *If $X^2 = V_{4k+1}$ is solvable, with X an integer and $k > 0$, and if (x,y), given by* (3.2), *is the corresponding solution to the Thue equation* (3.3), *then (x,y) is associated to ω_4.*

If $X^2 = V_{4k+3}$ is solvable, with X an integer and $k > 0$, and if (x,y), given by (3.4), *is the corresponding solution to the Thue equation* (3.3), *then (x,y) is associated to ω_1.*

Proof. Assume that $X^2 = V_{4k+3}$, as the other case is proved in the same way. The goal is to prove that

$$\left|\frac{\eta(\beta^{(1)},1)}{\xi(\beta^{(1)},1)} - \frac{\eta(x,y)}{\xi(x,y)}\right| < \frac{\pi}{12}\left|1 - \left(\frac{\eta(x,y)}{\xi(x,y)}\right)^4\right|.$$

Obtaining a common divisor for the left side, expanding and simplifying shows that the above inequality is the same as

$$\frac{|2\sqrt{-2t}|\,|x-\beta^{(1)}y|}{|\beta^{(1)}+\sqrt{-2t}|\,|x+y\sqrt{-2t}|} < \frac{|2P_t(x,y)|}{|\xi(x,y)^4|}.$$

Cross-multiplying the above and dividing through by $|y|^4$, we reduce the problem to proving the inequality

$$\frac{|2\sqrt{-2t}|\,|2-\sqrt{-2t}|\,|x/y+\sqrt{-2t}|^3}{|\beta^{(1)}+\sqrt{-2t}|} < |x/y-\beta^{(2)}|\,|x/y-\beta^{(3)}|\,|x/y-\beta^{(4)}|. \tag{5.3}$$

Lemma 3.1 shows that x/y is very close to $\beta^{(1)}$. Indeed, Lemma 3.1, together with the lower bound for $|y|$ determined in Section 4, shows that $|x/y-\beta^{(1)}| < 1/(2^8t^{10})$. This difference is sufficiently small that we can replace x/y in (5.3) by $\beta^{(1)}$, which entails that we need to prove the inequality

$$|2\sqrt{-2t}|\,|2-\sqrt{-2t}|\,|\beta^{(1)}+\sqrt{-2t}|^2 < |\beta^{(1)}-\beta^{(2)}|\,|\beta^{(1)}-\beta^{(3)}|\,|\beta^{(1)}-\beta^{(4)}|. \tag{5.4}$$

Expanding the left-hand side of (5.4) gives an estimate with leading terms $16t^2 + 20t$, while that for the right-hand side has leading terms $16t^2 + 24t$. ∎

6. The main lemma. The following represents the most crucial lemma of this paper, as it provides an absolute bound for the number of integer solutions to equation (1.1).

LEMMA 6.1. *There is at most one solution of* (5.1) *related to each fourth root of unity.*

Because of the lower bound for k obtained in Section 4, we may assume that $k > 6$. Furthermore, since $\xi_i^4 = 4(-\sqrt{-t/2} + 1)(x_i + \sqrt{-2t}\, y_i)^4$, via calculus one may conclude that

$$|\xi_1|^4 > t^{14}. \tag{6.1}$$

6.1. *Approximating polynomials.* The following lemma gives a family of dense approximations to ξ/η from rational function approximations to the binomial function $(1 - z)^{1/4}$.

LEMMA 6.2. *Let r be a positive integer and $g \in \{0, 1\}$. Put*

$$A_{r,g}(z) = \sum_{m=0}^{r} \binom{r - g + 1/4}{m} \binom{2r - g - m}{r - g} (-z)^m,$$

$$B_{r,g}(z) = \sum_{m=0}^{r-g} \binom{r - 1/4}{m} \binom{2r - g - m}{r} (-z)^m.$$

(i) *There exists a power series $F_{r,g}(z)$ such that for all complex numbers z with $|z| < 1$,*

$$A_{r,g}(z) - (1 - z)^{1/4} B_{r,g}(z) = z^{2r+1-g} F_{r,g}(z) \tag{6.2}$$

and

$$|F_{r,g}(z)| \le \frac{\binom{r-g+1/4}{r+1-g}\binom{r-1/4}{r}}{\binom{2r+1-g}{r}} (1 - |z|)^{-(2r+1-g)/2}. \tag{6.3}$$

(ii) *For all complex numbers z with $|1 - z| \le 1$ we have*

$$|A_{r,g}(z)| \le \binom{2r - g}{r}. \tag{6.4}$$

(iii) *For all complex numbers $z \ne 0$ and for $h \in \{1, 0\}$ we have*

$$A_{r,0}(z) B_{r+h,1,1}(z) \ne A_{r+h,1}(z) B_{r,0}(z). \tag{6.5}$$

Proof. See the proof of Lemma 4.1 of [1]. ∎

Combining the polynomials of Lemma 6.2 with the resolvent forms $\xi(x,y)$ and $\eta(x,y)$, we will consider the complex sequences $\Sigma_{r,g}$ given by

$$\Sigma_{r,g} = \frac{\eta_2}{\xi_2} A_{r,g}(z_1) - (-1)^r \frac{\eta_1}{\xi_1} B_{r,g}(z_1)$$

where $z_1 = 1 - \eta_1^4/\xi_1^4$. Define

$$\Lambda_{r,g} = \frac{\xi_1^{4r+1-g}\xi_2}{(-t/2-1)^{1/4}} \Sigma_{r,g}. \tag{6.6}$$

We will show that $\Lambda_{r,g}$ is either an integer in $\mathbb{Q}(\sqrt{-2t})$ or a fourth root of such an integer. If $\Lambda_{r,g} \neq 0$, this provides a lower bound upon $|\Lambda_{r,g}|$. In conjunction with the inequalities derived in Lemma 6.2, this will induce a strong "gap principle", guaranteeing that solutions to inequality (5.1) must, in a certain sense, increase rapidly in height.

Lemma 6.3. *If (x_1,y_1) and (x_2,y_2) are two pairs of rational integers then*

$$\frac{\xi(x_1,y_1)\eta(x_2,y_2)}{(-t/2-1)^{1/4}}, \quad \xi(x_1,y_1)^3\xi(x_2,y_2) \quad \text{and} \quad \eta(x_1,y_1)^3\eta(x_2,y_2)$$

are integers in $\mathbb{Q}(\sqrt{-2t})$.

