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On p-adic Siegel–Eisenstein series of weight k

by

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1. Introduction. In [3], Katsurada and Nagaoka introduced a p-adic Siegel–Eisenstein series $\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)}$ of weight k. In the present paper, we show that this series is a Siegel–Eisenstein series of degree two, weight k and level p. As a corollary it is a modular form. We also present a simple formula for the Fourier coefficients of a Siegel–Eisenstein series of degree two and prime levels.

Let $E^{(2)}_k$ be the Siegel–Eisenstein series of degree two, weight k and level one,

$$E^{(2)}_k(Z) = \sum_{\{C,D\}} \det(CZ+D)^{-k}, \qquad Z \in H_2,$$

where the sum is taken over all pairs $\{C, D\}$ which occur as the second matrix row of representatives of $\Gamma^{(2)}_\infty \backslash \mathrm{Sp}_2(\mathbb{Z})$ with the standard notations and $H_2 = \{Z = {}^tZ \in M_2(\mathbb{C});\ \Im Z > O\}$ is the Siegel upper half-space of degree two. This has a Fourier expansion with respect to $e(\mathrm{tr}(TZ))$ indexed by $T \in L_2$, the set of all half-integral positive semi-definite symmetric matrices of size two. Here $e(x) = e^{2\pi i x}$ as usual. Take a prime p and a natural number k such that $p > 2k$ and $k \equiv (p-1)/2 \pmod 2$. Put $k_m = k + p^{m-1}(p-1)/2$. If the p-adic convergence

$$\lim_{m\to\infty} \Big\{ \sum_{T \in L_2} A_m(T) e(\mathrm{tr}(TZ)) \Big\} = \sum_{T\in L_2} B(T) e(\mathrm{tr}(TZ)) \qquad (p\text{-adically})$$

is understood as

$$\inf_{T\in L_2} \{\mathrm{ord}_p(B(T) - A_m(T))\} \to \infty \qquad \text{as } m \to \infty,$$

then Katsurada–Nagaoka [3] defined a p-adic Siegel–Eisenstein series by

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$$\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)} = \lim_{m\to\infty} \frac{-B_{k_m}}{2k_m} E^{(2)}_{k_m} \quad (p\text{-adically}).$$

In this paper, we give a description of this p-adic Siegel–Eisenstein series as a Siegel–Eisenstein series of degree two, weight k and level p.

Let χ_p be the Legendre symbol. For any integer $k > 3$ such that $k \equiv (p-1)/2 \pmod 2$, a Siegel–Eisenstein series $E^{(2)}_{k,\chi_p}$ of degree two, weight k and character χ_p on $\Gamma^{(2)}_0(p)$ is defined in the standard way as

$$E^{(2)}_{k,\chi_p}(Z) = \sum_{\{C,D\}} \chi_p(\det D)\det(CZ+D)^{-k}, \qquad Z \in H_2,$$

where the sum is taken over all pairs $\{C,D\}$ which occur as the second matrix row of representatives of $\Gamma^{(2)}_\infty \backslash \Gamma^{(2)}_0(p)$,

$$\Gamma^{(2)}_0(p) = \{\gamma \in \mathrm{Sp}_2(\mathbb{Z});\, C \equiv O \pmod p\}, \qquad \Gamma^{(2)}_\infty = \{\gamma \in \mathrm{Sp}_2(\mathbb{Z});\, C = O\}.$$

Let $F^{(2)}_{k,\chi_p}$ be the twist of $E^{(2)}_{k,\chi_p}$,

$$F^{(2)}_{k,\chi_p}(Z) = p^{-k}\det Z^{-k} E^{(2)}_{k,\chi_p}(-(pZ)^{-1}).$$

Then $E^{(2)}_{k,\chi_p}, F^{(2)}_{k,\chi_p} \in M_k(\Gamma^{(2)}_0(p), \chi_p)$, the space of all Siegel modular forms of degree two, weight k and character χ_p on $\Gamma^{(2)}_0(p)$. The following identity will be proved in Section 3.

THEOREM 1. *For $k > 3$, one has*

$$\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)} = \frac{-B_{k,\chi_p}}{2k}\left\{E^{(2)}_{k,\chi_p} + (-1)^k \frac{p^{k-2}(1-p)}{p^{2k-3}-1} F^{(2)}_{k,\chi_p}\right\},$$

where B_{k,χ_p} is the kth generalized Bernoulli number.

