ACTA ARITHMETICA
142.2 (2010)

The density of S-integral points in projective space with respect to a quadric

by

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1. Introduction. Let $S = \{p_1, \dots, p_m, \infty\}$ be a finite set of places of $\mathbb{Q}$, including the archimedean one. The ring of S-integers is given by

$$\mathcal{O}_S = \{y \in \mathbb{Q} : |y|_p \le 1 \ \forall p \notin S\}.$$

Its multiplicative subgroup $\mathcal{O}_S^*$ consists of the S-units, i.e. those elements of $\mathcal{O}_S$ whose p-adic absolute value equals 1 for all $p \notin S$. Given a smooth projective algebraic variety X over $\mathbb{Q}$, and a smooth hypersurface $D \subset X$ defined by a form $F \in \mathbb{Z}[x_1, \dots, x_n]$, we say that $\mathbf{x} = (x_1, \dots, x_n) \in X$ is a (D, S)-*integral point*, or an *S-integral point with respect to* D, if $F(\mathbf{x}) \not\equiv 0 \pmod{p}$ for all $p \notin S$. We are interested in the asymptotic behaviour of the counting function

$$\mathcal{N}(P) = \#\{(\mathbf{x}, Y) \in (\mathbb{Z}^n_{\mathrm{prim}} \cap PB) \times \mathcal{O}_S^* : F(\mathbf{x}) = Y\}$$

for (D, S)-integral points of bounded height in the case where $X = \mathbb{P}^{n-1}$ and F is a quadratic form. Here P is a real parameter that tends to infinity, points in projective space are represented by primitive integral tuples, and B is some n-dimensional hyperrectangle in $\mathbb{R}^n$ centred at the origin. In this form the problem corresponds to the degree two case of a question raised by Tschinkel [5, Problem 5.6].

Before we can give a precise statement of our main result, it is necessary to introduce some notation. We write $\mathcal{F}$ for the matrix of F given by $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathcal{F} \mathbf{x}$, and let $\mathcal{M}$ be a real orthogonal matrix that diagonalises $\mathcal{F}$. Accordingly, we choose B such that the edges of $\mathcal{M}^T B$ are parallel to the coordinate axes. The set Λ shall consist of all primes p such that $p \mid 2 \det \mathcal{F}$ but $p \notin S$. We let Δ be the set of all $m+1$-tuples $\boldsymbol{\delta}$ with entries in $\{0, 1\}$, and write $\mathbf{p}^{\boldsymbol{\delta}} = (-1)^{\delta_0} p_1^{\delta_1} \cdots p_m^{\delta_m}$ for short. For a given prime $p \in \Lambda \cup S$ and

2010 *Mathematics Subject Classification*: Primary 14G99; Secondary 11D45, 11D72, 14J99, 14N25.

Key words and phrases: S-integral points, quadratic hypersurface, counting problem.

DOI: 10.4064/aa142-2-4

$\boldsymbol{\delta} \in \Delta$ we define t by $p^t \parallel 2\mathbf{p}^{\boldsymbol{\delta}}$. Finally, recall that for fixed F a place is called *(an)isotropic* if F does (not) represent zero non-trivially over the induced completion of $\mathbb{Q}$.

THEOREM 1.1. *Let F, B and S be given. Assume that F is indefinite and $n \geq 4$. Then $\mathcal{N}(P) = 0$ if for all $\boldsymbol{\delta} \in \Delta$ there exists $p \in \Lambda$ such that*

$$F(\mathbf{x}) \equiv \mathbf{p}^{\boldsymbol{\delta}} \pmod{p^{2t+1}} \tag{1}$$

has no solution. If there exists $\boldsymbol{\delta} \in \Delta$ such that the congruence is soluble for all $p \in \Lambda$, then

$$\mathcal{N}(P) \sim cP^{n-2}(\log P)^l \quad \text{as } P \to \infty, \tag{2}$$

where c is some positive constant and l is the number of non-archimedean isotropic places in S.