Proof. This is an immediate consequence of the definition of resolvent forms. ∎

For a polynomial $P(z)$ of degree n, we will denote by $P^*(x,y) = x^n P(y/x)$ an associated binary form. Let $A_{r,g}$ and $B_{r,g}$ be as in Lemma 6.2, and set

$$C_{r,g}(z) = A_{r,g}(1-z), \qquad D_{r,g}(z) = B_{r,g}(1-z).$$

For $z \neq 0$, we have $D_{r,0}(z) = z^r C_{r,0}(z^{-1})$, hence

$$\begin{aligned} A^*_{r,0}(z, z+\overline{z}) &= z^r A_{r,0}(1+\overline{z}/z) = z^r C_{r,0}(-\overline{z}/z) \\ &= (-1)^r \overline{z}^r D_{r,0}(-z/\overline{z}) = (-1)^r \overline{z}^r B_{r,0}(1+z/\overline{z}) \\ &= (-1)^r B^*_{r,0}(\overline{z}, \overline{z}+z) = (-1)^r \overline{B}^*_{r,0}(z, z+\overline{z}). \end{aligned}$$

Lemma 6.4. *For any pair of integers (x,y), both $A^*_{r,g}(\xi^4(x,y), \xi^4(x,y) - \eta^4(x,y))$ and $B^*_{r,g}(\xi^4(x,y), \xi^4(x,y) - \eta^4(x,y))$ are algebraic integers in $\mathbb{Q}(\sqrt{-2t})$.*

Proof. It is clear that $A^*_{r,g}(\xi^4(x,y), \xi^4(x,y) - \eta^4(x,y))$ and $B^*_{r,g}(\xi^4(x,y), \xi^4(x,y) - \eta^4(x,y))$ belong to $\mathbb{Q}(\sqrt{-2t})$; we need only show that they are algebraic integers. From the definitions of $A^*_{r,g}(x,y)$, $B^*_{r,g}(x,y)$, $\xi(x,y)$ and $\eta(x,y)$ (in particular, since $\xi^4(x,y) - \eta^4(x,y) = 8p_t(x,y)$), this is an immediate consequence of Lemma 4.1 of [6], which, in this case, implies that $\binom{a/4}{n}8^n$ is, for fixed nonnegative integers a and n, a rational integer. ∎

Proposition 6.1. *Let $\Lambda_{r,g}$ be the complex number defined in* (6.6). *Then $\Lambda_{r,0}$ belongs to $\mathbb{Z}[\sqrt{-2t}]$ and $\Lambda_{r,1}$ is a fourth root of an integer in $\mathbb{Q}(\sqrt{-2t})$.*

Proof. We have

$$\Lambda_{r,g} = \frac{\xi_1^{1-g}\eta_2}{(-t/2-1)^{1/4}}\, A^*_{r,g}(\xi_1^4, \xi_1^4 - \eta_1^4) - \frac{(-1)^r \xi_1^{2g}\xi_2\eta_1}{(-t/2-1)^{1/4}}\, B^*_{r,g}(\xi_1^4, \xi_1^4 - \eta_1^4).$$

By Lemmas 6.3 and 6.4, $\Lambda_{r,0} \in \mathbb{Z}\sqrt{-2t}$. Similarly, Lemmas 6.3 and 6.4 imply that $\Lambda_{r,1}^4$ is an algebraic integer in $\mathbb{Q}(\sqrt{-2t})$. We claim that it is not a rational integer. To see this, let us start by noting that

$$\begin{aligned}\frac{\Sigma_{r,g}}{(-t/2-1)^{1/4}} &= \frac{\eta_2}{\xi_2}\, A_{r,g}(z_1) - (-1)^r \frac{\eta_1}{\xi_1}\, B_{r,g}(z_1)\\ &= \frac{\eta}{\xi}\bigg(\frac{\eta_2/\eta}{\xi_2/\xi}\, A_{r,g}(z_1) - (-1)^r \frac{\eta_1/\eta}{\xi_1/\xi}\, B_{r,g}(z_1)\bigg),\end{aligned}$$

where $\eta = (\sqrt{-t/2} - 1)^{1/4}$ and $\xi = (\sqrt{-t/2} + 1)^{1/4}$. By Lemma 6.4,

$$\frac{\eta_2/\eta}{\xi_2/\xi}\, A_{r,g}(z_1) - (-1)^r \frac{\eta_1/\eta}{\xi_1/\xi}\, B_{r,g}(z_1) \in \mathbb{Q}(\sqrt{-2t})$$

and so

$$\begin{aligned}\mathfrak{f} &= \mathbb{Q}(\sqrt{-2t}, \Sigma_{r,g}) = \mathbb{Q}(\sqrt{-2t}, (-t/2-1)^{1/4}\eta/\xi)\\ &= \mathbb{Q}(\sqrt{-2t}, (-t/2+1-\sqrt{-2t})^{1/4}).\end{aligned}$$

If we choose a complex number X so that $\xi(X,1) = \eta(X,1)$ then $X \in \mathfrak{f}$ and

$$p_t(X,1) = \frac{1}{8}\,(\xi^4(X,1) - \eta^4(X,1)) = 0.$$

Since p_t is irreducible, X and $\Sigma_{r,g}$ both have degree 4 over $\mathbb{Q}(\sqrt{-2t})$.

Suppose that $\Lambda_{r,1}^4 \in \mathbb{Z}$. Then we deduce for some $\varrho, \varrho_1 \in \{\pm 1, \pm i\}$ that $\Lambda_{r,1} = \varrho\overline{\Lambda}_{r,1}$ and $(-t/2-1)^{1/4} = \varrho_1\overline{(-t/2-1)^{1/4}}$, whence, from Lemma 6.3,

$$\begin{aligned}\Sigma_{r,1} &= (-t/2-1)^{1/4}\xi_1^{-4r}\xi_2^{-1}\varrho\overline{\Lambda}_{r,1}\\ &= \xi_1^{-4r}\xi_2^{-1}\eta_1^{4r}\eta_2\varrho\varrho_1\bigg(\frac{\xi_2}{\eta_2}\, A_{r,1}\Big(1 - \frac{\xi^4}{\eta^4}\Big) - (-1)^r \frac{\xi_1}{\eta_1}\, B_{r,1}\Big(1 - \frac{\xi^4}{\eta^4}\Big)\bigg)\\ &= \varrho\varrho_1 \frac{\eta_1^{4r}}{\xi_1^{4r}}\bigg(A_{r,1}\Big(1 - \frac{\xi_1^4}{\eta_1^4}\Big) - (-1)^r \frac{\xi_1\eta_2}{\xi_2\eta_1}\, B_{r,1}\Big(1 - \frac{\xi_1^4}{\eta_1^4}\Big)\bigg).\end{aligned}$$