In [3], some genus theta series were employed rather than $E^{(2)}_{k,\chi_p}$ to show $\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)} \in M_k(\Gamma^{(2)}_0(p), \chi_p)$ and thus various explicit formulas for local densities of quadratic forms [2], [6] were needed. One can think that the use of $E^{(2)}_{k,\chi_p}$ is rather natural. However it could not be used because no explicit formula for its Fourier coefficients was available at that time. Such a formula has now been obtained in [4] and it is crucial to proving Theorem 1.

The proof of Theorem 1 contains a simple formula for the Fourier coefficients of a linear combination of Siegel–Eisenstein series defined by

$$\mathcal{E}^{(2)}_{k,\chi_p} = \frac{-B_{k,\chi_p}}{2k}\{E^{(2)}_{k,\chi_p} + (-1)^k p^{1-k} F^{(2)}_{k,\chi_p}\}.$$

This was introduced in [5] and a formula for its Fourier coefficients was given in case $p \equiv 1 \pmod 4$. Theorem 2 below generalizes it to any odd prime. Recall that the Bernoulli numbers B_m and the generalized Bernoulli numbers $B_{m,\chi}$ are defined by

$$\sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = \frac{te^t}{e^t-1}, \qquad \sum_{m=0}^{\infty} B_{m,\chi} \frac{t^m}{m!} = \sum_{a=1}^{f} \frac{\chi(a)te^{at}}{e^{ft}-1}$$

for any primitive character χ of conductor f.

THEOREM 2. *The Fourier expansion of* $\mathcal{E}^{(2)}_{k,\chi_p}$ *is given by*

$$\mathcal{E}^{(2)}_{k,\chi_p}(Z) = \sum_{\substack{T\in L_2\\ p\mid\det 2T}} \mathcal{A}_k(T)e(\operatorname{tr}(TZ)),$$

where the summation extends over all $T \in L_2$ (*the set of all half-integral positive semi-definite symmetric matrices of size two*) *such that* $\det 2T$ *is divisible by* p, *and the* $\mathcal{A}_k(T)$ *are given as follows. For* T *such that* $\operatorname{rk} T = 1$, *we have* $\mathcal{A}_k(T) = \sum_{d\mid e(T)} \chi_p(d)d^{k-1}$ *and* $\mathcal{A}_k(O) = -B_{k,\chi_p}/(2k)$. *For* $T > O$ *such that* $\det 2T$ *is divisible by* p, *we have*

$$\mathcal{A}_k(T) = \frac{2B_{k-1,\chi_{D_K}}}{B_{2k-2}} \sum_{d\mid e(T)} \chi_p(d)d^{k-1} \sum_{a\mid f/d} \mu(a)\chi_{D_K}(a)a^{k-2}\sigma_{2k-3}\left(\frac{f/d}{a}\right),$$

where $B_{k-1,\chi}$ *and* B_{2k-2} *are the Bernoulli numbers*, D_K *is the discriminant of* $K = \mathbb{Q}(\sqrt{(-\det 2T)/p^*})$ *with* $p^* = (-1)^{(p-1)/2}p$, *the natural number* f *is defined by* $-\det 2T = p^* D_K f^2$, χ_{D_K} *is the Kronecker symbol of* K, $e(T)$ *is the content of* T, $\sigma_s(n) = \sum_{d\mid n} d^s$ *and* μ *is the Möbius function.*

It seems interesting to compare this formula with [1, Corollary 2, p. 80].

2. Fourier coefficients of Siegel–Eisenstein series. In this section we prove Theorem 2. The proof contains formulas needed to show Theorem 1. We sketch it only to record the results, as it is similar to that given in [5].

The following two propositions give explicit forms for the Fourier coefficients of the Siegel–Eisenstein series $E^{(2)}_{k,\chi_p}$ and $F^{(2)}_{k,\chi_p}$. The formula for $F^{(2)}_{k,\chi_p}$ follows from an explicit formula for the Siegel series. See [3, Proposition 2.3, p. 104] for example.