We attack the problem of counting the zeros of $Q(\mathbf{x}, Y) = F(\mathbf{x}) - Y$ via the Hardy–Littlewood circle method. For non-zero integers N, it is a classical problem to find the number $R_B(P; N)$ of zeros $\mathbf{x} \in PB$ of $Q(\mathbf{x}, N)$. Indeed, the theorem below can be proved using Kloosterman's refinement of the circle method (see e.g. [4]).

THEOREM 1.2. *For any $C > 0$ one has*

$$R_B(P; N) = I_B(P; N)\mathfrak{S}(N) + O(P^{n-2}(\log P)^{-C}),$$

where

$$I_B(P; N) = P^{n-2} \int_{-\infty}^{\infty} \int_B e(zQ(\mathbf{x}, N/P^2))\, d\mathbf{x}\, dz \tag{3}$$

and

$$\mathfrak{S}(N) = \sum_{q=1}^{\infty} q^{-n} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \sum_{\mathbf{x} \,(\mathrm{mod}\, q)} e_q(aQ(\mathbf{x}, N))$$

are the usual singular integral and singular series, respectively. The singular integral is convergent for all $N \in \mathbb{Z}$, and the singular series converges absolutely.

Since numbers representable by $F(\mathbf{x})$ with $\mathbf{x} \in PB$ have size $O(P^2)$ and there are $\asymp (\log P)^m$ such S-units (by the integrality of F we may assume henceforth that Y is restricted to values in $\mathcal{O}_S^* \cap \mathbb{Z}$), we obtain

$$\mathcal{N}(P) = \sum_{Y \ll P^2} I_B(P; Y)\mathfrak{S}(Y) + O(P^{n-2}(\log P)^{-1}). \tag{4}$$

This leads us to analyse the behaviour of the singular integral and series with respect to changing Y. The key observation is that if $p \in S$ and F is anisotropic over $\mathbb{Q}_p$, then the local density for p tends to zero as the power p^r dividing Y increases. Conversely, if F is isotropic, i.e. represents 0, over $\mathbb{Q}_p$, then the corresponding local density converges to a positive constant as r grows. This is not surprising since as the power of p dividing Y increases, the p-adic size of Y approaches zero.

The proof of Theorem 1.1 is completed in Section 4. By summing over the S-units as indicated above, the factors of $\log P$ in (2) are generated. It should be noted that $l = m$ if $n \geq 5$ since any quadratic form in at least five variables is isotropic over all $\mathbb{Q}_p$. If $n = 4$ this is no longer true, and due to vanishing local densities any number of factors of $\log P$ may fail to occur. The exact shape of c will become apparent and we obtain an error $O(P^{n-2}(\log P)^{l-1} \log\log P)$ in the asymptotic formula.

One should contrast Theorem 1.1 with the case of (positive) definite F (and $k < 0$). It is well-known that the total number of integral representations of a positive integer N by F is given by

$$(5) \qquad \frac{(2\pi)^{n/2}}{\Gamma(n/2)\sqrt{\det \mathcal{F}}}\,\mathfrak{S}(N) N^{n/2-1} + O(N^{(n-1)/4+\varepsilon})$$

as $N \to \infty$. On using this formula in Section 4, the reader can convince themselves that now $\mathcal{N}(P) \asymp P^{n-2}(\log P)^{l-1}$ if (1) has a solution for all $p \in \Lambda$ and $l \geq 1$. If $l = 0$ however, the singular series tends to zero. In fact, it dwindles rapidly enough to neutralise the growth of the singular integral, leaving us with an error term only for $\mathcal{N}(P)$. Indeed, since for anisotropic p the form F does not represent the integer N if the greatest power of p dividing N is $\gg 1$, one easily sees that $\mathcal{N}(P)$ is constant for all sufficiently large P. Curiously, this constant may be zero as is illustrated by the example

$$3(x_1^2 + x_2^2) + 14(x_3^2 + x_4^2) - 7^r = 0.$$

There are no integral solutions to this equation although (1) is soluble for $p = 2, 3$ when $r = 0$.

Lastly, we should note that almost everything done below goes through as soon as $n \geq 3$. Furthermore, at least since the work by Duke [2] and Iwaniec [3] on modular forms of half-integral weight it has been conjectured that Theorem 1.2 also holds when $n = 3$. We will see in Section 3 that in this case the singular series is essentially a Dirichlet L-function evaluated at $s = 1$, which diverges if and only if the character involved is principal. However, the singular series is originally a finite sum truncated at P, so that a simple pole at $s = 1$ would lead to an extra factor of $\log P$ in (2).