This together with Lemmas 6.3 and 6.4 implies that $\Sigma_{r,1} \in \mathbb{Q}(\sqrt{-2t}, \varrho\varrho_1)$, which contradicts the fact that $\Sigma_{r,1}$ has degree 4 over $\mathbb{Q}(\sqrt{-2t})$. We conclude that $\Lambda_{r,1}$ cannot be a rational integer. ∎

From the well-known characterization of algebraic integers in quadratic fields, we know that for any square-free integer d, the ring of integers

$\mathbb{Q}(\sqrt{d})$ is

$$\{m + n\sqrt{d} \mid m, n \in \mathbb{Z}\} \quad \text{if } d \equiv 2, 3 \pmod 4$$

and

$$\left\{ m + n\,\frac{1+\sqrt{d}}{2} \;\middle|\; m, n \in \mathbb{Z} \right\} \quad \text{if } d \equiv 1 \pmod 4.$$

We may therefore conclude that, if $\Lambda_{r,g} \neq 0$, $g \in \{0, 1\}$, then

$$|\Lambda_{r,g}| \geq 2^{-g/4}(2t)^{1/2-3g/8}. \tag{6.7}$$

6.2. *Gap principles.* Lemma 5.1 shows that each integer pair (x, y) is related to precisely one fourth root of unity. Let us fix such a fourth root, say ω, and suppose that we have distinct coprime positive solutions (x_1, y_1) and (x_2, y_2) to inequality (5.1), each related to ω. We will assume that $|\xi(x_2, y_2)| \geq |\xi(x_1, y_1)|$. Let us write $\eta_i = \eta(x_i, y_i)$ and $\xi_i = \xi(x_i, y_i)$. We will use the following results to prove that (x_1, y_1) and (x_2, y_2) are far apart in height.

Since

$$|z| = \frac{8p_t(x, y)}{|\xi|^4} \leq \frac{32t^2}{|\xi|^4}, \tag{6.8}$$

it follows from (5.2) that

$$\begin{aligned} |\xi_1\eta_2 - \xi_2\eta_1| &= |\xi_1(\eta_2 - \omega\xi_2) - \xi_2(\eta_1 - \omega\xi_1)| \\ &\leq \frac{8\pi t^2}{3}\left(\frac{|\xi_1|}{|\xi_2^3|} + \frac{|\xi_2|}{|\xi_1^3|}\right) \leq \frac{16\pi t^2|\xi_2|}{3|\xi_1^3|}. \end{aligned} \tag{6.9}$$

On the other hand, choosing our fourth root appropriately, we have

$$\begin{pmatrix} \sqrt{2}(-\sqrt{-t/2}+1)^{1/4} & \sqrt{2}(-\sqrt{-t/2}+1)^{1/4}\sqrt{-2t} \\ \sqrt{2}(-\sqrt{-t/2}-1)^{1/4} & -\sqrt{2}(-\sqrt{-t/2}-1)^{1/4}\sqrt{-2t} \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi_2 \\ \eta_1 & \eta_2 \end{pmatrix}$$

and so

$$|\xi_1\eta_2 - \xi_2\eta_1| = |4(t/2+1)^{1/4}\sqrt{2t}\,(x_1y_2 - x_2y_1)|.$$

Since $x_1y_2 - x_2y_1$ is a nonzero integer (recall that we assumed $\gcd(x_i, y_i) = 1$), we have

$$|\xi_1\eta_2 - \xi_2\eta_1| \geq 4\sqrt{2t}(t/2+1)^{1/4}, \tag{6.10}$$

and thus, combining (6.9) and (6.10), we conclude that if (x_1, y_1) and (x_2, y_2) are distinct solutions to (5.1), related to ω, with $|\xi(x_2, y_2)| \geq |\xi(x_1, y_1)|$ then

$$|\xi_2| > \frac{3\,t^{-5/4}}{4\pi}\,|\xi_1|^3. \tag{6.11}$$

We will now combine inequality (6.7) with upper bounds from Lemma 6.2 to show that solutions to (5.1) are widely spaced:

LEMMA 6.5. *If $\Sigma_{r,g} \neq 0$, then*

$$c_1(r,g)|\xi_1|^{4r+1-g}|\xi_2|^{-3} + c_2(r,g)|\xi_1|^{-4r-3(1-g)}|\xi_2| > 1,$$

where we may take

$$c_1(r,g) = \frac{\sqrt{\pi}\, 2^{2r+11/4+5g/8}}{3\sqrt{r}}\, t^{5/4+3g/8},$$

$$c_2(r,g) = \frac{2^{1/4+5g/8-2r}33^{2r+1-g}}{\pi\sqrt{r}}\, t^{4r+5/4-13g/8}.$$

Proof. By (6.2), we can write

$$|(t/2+1)^{1/4}\Lambda_{r,g}| = |\xi_1|^{4r+1-g}|\xi_2|\,|(\eta_2/\xi_2-\omega)A_{r,g}(z_1) + \omega z_1^{2r+1-g}F_{r,g}(z_1)|.$$

Since $|1-z_1| = 1$ and $|z_1| \leq 1$, from (5.2), (6.8), (6.3), (6.4), and the inequality

$$|\xi_1|^4 > 8\sqrt{2}\, t^{5/2},$$

we get

$$|(t/2+1)^{1/4}\Lambda_{r,g}| \leq |\xi_1|^{4r+1-g}|\xi_2|\left(\binom{2r-g}{r}\frac{2\pi t^2}{3|\xi_2^4|} + \frac{\binom{r-g+1/4}{r+1-g}\binom{r-1/4}{r}}{\binom{2r+1-g}{r}}\left(\frac{9t^2}{|\xi_1^4|}\right)^{2r+1-g}\right).$$

Comparing this with (6.7), we obtain

$$c_1(r,g)|\xi_1|^{4r+1-g}|\xi_2|^{-3} + c_2(r,g)|\xi_1|^{-4r-3(1-g)}|\xi_2| > 1,$$

where we may take c_1 and c_2 so that

$$c_1(r,g) \geq \frac{2^{11/4+5g/8}t^{5/4+3g/8}\pi}{3}\binom{2r}{r},$$

$$c_2(r,g) \geq 2^{5g/8-1/4}33^{2r+1-g}t^{4r+5/4-13g/8}\,\frac{\binom{r-g+1/4}{r+1-g}\binom{r-1/4}{r}}{\binom{2r+1-g}{r}}.$$

Applying the following version of Stirling's formula (see Theorem (5.44) of [16]):

$$\frac{1}{2\sqrt{k}}\,4^k \leq \binom{2k}{k} < \frac{1}{\sqrt{\pi k}}\,4^k$$

(valid for $k \in \mathbb{N}$) leads immediately to the stated choice of c_1. One can also show via Stirling's formula that for $r \in \mathbb{N}$ and $g \in \{0,1\}$, we have

$$\frac{\binom{r-g+1/4}{r+1-g}\binom{r-1/4}{r}}{\binom{2r+1-g}{r}} < \frac{\sqrt{2}}{\sqrt{r}\,\pi 4^r}$$

(see the proof of Lemma 6.1 of [1] for more details). This gives the desired value for $c_2(r,g)$. ∎

6.3. *The proof of Lemma 6.1.* We will start this section with the statements of two lemmas from [1]. These lemmas allow us to apply the strong gap principle provided by Lemma 6.5. We note here that although $\xi(x,y)$ and $\eta(x,y)$ are defined differently in [1], those properties of ξ and η used in the proofs of Lemmas 6.2 and 6.3 of [1] hold for our choice of ξ and η in the present paper.