PROPOSITION 1. *The Fourier expansion of* $F^{(2)}_{k,\chi_p}$ *is given by*

$$\frac{p^k\Gamma(k)L(k,\chi_p)}{(2\pi i)^k} F^{(2)}_{k,\chi_p}(Z)$$
$$= \sum_{T\in L_2,\, T>O} \left\{\sum_{d\mid e(T)} \chi_p(d)d^{k-1}e^0_{\chi_p}\left(\frac{-\det 2T}{d^2}\right)\right\} e(\operatorname{tr}(TZ)),$$

$$e^0_{\chi_p}(D) = (-1)^k \alpha_k |D|^{k-3/2} \frac{L(k-1, \chi_{D_K}\chi_p)}{L(2k-2, \chi_p^2)} \Upsilon^{k-1}_{D_K,\chi_p}(f), \tag{1}$$

where $L(s,\chi)$ *is the Dirichlet* L*-function,* $\Gamma(s)$ *is the gamma function,* D_K *is the discriminant of* $K = \mathbb{Q}(\sqrt{D})$*, the natural number* f *is defined by* $D = D_K f^2$*,* χ_{D_K} *is the Kronecker symbol of* K*,* $\alpha_k = 2^{2-k} i^{-k} \pi^{k-1/2}/\Gamma(k-1/2)$*,*

$$\Upsilon^s_{D_K,\chi}(f) = \sum_{d|f} \mu(d)\chi_{D_K}(d)\chi(d)d^{-s}\sigma_{1-2s,\chi^2}(f/d), \tag{2}$$

$\sigma_{s,\chi^2}(f) = \sum_{d|f} \chi^2(d)d^s$ *and* μ *is the Möbius function.*

Theorem 1 in [4] yields the following result.

Proposition 2. *The Fourier expansion of* $E^{(2)}_{k,\chi_p}$ *is given by*

$$\frac{p^k \Gamma(k) L(k,\chi_p)}{\tau_p(\chi_p)(-2\pi i)^k} E^{(2)}_{k,\chi_p}\left(\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}\right) = \frac{-B_{k,\chi_p}}{2k}$$
$$+ \sum_{\substack{m\geq 0,\, n\geq 0,\, r\in\mathbb{Z} \\ 4mn\geq r^2,\, (n,r,m)\neq(0,0,0)}} \left(\sum_{d|(n,r,m)} \chi_p(d) d^{k-1} e^\infty_{\chi_p}\left(\frac{r^2-4mn}{d^2}\right)\right) e(n\tau + rz + m\tau')$$

with $\tau_p(\chi_p) = \sum_{r=1}^p \chi_p(r)e^{2\pi i r/p}$*,* $e^\infty_{\chi_p}(0) = 1$ *and* $e^\infty_{\chi_p}(D)$ *given for* $D<0$ *by*

$$e^\infty_{\chi_p}(D) = (-1)^k \alpha_k |D|^{k-3/2} \frac{L(k-1, \chi_{D_K}\chi_p)}{L(2k-2, \chi_p^2)} \Upsilon^{k-1}_{D_K,\chi_p}(f) \times \sum_{e\geq 1} p^{-(k-1/2)e} \varepsilon^3_{p^e} C^\infty_{\chi_p,p}(D, p^e), \tag{3}$$

where D_K*,* f*,* α_k *and* $\Upsilon^{k-1}_{D_K,\chi_p}(f)$ *are as in Proposition* 1*,* $\varepsilon_d = 1$ *or* i *according as* $d \equiv 1 \pmod 4$ *or* $3 \pmod 4$*, and*

$$C^\infty_{\chi_p,p}(D,p^e) = \sum_{d\in(\mathbb{Z}/p^e\mathbb{Z})^*} \chi_p(d)^{e+1} e_{p^e}(dD), \qquad e_m(x) = e^{2\pi i x/m}.$$

Put $p^* = (-1)^{(p-1)/2}p$ and take $\delta_p \in \{0,1\}$ such that $k \equiv \delta_p \pmod 2$. Hence p^* is a prime discriminant. A key proposition to prove Theorem 2 is

Proposition 3. *For* $D<0$ *such that* $D \equiv 0,1 \pmod 4$ *we have*

$$e^\infty_{\chi_p}(D) + (-1)^k i^{\delta_p} p^{-(k-1/2)} e^0_{\chi_p}(D) = i^{\delta_p} \alpha_k |D|^{k-3/2} p^{-(k-3/2)} \frac{L_{D/p^*}(k-1)}{\zeta(2k-2)}.$$