Conjecture 1.3. *Let F, B and S be given. Assume that F is indefinite and $n = 3$. Then $\mathcal{N}(P) = 0$ if for all $\boldsymbol{\delta} \in \Delta$ there exists $p \in \Lambda \cup S$ such that* (1) *has no solution. If there exists $\boldsymbol{\delta} \in \Delta$ such that* (1) *is soluble for all $p \in \Lambda \cup S$, then*

$$\mathcal{N}(P) \sim cP^{n-2}(\log P)^l, \qquad c > 0,$$

if $-2\mathbf{p}^{\boldsymbol{\delta}}\det F$ *is not a perfect square for all such* $\boldsymbol{\delta}$, *and*

$$\mathcal{N}(P) \sim cP^{n-2}(\log P)^{m+1}, \qquad c > 0,$$

otherwise.

Subject to the circle method succeeding, what we have said above in principle also applies to higher degree forms. Our observations highlight the dependence of the asymptotic behaviour of $\mathcal{N}(P)$ on the arithmetic of F.

Throughout we shall use ε to represent any positive real number. No importance is attached to its exact size. Indeed, we allow different instances of ε to take different values. Implicit constants in big-O and $\ll$ notation may depend upon F, B, S and ε.

2. The singular integral. Our first result shows that $I_B(P;Y)$ is well-approximated by $I_B(P;0)$ whenever $Y = o(P)$, and allows us to infer that the singular integral is of order of magnitude $O(P^{n-2})$ for all $Y \ll P^2$.

LEMMA 2.1. *We have*

$$I_B(P;Y) = I_B(P;0) + O(P^{n-2}((|Y|/P^2)^{(n-2)/5} + (|Y|/P^2)^{1/5})).$$

Proof. On writing

$$I_B(P;Y) - I_B(P;0) = P^{n-2}\int_{-\infty}^{\infty}\int_B e(zF(\mathbf{x}))(e(zY)-1)\,d\mathbf{x}\,dz$$

we see that the part of the integration for which $|z| < R := (|Y|/P^2)^{-2/5}$ is

$$\ll \int_{|z|<R} |e(z(Y/P^2)) - 1|\,dz \ll \frac{|Y|}{P^2}\int_{|z|<R} |z|\,dz \ll (|Y|/P^2)^{1/5}.$$

Next we note that

$$\int_a^b e(zv^2)\,dv \ll |z|^{-1/2}$$

uniformly in $a,b \in \mathbb{R}$. This follows from observing that the integral over the range $[-|z|^{-1/2}, |z|^{-1/2}] \cap [a,b]$ is trivially $\ll |z|^{-1/2}$, whence we may assume without loss of generality that $|z|^{-1/2} \le a < b$. We then have

$$\int_a^b e(zv^2)\,dv = \int_a^b 4\pi izv\,e(zv^2)\,\frac{1}{4\pi izv}\,dv \ll |z|^{-1/2}$$

by partial integration. Therefore the contribution from $|z| > R$ to $I_B(P;Y)$ is, after diagonalising F via the substitution $\mathbf{x} \mapsto \mathcal{M}\mathbf{x}$,

$$\ll (|Y|/P^2)^{(n-2)/5}.$$

The same applies for $I_B(P;0)$. ∎

The lemma above is complemented by the one below. Together they show that the singular integral is positive if $Y = o(P)$ and P is sufficiently large.

LEMMA 2.2. *Let B be centred at a non-singular zero of F and sufficiently small. Then*

$$I_B(P;0) = H_B P^{n-2}$$

for some positive constant H_B.