LEMMA 6.6. *If $r \in \mathbb{N}$ and $h \in \{0,1\}$, then at most one of $\{\Sigma_{r,0}, \Sigma_{r+h,1}\}$ can vanish.*

LEMMA 6.7. *Suppose that $t > 1200$. For $r \in \{1,\ldots,5\}$, we have $\Sigma_{r,0} \neq 0$.*

Assume that there are two distinct coprime solutions (x_1,y_1) and (x_2,y_2) to inequality (5.1) with $|\xi_2| > |\xi_1|$. We will show that $|\xi_2|$ is arbitrarily large in relation to $|\xi_1|$. In particular, we will demonstrate via induction that

$$|\xi_2| > \frac{\sqrt{r}}{t^{4r+7/4}}\left(\frac{4}{33^2}\right)^r |\xi_1|^{4r+3} \tag{6.12}$$

for each positive integer r. So by (6.1),

$$|\xi_2| > t^{13r}$$

for arbitrary r, a contradiction.

To prove inequality (6.12) for $r = 1$, we use (6.11) and (6.1) to get

$$c_1(1,0)|\xi_1|^5|\xi_2|^{-3} < \frac{2^{43/4}\sqrt{\pi}}{3}\, t^5 |\xi_1|^{-4} < 0.01,$$

and hence, since $\Sigma_{1,0} \neq 0$, Lemma 6.5 yields

$$c_2(1,0)|\xi_1|^{-7}|\xi_2| > 0.09,$$

which immediately implies (6.12). We now proceed by induction. Suppose that (6.12) holds for some $r \geq 1$. Then

$$c_1(r+1,0)|\xi_1|^{4r+5}|\xi_2|^{-3} < \frac{\sqrt{\pi}}{\sqrt{3}\,r^2}\, t^{12r+26/4}\left(\frac{33^6}{2^6}\right)^r |\xi_1|^{-8r-4},$$

and hence from (6.1), and the fact that $t \geq 1200$,

$$c_1(r+1,0)|\xi_1|^{4r+5}|\xi_2|^{-3} < 0.1.$$

If $\Sigma_{r+1,0} \neq 0$, then by Lemma 6.5,

$$c_2(r+1,0)|\xi_1|^{-4(r+1)-3}|\xi_2| > 0.9,$$

which again leads to (6.12). If, however, $\Sigma_{r+1,0} = 0$, then by Lemmas 6.6 and 6.7, both $\Sigma_{r+1,1}$ and $\Sigma_{r+2,1}$ are nonzero and $r \geq 2$. Using the induction hypothesis, we find as previously that

$$c_1(r+1,1)|\xi_1|^{4r+4}|\xi_2|^{-3} < 0.001,$$

and thus by Lemma 6.5 conclude that

$$c_2(r+1,1)|\xi_1|^{-4r-4}|\xi_2| > 0.999.$$

It follows that

$$|\xi_2| > \frac{\sqrt{r+1}}{2^{7/8}t^{4r+29/8}}\left(\frac{4}{33^2}\right)^{r+1}|\xi_1|^{4r+4}.$$

Consequently, from (6.1), $r \geq 6$ and $t > 1200$,

$$c_1(r+2,1)|\xi_1|^{4r+8}|\xi_2|^{-3} < 0.001.$$

Therefore Lemma 6.5 implies that

$$c_2(r+2,1)|\xi_1|^{-4r-8}|\xi_2| > 0.999,$$

and so

$$|\xi_2| > \frac{\pi}{2^{7/8}}\sqrt{r+2}\left(\frac{4}{33^2}\right)^{r+2} t^{-4r-61/8}|\xi_1|^{4r+8}.$$

From (6.1), it follows that

$$|\xi_2| > \frac{\sqrt{r+1}}{t^{4r+4+7/4}}\left(\frac{4}{33^2}\right)^{r+1}|\xi_1|^{4r+7},$$

as desired. This completes the proof of inequality (6.12), and hence we conclude that there is at most one solution to (5.1) related to each fourth root of unity. ∎

We conclude from the above, in conjunction with Lemma 5.2, that there are at most three solutions in positive integers to equation (1.1). In particular, there is the solution $(X,Y) = (1,1)$, and for both $i = 1$ and $i = 3$, at most one integer k for which $X^2 = V_{4k+i}$ is solvable. We now proceed to the proof that for t large enough, the equation $X^2 = V_{4k+1}$ is not solvable for all $k > 0$.

7. An effective measure of approximation. In this section we will apply the hypergeometric method to obtain effective measures of approximation to the two roots $\beta^{(3)}$ and $\beta^{(4)}$. Because of the relation $\beta^{(3)}\beta^{(4)} = -2t$, we will only need to deal with one of the roots, say $\beta^{(3)}$.

Our first lemma is Thue's "Fundamentaltheorem" [17] together with its relation to the hypergeometric function, as discovered by Siegel. The reader is also referred to Proposition 1 in [10], or Lemma 3.1 in [18].