Here α_k *is as in Proposition* 1 *and* $L_D(s) = L(s,\chi_{D_K})\Upsilon^s_{D_K,\chi_0}(f)$*, where we use the symbol* (2) *for the principal character* χ_0 ($\chi_0(\mathbb{Z}) = \{1\}$)*, the natural number* f *is defined by* $D = D_K f^2$ *with the discriminant* D_K *of* $K = \mathbb{Q}(\sqrt{D})$*,* χ_{D_K} *is the Kronecker symbol of* K*, and we set* $L_{D/p^*}(s) = 0$ *if* D *is not divisible by* p*.*

We sketch the proof. It follows from (1) and (3) that for $D = D_K f^2$,

$$\begin{aligned} &e^{\infty}_{\chi_p}(D) + (-1)^k i^{\delta_p} p^{-(k-1/2)} e^{0}_{\chi_p}(D) \\ &\qquad = (-1)^k \alpha_k |D|^{k-3/2} \frac{L(k-1, \chi_{D_K}\chi_p)}{L(2k-2, \chi_p^2)} \Upsilon^{k-1}_{D_K,\chi_p}(f) \\ &\qquad\quad \times \Big\{ \sum_{e\geq 1} p^{-(k-1/2)e} \varepsilon^3_{p^e} C^{\infty}_{\chi_p,p}(D, p^e) + (-1)^k i^{\delta_p} p^{-(k-1/2)} \Big\}. \end{aligned}$$

Then $C^{\infty}_{\chi_p,p}(D, p^e)$ was evaluated in Proposition 4 of [4]:

LEMMA 1.

(I) *If D is not divisible by p, then*

$$C^{\infty}_{\chi_p,p}(D, p^e) = \begin{cases} -1, & e = 1, \\ 0, & e \geq 2. \end{cases}$$

(II) *If D is divisible by p, let p^m be the exact power of p dividing D. Then*

(a) *for $e \leq m$,*

$$C^{\infty}_{\chi_p,p}(D, p^e) = \begin{cases} p^{e-1}(p-1), & e \text{ is odd}, \\ 0, & e \text{ is even}, \end{cases}$$

(b) *for $e = m+1$,*

$$C^{\infty}_{\chi_p,p}(D, p^e) = \begin{cases} -p^m, & m \text{ is even}, \\ \chi_p(D/p^m) i^{\delta_p} p^{m+1/2}, & m \text{ is odd}, \end{cases}$$

(c) *for $e \geq m+2$, $C^{\infty}_{\chi_p,p}(D, p^e) = 0$.*

From this, we have the following two lemmas.

LEMMA 2. *For $D = D_K f^2$, suppose that D_K is divisible by p. Then*

$$\begin{aligned} &e^{\infty}_{\chi_p}(D) + (-1)^k i^{\delta_p} p^{-(k-1/2)} e^{0}_{\chi_p}(D) \\ &= i^{\delta_p} \alpha_k |D|^{k-3/2} \frac{L(k-1, \chi_{D_K/p^*})}{\zeta(2k-2)} \Upsilon^{k-1}_{D_K,\chi_p}(f) \\ &\quad \times p^{-(k-3/2)} \frac{1 - p^{(3-2k)(m_p+1)} - \chi_{D_K/p^*}(p) p^{-(k-1)} (1 - p^{(3-2k)m_p})}{1 - p^{3-2k}}, \end{aligned}$$

where χ_{D_K/p^*} is the Kronecker symbol $\left(\frac{D_K/p^*}{}\right)$ and p^{m_p} is the exact power of p dividing f.

LEMMA 3. *For $D = D_K f^2$, suppose that D_K is not divisible by p. Let p^{m_p} be the exact power of p dividing f. Then*

$$e^{\infty}_{\chi_p}(D) + (-1)^k i^{\delta_p} p^{-(k-1/2)} e^{0}_{\chi_p}(D) = i^{\delta_p}\alpha_k |D|^{k-3/2} \frac{L(k-1,\chi_{D_K p^*})}{\zeta(2k-2)} \Upsilon^{k-1}_{D_K,\chi_p}(f) p^{-(k-3/2)} \frac{1-p^{(3-2k)m_p}}{1-p^{3-2k}},$$

where $\chi_{D_K p^*}$ is the Kronecker symbol $\left(\frac{D_K p^*}{}\right)$.