Proof. This can be proved by an application of the implicit function theorem followed by an instance of Fourier inversion (compare for example Chapter 16 of [1]). In this context one only needs to note that any non-trivial zero $\mathbf{x} \in B$ of F is necessarily non-singular. ∎

3. The singular series. We begin our investigation of the singular series by performing the usual analysis which shows that it has an Euler product whose factors, the local densities

$$\hat{\sigma}_p(Y) = \sum_{h=0}^{\infty} p^{-hn} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^h} \sum_{\mathbf{x} \,(\mathrm{mod}\, p^h)} e_{p^h}(aQ(\mathbf{x}, Y)),$$

can be expressed via the number of solutions to certain congruences. Subsequently we will see when the singular series vanishes and how it behaves with respect to changing Y.

LEMMA 3.1. *We have*

$$\mathfrak{S}(Y) = \prod_p \hat{\sigma}_p(Y).$$

Proof. For coprime integers r and s it is elementary to verify the multiplicative property

$$S_0(rs, \mathbf{0}) = S_0(r, \mathbf{0}) S_0(s, \mathbf{0}).$$

Together with the absolute convergence of the $\hat{\sigma}_p(Y)$ this proves that the Euler product representation of $\mathfrak{S}(Y)$ is valid. ∎

A standard argument shows that the local densities satisfy

$$\hat{\sigma}_p(Y) = \lim_{l \to \infty} p^{(1-n)l} \# \hat{M}(Y, p^l),$$

where

$$\hat{M}(Y, p^l) = \{\mathbf{x} \ (\mathrm{mod}\ p^l) : F(\mathbf{x}) \equiv Y \ (\mathrm{mod}\ p^l)\}.$$

This implies that the arithmetic functions $\hat{\sigma}_p$, and thus $\mathfrak{S}$, map into the non-negative real numbers.

Any S-unit is of the form $Y = p_1^{r_1} \cdots p_m^{r_m}$. Thus we shall say that Y is *of type* $\boldsymbol{\delta}$ if

$$((\mathrm{sign}(Y) - 1)/2, r_1, \ldots, r_m) \equiv \boldsymbol{\delta} \ (\mathrm{mod}\ 2).$$

For a given prime p we define τ to be the integer that satisfies $p^\tau \parallel 2Y$. Since there is a bijection from $\hat{M}(Y,p^l)$ to $\hat{M}(\mathbf{p}^{\boldsymbol{\delta}}p^{2k},p^l)$, where $k=\lfloor \tau/2\rfloor$ if p is odd and $k=\lfloor(\tau-1)/2\rfloor$ if $p=2$, the cardinality of $\hat{M}(Y,p^l)$ depends only on $\boldsymbol{\delta}$, k and l. We shall make this explicit in our notation by setting

$$M_p(\boldsymbol{\delta},k,l)=\hat{M}(\mathbf{p}^{\boldsymbol{\delta}}p^{2k},p^l) \quad \text{and} \quad \sigma_p(\boldsymbol{\delta},k)=\hat{\sigma}_p(\mathbf{p}^{\boldsymbol{\delta}}p^{2k}).$$

Now Euler's identity $F(\mathbf{x})=\frac{1}{2}\mathbf{x}.\nabla F(\mathbf{x})$ shows that for any $\mathbf{x}\in M_p(\boldsymbol{\delta},k,l)$ with $l>\tau$ we have $p^h \parallel \nabla F(\mathbf{x})$ for some $h\in\{0,\ldots,\tau\}$. Therefore each $\mathbf{x}\in M_p(\boldsymbol{\delta},k,l)$ falls into one of $\min(\tau+1,l+1)$ disjoint sets $M_p^h(\boldsymbol{\delta},k,l)$ according to which

$$h\in\begin{cases}\{0,\ldots,l-1,\infty\} & \text{if } l\le\tau,\\ \{0,\ldots,\tau\} & \text{if } l>\tau,\end{cases}$$

satisfies $p^h \parallel \nabla F(\mathbf{x})$.