LEMMA 7.1. *Let α_1, α_2, c_1 and c_2 be complex numbers with $\alpha_1 \neq \alpha_2$. For $n \geq 2$, we define the following polynomials*:

$$a(X) = \frac{n^2-1}{6}(\alpha_1-\alpha_2)(X-\alpha_2), \qquad c(X) = \frac{n^2-1}{6}\alpha_1(\alpha_1-\alpha_2)(X-\alpha_2),$$
$$b(X) = \frac{n^2-1}{6}(\alpha_2-\alpha_1)(X-\alpha_1), \qquad d(X) = \frac{n^2-1}{6}\alpha_2(\alpha_2-\alpha_1)(X-\alpha_1),$$

and

$$u(X) = -c_2(X-\alpha_2)^n, \qquad z(X) = c_1(X-\alpha_1)^n.$$

Putting $\lambda = (\alpha_1 - \alpha_2)^2/4$, *for any positive integer* r, *we define*

$$(\sqrt{\lambda})^r A_r(X) = a(X)X^*_{n,r}(z,u) + b(X)X^*_{n,r}(u,z),$$
$$(\sqrt{\lambda})^r B_r(X) = c(X)X^*_{n,r}(z,u) + d(X)X^*_{n,r}(u,z).$$

Then, for any root β *of* $P(X) = z(X) - u(X)$, *the polynomial*

$$C_r(X) = \beta A_r(X) - B_r(X)$$

is divisible by $(X-\beta)^{2r+1}$.

Proof. This is a simplified version of Lemma 2.1 from [5], obtained by noting that if $P(X)$ satisfies the differential equation given there, with $U(X) = (X-\alpha_1)(X-\alpha_2)$, then $P(X)$ must be of the form given here, which allows us to determine the above expressions. ■

LEMMA 7.2. *With the above notation, put* $w(x) = z(x)/u(x)$ *and write* $w(x) = \mu e^{i\varphi}$ *with* $\mu \ge 0$ *and* $-\pi < \varphi \le \pi$. *Put* $w(x)^{1/n} = \mu^{1/n}e^{i\varphi/n}$.

(i) *For any* $x \in \mathbb{C}$ *such that* $w = w(x)$ *is not a negative real number or zero*,

$$\begin{aligned}(\sqrt{\lambda})^r C_r(x) &= \{\beta(a(x)w(x)^{1/n} + b(x)) - (c(x)w(x)^{1/n} + d(x))\}X_{n,r}(u,z)\\ &\quad - (\beta a(x) - c(x))u(x)^r R_{n,r}(w),\end{aligned}$$

with

$$R_{n,r}(w) = \frac{\Gamma(r+1+1/n)}{r!\Gamma(1/n)} \int_1^w ((1-t)(t-w))^r t^{1/n-r-1}\,dt,$$

where the integration path is the straight line from 1 *to* w.

(ii) *Let* $w = e^{i\varphi}, 0 < \varphi < \pi$ *and put* $\sqrt{w} = e^{i\varphi/2}$. *Then*

$$|R_{n,r}(w)| \le \frac{n\Gamma(r+1+1/n)}{r!\Gamma(1/n)}\varphi|1-\sqrt{w}|^{2r}.$$

Proof. This is Lemma 2.5 of [5]. ■

LEMMA 7.3. *Let* u, w *and* z *be as above. Then*

$$|X^*_{n,r}(u,z)| \le 4|u|^r \frac{\Gamma(1-1/n)r!}{\Gamma(r+1-1/n)}|1+\sqrt{w}|^{2r-2}.$$

Proof. This is Lemma 2.6 of [5]. ■

LEMMA 7.4. *Let* $N_{4,r}$ *be the greatest common divisor of the numerators of the coefficients of* $X_{4,r}(1-2x)$ *and let* $D_{4,r}$ *be the least common multiple of the denominators of the coefficients of* $X_{4,r}(x)$. *Then the polynomial*

$(D_{4,r}/N_{4,r})\, X_{4,r}(1-2x)$ *has integral coefficients. Moreover*, $N_{4,r} = 2^r$ *and*

$$D_{4,r}\,\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)} < 0.8397\cdot 5.342^r \quad \textit{and} \quad D_{4,r}\,\frac{\Gamma(r+5/4)}{\Gamma(1/4)r!} < 0.1924\cdot 5.342^r.$$

Proof. Using the so-called Kummer transformation, we can write

$$X_{4,r}(1-2x) = \frac{r(r+1)\cdots(2r)}{(3/4)(7/4)\cdots(r-1/4)} = {}_2F_1(-r,-r-1/4;-2r;2x).$$

Expanding the right-hand side, we find that

$$X_{4,r}(1-2x) = \sum_{i=0}^{r}(-1)^i\,\frac{(r+1)\cdots(2r-i)}{3\cdot 7\cdots(4r-1)}\binom{r}{i}(4r-4i+1)\cdots(4r+1)2^{2r-i}x^i.$$

Therefore, 2^r divides $N_{4,r}$, and by examining the coefficient of x^r, we see that $N_{4,r} = 2^r$. We now turn to the inequalities.

From the arguments in the proof of Proposition 2(c) from [10], we obtain

$$D_{4,r} < \exp(1.6708r + 3.43\sqrt[3]{r}) < 5.341227^r$$

for $r \geq 20\,000$. Since $\exp(0.000073r) > \exp(1.46) > 2$ for such values of r, the upper bound for $D_{4,r}$ holds for $r \geq 20\,000$.

For $r \geq 2$,

$$\frac{\Gamma(r+5/4)}{\Gamma(1/4)r!} = \frac{5}{16}\prod_{i=2}^{r}\frac{i+1/4}{i} < \frac{5}{16}\exp\left(\int_1^r \log\left(\frac{x+1/4}{x}\right)dx\right)$$
$$< \frac{5}{16}\exp\left(\int_1^r \frac{dx}{4x}\right) \leq \frac{5}{16}\,r^{1/4}.$$

As a consequence, the inequalities in the statement of the lemma hold for $r \geq 20\,000$. A computation, similar to those described in the proof of Proposition 2 from [10], shows that the same inequalities holds for all smaller values of r. ∎

LEMMA 7.5. *Let* α_1, α_2, $A_r(X)$, $B_r(X)$ *and* $P(X)$ *be defined as in Lemma* 7.1 *and let* a, b, c *and* d *be complex numbers satisfying* $ad - bc \neq 0$. *Define*

$$K_r(X) = aA_r(X) + bB_r(X), \qquad L_r(X) = cA_r(X) + dB_r(X).$$

If $(x-\alpha_1)(x-\alpha_2)P(x) \neq 0$, *then*

$$K_{r+1}(x)L_r(x) \neq K_r(x)L_{r+1}(x)$$

for all $r \geq 0$.