Proposition 3 follows from these lemmas in the same way as in [5, Section 2]. Propositions 1–3 imply Theorem 2.

3. $\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)}$ is a Siegel–Eisenstein series.

Put

$$\gamma_{k,p} = \frac{B_{k,\chi_p}}{2k} \frac{1-p^{2-2k}}{p^{k-2}(1-p^{3-2k})}.$$

The next proposition implies Theorem 1.

PROPOSITION 4. *For $k > 3$, one has*

$$\widetilde{G}^{(2)}_{(k,(p+2k-1)/2)} = \mathcal{E}^{(2)}_{k,\chi_p} + (-1)^k \gamma_{k,p} F^{(2)}_{k,\chi_p}.$$

Proof. Applying the Siegel operator shows the coincidence of the constant terms and rank one parts. In fact both sides have the same image

$$-\frac{B_{k,\chi_p}}{2k} + \sum_{n\geq 1}\sum_{d|n} \chi_p(d) d^{k-1} e(n\tau).$$

To show the coincidence of the rank two parts, we denote by $a(T)$ the Tth Fourier coefficient of the left-hand side, by $b(T)$ that of the right-hand side and by $C(T)$ that of $F^{(2)}_{k,\chi_p}$. Hence $b(T) = \mathcal{A}_k(T) + (-1)^k \gamma_{k,p} C(T)$. An explicit formula for $a(T)$ is given in [3, (6.4), p. 113], and $\mathcal{A}_k(T)$ and $C(T)$ follow from Propositions 1 and 2. By these results, $b(T)$ has the form

$$b(T) = \sum_{d|e(T)} \chi_p(d) d^{k-1} c\left(\frac{-\det 2T}{d^2}\right), \tag{4}$$

where $e(T)$ is the content of T and $c(D)$ for $D < 0$ is given by

$$c(D) = \{e^{\infty}_{\chi_p}(D) + (-1)^k i^{\delta_p} p^{-(k-1/2)} e^{0}_{\chi_p}(D)\} - (-1)^k i^{\delta_p} p^{3/2-k} \frac{1-p^{2-2k}}{1-p^{3-2k}} e^{0}_{\chi_p}(D)$$

with the notations of Section 2. Let D_K be the discriminant of $K = \mathbb{Q}(\sqrt{D})$ and f a natural number defined by $D = D_K f^2$. Put $m_p = \operatorname{ord}_p f$. Then Lemmas 2 and 3 imply the following simple expression for $c(D)$, depending on divisibility of D_K by p. If D_K is not divisible by p, then

$$c(D) = \frac{2B_{k-1,\chi_{D_K p^*}}}{B_{2k-2}} f^{2k-3} \Upsilon^{k-1}_{D_K,\chi_p}(f) \frac{p^{(3-2k)m_p}}{1-p^{2k-3}}.$$

If D_K is divisible by p, then

$$c(D) = \frac{2B_{k-1,\chi_{D_K/p^*}}}{B_{2k-2}} f^{2k-3} \Upsilon^{k-1}_{D_K,\chi_p}(f) p^{(3-2k)m_p} \frac{1-\chi_{D_K/p^*}(p)p^{k-2}}{1-p^{2k-3}}.$$

It is easy to see that $a(T)$ given in [3, (6.4), p. 113] has the same form as in (4). Hence the desired result follows.

This proposition implies Theorem 1 and completes the proof.

References

[1] M. Eichler and D. Zagier, *The Theory of Jacobi Forms*, Progr. Math. 55, Birkhäuser Boston, Boston, MA, 1985.

[2] H. Katsurada, *An explicit formula for Siegel series*, Amer. J. Math. 121 (1999), 415–452.

[3] H. Katsurada and S. Nagaoka, *On some p-adic properties of Siegel–Eisenstein series*, J. Number Theory 104 (2004), 100–117.

[4] Y. Mizuno, *An explicit arithmetic formula for Fourier coefficients of Siegel–Eisenstein series of degree two with square free odd levels*, preprint.

[5] —, *A p-adic limit of Siegel–Eisenstein series of prime level q*, Int. J. Number Theory, to appear.

[6] T. Yang, *An explicit formula for local densities of quadratic forms*, J. Number Theory 72 (1998), 309–356.

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