LEMMA 3.2. *If $h\neq\infty$ and $l\ge 2h+1$, then*

$$\#M_p^h(\boldsymbol{\delta},k,l+1)=p^{n-1}\#M_p^h(\boldsymbol{\delta},k,l).$$

Proof. It is clear that the two sets $M_p^h(\boldsymbol{\delta},k,l+i)$, $i\in\{0,1\}$, can be partitioned into equivalence classes $E_{\mathbf{a}}^i$ according to the reduction $\mathbf{a}$ of their elements mod p^{l-h}. More precisely, as a disjoint union we have

$$M_p^h(\boldsymbol{\delta},k,l+i)=\bigcup_{\mathbf{a}\in A^i}E_{\mathbf{a}}^i$$

where

$$E_{\mathbf{a}}^i=\{\mathbf{x}\in M_p^h(\boldsymbol{\delta},k,l+i):\mathbf{x}\equiv\mathbf{a}\ (\mathrm{mod}\ p^{l-h})\},$$
$$A^i=\{\mathbf{a}\ (\mathrm{mod}\ p^{l-h}):\exists\mathbf{x}\in M_p^h(\boldsymbol{\delta},k,l+i)\text{ such that }\mathbf{a}\equiv\mathbf{x}\ (\mathrm{mod}\ p^{l-h})\}.$$

First we shall show that

$$\#E_{\mathbf{a}}^0=p^{hn}\qquad\forall\mathbf{a}\in A^0. \tag{6}$$

To see this fix an $\mathbf{a}\in A^0$ and choose an element $\mathbf{x}\in E_{\mathbf{a}}^0$. Now let $\mathbf{y}$ be any element $(\mathrm{mod}\,p^l)$ with $\mathbf{y}\equiv\mathbf{a}\ (\mathrm{mod}\ p^{l-h})$. We can uniquely write $\mathbf{y}=\mathbf{x}+p^{l-h}\mathbf{z}$ with $\mathbf{z}\in\{0,\ldots,p^h-1\}^n$. By linearity we have

$$\nabla F(\mathbf{y})=\nabla F(\mathbf{x})+p^{l-h}\nabla F(\mathbf{z}),$$

so that $p^h \parallel \nabla F(\mathbf{x})$ implies

$$p^h \parallel \nabla F(\mathbf{y}). \tag{7}$$

Hence by Taylor-expanding we also have

$$\begin{aligned}F(\mathbf{y})&\equiv F(\mathbf{x})+p^l\mathbf{z}\,\frac{\nabla F(\mathbf{x})}{p^h}+p^{2(l-h)}F(\mathbf{z})\ (\mathrm{mod}\ p^l)\\&\equiv F(\mathbf{x})\equiv\mathbf{p}^{\boldsymbol{\delta}}p^{2k}\ (\mathrm{mod}\ p^l),\end{aligned} \tag{8}$$

which together with (7) shows that $\mathbf{y} \in E^0_{\mathbf{a}}$. Since there are p^{hn} possible $\mathbf{y}$, equation (6) holds.

Next we note that $A^1 \subseteq A^0$. To see this, let $\mathbf{a} \in A^1$. Then there exists an $\mathbf{x} \in M^h_p(\boldsymbol{\delta}, k, l+1)$ such that $\mathbf{x} \equiv \mathbf{a} \pmod{p^{l-h}}$. If we define $\mathbf{y} = \mathbf{x} \pmod{p^l}$ we have $\mathbf{x} = \mathbf{y} + p^l\mathbf{z}$ for some $\mathbf{z} \in \{0, \dots, p-1\}$, whence the linearity of $\nabla F(\mathbf{x})$ gives $p^h \parallel \nabla F(\mathbf{y})$. It is now easily seen that $\mathbf{y} \in M^h_p(\boldsymbol{\delta}, k, l)$ and that $\mathbf{y} \equiv \mathbf{a} \pmod{p^{l-h}}$. Therefore $\mathbf{a} \in A^0$.

Finally we prove that

$$\#E^1_{\mathbf{a}} = p^{hn+n-1} \quad \forall \mathbf{a} \in A^0. \tag{9}$$

Fix an $\mathbf{a} \in A^0$ and consider any element $\mathbf{y} \pmod{p^{l+1}}$ with $\mathbf{y} \equiv \mathbf{a} \pmod{p^{l-h}}$. We uniquely write $\mathbf{y} = \mathbf{a} + p^{l-h}\mathbf{z}$ with $\mathbf{z} \in \{0, \dots, p^{h+1} - 1\}^n$. From (6) it follows that $\mathbf{a} \in E^0_{\mathbf{a}}$, which combined with the linearity of $\nabla F(\mathbf{y})$ shows that $p^h \parallel \nabla F(\mathbf{y})$. Moreover, we can expand

$$F(\mathbf{y}) \equiv F(\mathbf{a}) + p^l\mathbf{z}\,\frac{\nabla F(\mathbf{a})}{p^h} + p^{2(l-h)}F(\mathbf{z}) \pmod{p^{l+1}}.$$