Proof. This is Lemma 2.7 of [5]. ∎

LEMMA 7.6. *Let $\theta \in \mathbb{R}$. Suppose that there exist $k_0, l_0 > 0$ and $E, Q > 1$ such that for all $r \in \mathbb{N}$, there are rational integers p_r and q_r with $|q_r| < k_0 Q^r$ and $|q_r\theta - p_r| \le l_0 E^{-r}$ satisfying $p_r q_{r+1} \ne p_{r+1} q_r$. Then for any rational integers p and q with $|q| \ge 1/(2l_0)$, we have*

$$\left|\theta - \frac{p}{q}\right| > \frac{1}{c|q|^{\kappa+1}}, \qquad \text{where } c = 2k_0 Q(2l_0E)^\kappa \text{ and } \kappa = \frac{\log Q}{\log E}.$$

Proof. This is Lemma 2.8 from [5]. ∎

For the remainder of this section, we shall assume that t is a fixed integer greater than 37. We shall also simplify our notation here to reflect the fact that we have $n = 4$. We shall use R_r and X_r instead of $R_{4,r}$ and $X_{4,r}$.

We now determine the quantities defined in Lemma 7.1. Put

$$\alpha_1 = \sqrt{-2t}, \quad \alpha_2 = -\sqrt{-2t}, \quad c_1 = (1 + \sqrt{-t/2})/2, \quad c_2 = (1 - \sqrt{-t/2})/2.$$

Then

$$p_t(X) = X^4 + 4tX^3 - 12tX^2 - 8t^2X + 4t^2.$$

We define

$$\tau = \frac{\sqrt{t} + \sqrt{t+2}}{\sqrt{2}}, \qquad \varrho = \sqrt{\tau^2 + 1} = \sqrt{t + 2 + \sqrt{t^2 + 2t}}$$

for any positive integer t.

The preliminary results above will now be used in order to obtain an effective measure of approximation to $\beta^{(3)}$. By Lemma 7.2, we want to choose x so that

$$\beta^{(3)} = \frac{c(x)w(x)^{1/4} + d(x)}{a(x)w(x)^{1/4} + b(x)},$$

and for this purpose we will select $x = 0$. We have

$$w = w(0) = \frac{2 + \sqrt{-2t}}{-2 + \sqrt{-2t}}, \qquad \left(\frac{\tau - i}{\tau + i}\right)^2 = w, \qquad \left(\frac{\tau - i}{\varrho}\right)^2 = \frac{\tau - i}{\tau + i},$$

and so

$$w^{1/4} = \frac{\tau - i}{\varrho}.$$

Using the fact that $\varrho^2 = \tau^2 + 1$, one can check that

$$\frac{-i\tau - 1 + i\varrho}{-\tau + i - \varrho} = -\tau + \varrho,$$

and since

$$a(0) = -10t, \qquad b(0) = -10t, \qquad c(0) = -10t\sqrt{-2t}, \qquad d(0) = 10t\sqrt{-2t},$$

it follows that

$$\beta^{(3)} = \frac{c(0)w^{1/4} + d(0)}{a(0)w^{1/4} + b(0)}.$$

Therefore, the first term in the expression for $(-2t)^{r/2}C_r(0)$ in Lemma 7.2 disappears.

We now construct our sequence of rational approximations to $\beta^{(3)}$. By Lemmas 7.1 and 7.2, we have $\lambda = -2t$, and moreover,

$$\begin{aligned}(-2t)^{r/2}A_r(0) &= a(0)X_r^*(z(0),u(0)) + b(0)X_r^*(u(0),z(0)),\\ (-2t)^{r/2}B_r(0) &= c(0)X_r^*(z(0),u(0)) + d(0)X_r^*(u(0),z(0)),\\ (-2t)^{r/2}C_r(0) &= -(\beta^{(3)}a(0)-c(0))u(0)^r R_r(w).\end{aligned}$$

Therefore,

$$\begin{aligned}(-2t)^{r/2}A_r(0) &= -10t[X_r^*(z(0),u(0)) + X_r^*(u(0),z(0))],\\ (-2t)^{r/2}B_r(0) &= 5(-2t)^{3/2}[X_r^*(z(0),u(0)) - X_r^*(u(0),z(0))],\\ (-2t)^{r/2}C_r(0) &= 10t^{2r+1}[\beta^{(3)} - \sqrt{-2t}](-2+\sqrt{-2t})^r R_r(w).\end{aligned}$$

These quantities will form the basis for our approximations. We first eliminate some common factors. We can write $u(0) = t^2(-2+\sqrt{-2t})$ and $z(0) = t^2(2+\sqrt{-2t})$. Using Lemmas 7.2 and 7.3, and the triangular inequality, we obtain

$$\begin{aligned}|A_r(0)| &\le 80t^{(3r+2)/2}(2+t)^{r/2}\,\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,|1+\sqrt{w}|^{2r-2},\\ |B_r(0)| &\le 80\sqrt{2}\,t^{(3r+3)/2}(2+t)^{r/2}\,\frac{\Gamma(3/4)r!}{\Gamma(r+3/4)}\,|1+\sqrt{w}|^{2r-2},\\ |C_r(0)| &\le 40t^{(3r+2)/2}\varphi|\beta^{(3)} - \sqrt{-2t}|(2+t)^{r/2}\,\frac{\Gamma(r+5/4)}{r!\Gamma(1/4)}\,|1-\sqrt{w}|^{2r}.\end{aligned}$$

On the other hand, after some routine manipulations, we find that

$$\begin{aligned}(-2t)^{r/2}A_r(0) = \frac{-10t^{2r+1}N_{4,r}}{D_{4,r}}\Bigg\{\frac{D_{4,r}}{N_{4,r}}\,&[(2+\sqrt{-2t})^r X_r(1-2\eta)\\ &+(-2+\sqrt{-2t})^r X_r(1-2\overline{\eta})]\Bigg\},\end{aligned}$$

and

$$\begin{aligned}(-2t)^{r/2}B_r(0) = \frac{5(-2)^{3/2}t^{2r+3/2}N_{4,r}}{D_{4,r}}\Bigg\{\frac{D_{4,r}}{N_{4,r}}\,&[(-2+\sqrt{-2t})^r X_r(1-2\overline{\eta})\\ &-(2+\sqrt{-2t})^r X_r(1-2\eta)]\Bigg\},\end{aligned}$$

where $\eta = 2/(2+\sqrt{-2t}\,)$.

By Lemma 7.4, the quantities inside the braces can be expressed as $(-1)^r(e - f\sqrt{-2t}) \pm (e - f\sqrt{-2t})$, where e and f are rational integers, and recalling from Lemma 7.4 that $N_{4,r} = 2^r$, considering the cases of r being even or odd separately, we find that

$$P_r = \frac{D_{4,r}B_r(0)}{20 \cdot t^{[(3r+3)/2]}}, \qquad Q_r = \frac{D_{4,r}A_r(0)}{20 \cdot t^{[(3r+3)/2]}} \tag{7.1}$$

are rational integers. We note for future reference that if r is even, then P_r is divisible by t. The numbers P_r/Q_r are those that will be used as the rational approximations to $\beta^{(3)}$. We have

$$Q_r\beta^{(3)} - P_r = S_r, \quad \text{where} \quad S_r = \frac{D_{4,r}C_r(0)}{20 \cdot t^{[(3r+3)/2]}}.$$

We wish to show that these are good approximations. This will be done by estimating $|P_r|, |Q_r|$ and $|S_r|$ from above. It is readily verified that

$$|1 + \sqrt{w(0)}| = \frac{2\tau}{\sqrt{\tau^2+1}} = 2 - \frac{1}{2t} + O\left(\frac{11}{16t^2}\right);$$

in particular,

$$|1 + \sqrt{w(0)}| < 2.$$

Therefore, we have, for $t \geq 37$,

$$|Q_r| \leq 3.36(22\sqrt{t})^r.$$

Similarly, for $t \geq 37$ one obtains

$$|P_r| \leq 4.75\sqrt{t}\,(22\sqrt{t})^r.$$

Also,

$$|1 - \sqrt{w(0)}|^2 = \frac{4}{\tau^2+1} \leq \frac{2}{t}.$$

With φ as in Lemma 7.2, it can be shown that $2\varphi/\pi \leq \sin\varphi$ and

$$\sin\varphi = \operatorname{Im} w(0) = -2\sqrt{2t}/(t+2) \geq -2\sqrt{2/t}.$$

From the estimates for the roots of $p_t(X)$ given in Section 3, we know that $0 < \beta^{(3)} < 0.5$, and so

$$\varphi|\beta^{(3)} - \sqrt{-2t}| \leq \pi(2 + 1/\sqrt{2t}) \leq \pi(2 + 1/\sqrt{2}).$$

Combining these inequalities with Lemma 7.4, we obtain

$$|S_r| < 3.3(11/\sqrt{t})^r$$

for $t \geq 37$. Note also that since $\beta^{(3)}\beta^{(4)} = -2t$, we have

$$2tQ_r + \beta^{(4)}P_r = -\beta^{(4)}S_r.$$

With these estimates, Lemma 7.6 gives the following.

LEMMA 7.7. *Suppose that $t \geq 37$. Define*

$$\kappa = \frac{\log(22\sqrt{t})}{\log(\sqrt{t}/11)}.$$

For $j = 3, 4$ and any rational integers p and q, we have

$$|p - \beta^{(j)} q| > \frac{1}{c_j |q|^{\kappa}}$$

for $|q| \geq 1$, where

$$c_3 = 147.84\sqrt{t}\,(0.6\sqrt{t})^{\kappa}, \qquad c_4 = 4598\sqrt{t}\,(0.44t)^{\kappa}.$$

Proof. In each case we will apply Lemmas 7.5 and 7.6. First notice that $P_r Q_{r+1} - P_{r+1} Q_r$ is a nonzero multiple of $A_{r+1}(0)B_r(0) - A_r(0)B_{r+1}(0)$. Applying Lemma 7.5, with $a = d = 1$, $b = c = 0$ and $x = 0$, we see that $P_r Q_{r+1} \neq P_{r+1} Q_r$. For $\beta^{(3)}$, we put $p_r = P_r$ and $q_r = Q_r$, and apply Lemma 7.6 with $k_0 = 3.36$, $l_0 = 3.3$, $E = \sqrt{t}/11$ and $Q = 22\sqrt{t}$. For $\beta^{(4)}$, we take advantage of the fact that P_{2r} is divisible by t. In this case, we set $p_r = -2Q_{2r}$, $q_r = P_{2r}/t$, $s_r = S_{2r}\beta^{(4)}/t$, and apply Lemma 7.6 accordingly. Since $-4t-2 < \beta^{(4)} < -4t$, we can put $k_0 = 4.75/\sqrt{t}$, $l_0 = 13.2$, $E = t/11^2 = (\sqrt{t}/11)^2$, and $Q = 484t = (22\sqrt{t})^2$. We see therefore that the value of κ in this case is the same as in the case of $\beta^{(3)}$. ∎

8. Completion of the proof of Theorem 1.1. The goal now is to solve equation (2.1) for all $t > 40\,000$. We remark that the analysis here will use the fact that, by not restricting that a solution (x, y) have the property that x/y is close to $\beta^{(4)}$, as asserted in Lemma 3.1, we can restrict to the case that s in equation (3.3) satisfies the inequality $s < \sqrt{2t}$. This can be seen by considering once again the equation

$$s^2 B^4 - 4tsAB^3 - 12tA^2B^2 + 4rtBA^3 + r^2A^4 = 1$$

appearing in Section 3. Since $rs = 2t$, we define $s_0 = \min(r, s)$, and multiply the above equation through by s_0^2. By defining $(x, y) = (-sB, A)$ if $s_0 = s$ and $(x, y) = (rA, B)$ if $s_0 = r$, one obtains the Thue equation

$$x^4 + 4tx^3y - 12tx^2y^2 - 8t^2xy^3 + 4t^2y^4 = s_0^2,$$

where, as discussed above, s_0 divides $2t$, and also, $s_0 \leq \sqrt{2t}$. It is not difficult to verify that if $s_0 = s$, then the closest root of $p_t(X)$ to x/y is $\beta^{(4)}$, while if $s_0 = r$, then the closest root of $p_t(X)$ to x/y is $\beta^{(3)}$. The consequence of this remark is that one then only needs to solve the Thue inequality

$$|p_t(x, y)| \leq 2t,$$

as opposed to having $4t^2$ on the right-hand side.

We first obtain a lower bound for $|y|$ for any solution to the Thue inequality. We will assume, in the construction of p_t given in Section 3, that $r < \sqrt{2t}$, as the case $s < \sqrt{2t}$ actually gives a larger lower bound for $|y|$. Recall that $y = A$, where

$$X - V_{2k+1} = \sqrt{V_{4k+1}} - V_{2k+1} = 2rA^2.$$

Recall also that $r < 2t$. We make use of the inequality $U_{2k} > (2t)^{k-1}$, which is easily proved by induction. We note that because of the relation $V_{4k+1} = V_{2k+1}^2 + 2tU_{2k}^2$, we can deduce the following expression:

$$\sqrt{V_{4k+1}} - V_{2k+1} = \frac{\sqrt{2t}\,U_{2k}}{\sqrt{V_{2k+1}/\sqrt{2t}\,U_{2k} + 1} + V_{2k+1}/\sqrt{2t}\,U_{2k}}.$$