Now $F(\mathbf{a}) = \mathbf{p}^{\boldsymbol{\delta}}p^{2k} + tp^l$ for some $t \in \mathbb{Z}$ since $\mathbf{a} \in M^h_p(\boldsymbol{\delta}, k, l)$, so that

$$F(\mathbf{y}) \equiv \mathbf{p}^{\boldsymbol{\delta}}p^{2k} \pmod{p^{l+1}}$$

if and only if

$$t + \mathbf{z}\,\frac{\nabla F(\mathbf{a})}{p^h} \equiv 0 \pmod{p}.$$

In the last congruence above, $\mathbf{z}$ is a solution if and only if $\mathbf{z} \pmod{p}$ takes one of exactly p^{n-1} values since $p^h \parallel \nabla F(\mathbf{a})$, giving a total of p^{hn+n-1} possible solutions $\mathbf{z}$.

Now equation (9) implicitly gives the inclusion $A^0 \subseteq A^1$, so that

$$\begin{aligned}\#M^h_p(\boldsymbol{\delta}, k, l+1) &= \sum_{\mathbf{a}\in A^1} \#E^1_{\mathbf{a}} = \#A^0 p^{hn+n-1}\\ &= p^{n-1}\#A^0\#E^0_{\mathbf{a}} = p^{n-1}\sum_{\mathbf{a}\in A^0}\#E^0_{\mathbf{a}} = p^{n-1}\#M^h_p(\boldsymbol{\delta}, k, l),\end{aligned}$$

as claimed. ∎

Corollary 3.3. *We have*

$$\sigma_p(\boldsymbol{\delta}, k) = p^{1-n}\sum_{h=0}^{\tau} p^{-2h(n-1)}\#M^h_p(\boldsymbol{\delta}, k, 2h+1).$$

Proof. This follows immediately from the lemma. ∎

If $p \notin S$, then τ is fixed, and the corollary implies that $\hat{\sigma}_p(Y) = \sigma_p(\boldsymbol{\delta}, 0)$ only depends on the type of Y. Furthermore, it transpires that for $p \in \Lambda$ the local density $\sigma_p(\boldsymbol{\delta}, 0)$ is positive if the congruence (1) is soluble. If $p \notin \Lambda \cup S$, then $\tau = 0$ and there is the classical identity

$$\sum_{\mathbf{x}\,(\mathrm{mod}\,p)} e_p(aF(\mathbf{x})) = p^{n/2}\varepsilon_p^n\left(\frac{2^n \det \mathcal{F}}{p}\right), \qquad \varepsilon_p = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 4, \\ i & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

for the quadratic Gauss sum when $(2a \det \mathcal{F}, p) = 1$. Therefore

$$M_p(\boldsymbol{\delta}, 0, 1) = p^{n-1} + p^{n/2-1}\varepsilon_p^n\left(\frac{\det \mathcal{F}}{p}\right)\sum_{a=1}^{p-1}\left(\frac{2a}{p}\right)^n e_p(-a\mathbf{p}^{\boldsymbol{\delta}}),$$

which by Corollary 3.3 yields

$$\sigma_p(\boldsymbol{\delta}, 0) = \begin{cases} 1 - p^{-n/2}\varepsilon_p^n\left(\dfrac{\det \mathcal{F}}{p}\right) & \text{if } n \text{ is even and } \geq 2, \\ 1 + p^{(1-n)/2}\varepsilon_p^{n+1}\left(\dfrac{-2\mathbf{p}^{\boldsymbol{\delta}} \det \mathcal{F}}{p}\right) & \text{if } n \text{ is odd.} \end{cases}$$