Therefore,

$$y^2 = A^2 > \frac{\sqrt{2t}\,U_{2k}}{4t} > (\sqrt{2t})^{2k-1}.$$

By the fact that $k > 24$, we deduce that

$$|y| > (2t)^{47/4}.$$

As remarked earlier, the assumption $s < \sqrt{2t}$ implies that x/y is closest to $\beta^{(3)}$. In other words, $|x-\beta^{(3)}y| = \min_{i=1,2,3,4} |x-\beta^{(i)}y|$, and since $|P_t(x,y)| \le 2t$, it follows that $|x - \beta^{(3)}y| < (2t)^{1/4}$. Therefore, as $y > 4$, $x/y > \beta^{(3)} - (2t)^{1/4}/4$, and so

$$|x/y - \beta^{(4)}| > \beta^{(3)} - (2t)^{1/4}/4 - \beta^{(4)} > 4t - (2t)^{1/4}/4 + 3 - 21/(16t) + \cdots .$$

Similarly, for $i = 1, 2$,

$$|x/y - \beta^{(i)}| > \sqrt{2t} - (2t)^{1/4}/4 - 1/2 + \cdots .$$

Therefore, because $t > 40\,000$, it is readily deduced that

$$|x/y - \beta^{(3)}| < \frac{1}{15.9ty^4}.$$

A similar argument for $\beta^{(4)}$ gives

$$|x/y - \beta^{(4)}| < \frac{1}{31.9t^2y^4}.$$

Combining the above upper bound for $|x/y - \beta^{(3)}|$ with the lower bound proved in Lemma 7.7, we find that

$$|y|^{3-\kappa} < \frac{c_3(t)}{15.9t}.$$

For $t > 40\,000$, $\kappa < 2.9$, and we conclude that

$$3t^2 > \frac{147.84\sqrt{t}\,(0.6\sqrt{t})^3}{15.9t} > \frac{147.84\sqrt{t}\,(0.6\sqrt{t})^\kappa}{15.9t} > |y|^{0.1} > (2^{21}t^{23})^{0.1} > 4t^2,$$

which is not possible.

Similarly, for $\beta^{(4)}$, combining the upper and lower bounds for $|x/y-\beta^{(4)}|$ gives

$$|y|^{3-\kappa} < \frac{c_3(t)}{31.9t^2}.$$

Again since $t > 40\,000$, we have that $\kappa < 2.9$, and we therefore conclude that

$$13t^{1.5} > \frac{4598\sqrt{t}\,(0.44t)^3}{31.9t^2} > \frac{4598\sqrt{t}\,(0.44t)^\kappa}{31.9t^2} > |y|^{0.1} > (2^{21}t^{23})^{0.1} > 4t^2,$$

which is not possible for $t > 10$. ∎

Acknowledgements. The authors gratefully acknowledge the help in computation from the LORIA research group. The first author would like to thank Michael Bennett for his guidance and support. The third author gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada.

References

[1] S. Akhtari, *The equation* $aX^4 - bY^2 = 1$, submitted.
[2] M. A. Bennett, *On the representation of unity by binary cubic forms*, Trans. Amer. Math. Soc. 353 (2001), 1507–1534.
[3] M. A. Bennett and P. G. Walsh, *The Diophantine equation* $b^2X^4 - dY^2 = 1$, Proc. Amer. Math. Soc. 127 (1999), 3481–3491.
[4] F. Beukers and Sz. Tengely, *An implementation of Runge's method for Diophantine equations*, preprint, 2005.
[5] J. H. Chen and P. M. Voutier, *A complete solution of the Diophantine equation* $x^2 + 1 = dy^4$ *and a related family of quartic Thue equations*, J. Number Theory 62 (1997), 71–99.
[6] G. V. Chudnovsky, *On the method of Thue–Siegel*, Ann. of Math. 117 (1983), 325–382.
[7] J.-H. Evertse, *On the representation of positive integers by binary cubic forms of positive discriminant*, Invent. Math. 73 (1983), 117–138.
[8] C. Heuberger, A. Togbé and V. Ziegler, *Automatic solutions of families of Thue equations and an example of degree* 8, J. Symbolic Comput. 38 (2004), 1145–1163.
[9] D. H. Lehmer, *An extended theory of Lucas' functions*, Ann. of Math. 31 (1930), 419–448.
[10] G. Lettl, A. Pethő and P. M. Voutier, *Simple families of Thue inequalities*, Trans. Amer. Math. Soc. 351 (1999), 1871–1894.
[11] W. Ljunggren, *Zur Theorie der Gleichung* $x^2 + 1 = Dy^4$, Avh. Norsk. Vid. Akad. Oslo, 1942, no. 5, 27 pp.
[12] —, *Über die Gleichung* $x^4 - Dy^2 = 1$, Arch. Math. Naturvid. 45 (1942), no. 5, 61–70.
[13] —, *Einige Eigenschaften der Einheiten reeller quadratischer und rein-biquadratischer Zahlkörper usw.*, Skr. Norske Vid.-Akad. 1936, no. 12, 73 pp.
[14] —, *Ein Satz über die diophantische Gleichung* $Ax^2 - By^4 = C$ $(C = 1, 2, 4)$, Tolfte Skand. Matematikerkongressen (Lund, 1953), Lunds Univ. Mat. Inst., Lund, 1954, 188–194.
[15] —, *On the Diophantine equation* $Ax^4 - By^2 = C$ $(C = 1, 4)$, Math. Scand. 21 (1967), 149–158.
[16] K. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, Belmont, CA, 1981.

[17] A. Thue, *Ein Fundamentaltheorem zur Bestimmung von Annäherungswerten aller Wurzeln gewisser ganzer Funktionen*, J. Reine Angew. Math. 138 (1910), 96–108.

[18] A. Togbé, P. M. Voutier and P. G. Walsh, *Solving a family of Thue equations with an application to the equation $x^2 - Dy^4 = 1$*, Acta Arith. 120 (2005), 39–58.

[19] P. G. Walsh, *On a family of quartic equations and a Diophantine problem of Martin Gardner*, Glas. Mat. 41 (2006), 217–221.

Department of Mathematics
University of British Columbia
Vancouver, BC, Canada V6T 1Z2
E-mail: akhtari@math.ubc.ca

Department of Mathematics
Purdue University North Central
1401 S. U.S. 421
Westville, IN 46391, U.S.A.
E-mail: atogbe@pnc.edu

Department of Mathematics
University of Ottawa
585 King Edward St.
Ottawa, ON, Canada K1N 6N5
E-mail: gwalsh@uottawa.ca

Received on 29.3.2007
and in revised form on 29.10.2007 (5420)