This shows that

$$L_{\boldsymbol{\delta}} = \prod_{p \notin S} \hat{\sigma}_p(Y) \tag{10}$$

is a non-negative constant only dependent on the type of Y if $n \geq 4$. It is zero if and only if the congruence (1) does not have a solution for some $p \in \Lambda$. With regard to our conjecture we observe that if $n = 3$, then

$$L_{\boldsymbol{\delta}} = \frac{L(1,\chi)}{L(2,\chi^2)} \prod_{p \notin S} \frac{\hat{\sigma}_p(Y)}{1 + \chi(p)p^{-1}}$$

equally depends only on the type of Y. Here χ is the Dirichlet character given by $\chi(p) = (-2\mathbf{p}^{\boldsymbol{\delta}} \det \mathcal{F}/p)$; it is principal if and only if $-2\mathbf{p}^{\boldsymbol{\delta}} \det \mathcal{F}$ is a square. Moreover, if (1) is soluble for all $p \in \Lambda \cup S$, then the quadratic form $f(x_1, \ldots, x_4) = F(\mathbf{x}) - \mathbf{p}^{\boldsymbol{\delta}} x_4^2$ is isotropic over all $\mathbb{Q}_p$ by Lemma 3.2. When this coincides with χ being principal, then for any prime p Hasse's invariant $\varepsilon_p(f)$ must satisfy

$$\varepsilon_p(f) = (-1, -1)_p, \tag{11}$$

where $(\cdot, \cdot)_p$ is the Hilbert symbol relative to $\mathbb{Q}_p$. Because of the identities $\varepsilon_p(f) = \varepsilon_p(F)(-\mathbf{p}^{\boldsymbol{\delta}}, 2 \det \mathcal{F})$ and

$$(-1, -1)_p(-\mathbf{p}^{\boldsymbol{\delta}}, 2 \det \mathcal{F})_p(-1, -2 \det \mathcal{F})_p = (\mathbf{p}^{\boldsymbol{\delta}}, -2\mathbf{p}^{\boldsymbol{\delta}} \det \mathcal{F}) = 1,$$

equation (11) is equivalent to $\varepsilon_p(F) = (-1, -2 \det \mathcal{F})_p$, which in turn implies that F is isotropic over $\mathbb{Q}_p$.

If $p \in S$, then τ may grow with P. Subject to existence we define

$$\rho_p = \lim_{k \to \infty} \sigma_p(\boldsymbol{\delta}, k),$$

whose key properties are summarised as follows.

THEOREM 3.4. *The limit ρ_p exists, is non-negative and independent of $\boldsymbol{\delta}$, satisfies*

$$\rho_p - \sigma_p(\boldsymbol{\delta}, k) \ll p^{-k}, \tag{12}$$

and equals zero if and only if F is anisotropic over $\mathbb{Q}_p$.

Proof. Let us write

$$S_p^h = \{\mathbf{x} \pmod{p^{2h+1}} : F(\mathbf{x}) \equiv 0 \pmod{p^{2h+1}} \wedge p^h \parallel \nabla F(\mathbf{x})\}.$$

Since F is non-singular, one has

$$\#S_p^h, \#M_p^h(\boldsymbol{\delta}, k, 2h+1) \le \#\{\mathbf{x} \pmod{p^{2h+1}} : p^h \mid \nabla F(\mathbf{x})\} \ll p^{hn}.$$

So setting

$$\rho_p = p^{1-n} \sum_{h=0}^{\infty} p^{-2h(n-1)} \#S_p^h$$

gives

$$\rho_p - \sigma_p(\boldsymbol{\delta}, k) \ll \sum_{h>\tau/2} p^{-2h(n-1)+hn} \ll p^{-k}.$$

It is also clear now that $\rho_p = 0$ if F is anisotropic over $\mathbb{Q}_p$ since any element of S_p^h can be lifted to a non-trivial p-adic zero by Lemma 3.2. If F does represent zero p-adically, then there exists a zero $\mathbf{x} \in \mathbb{Z}_p^n$ of F with $p^h \parallel \nabla F(\mathbf{x})$ for some integer h. Therefore $\mathbf{x} \in S_p^h$, and ρ_p must be positive. ∎

4. Proof of Theorem 1.1. Without loss of generality we assume from now on that $p_1, \ldots, p_l$ are isotropic and $p_{l+1}, \ldots, p_m$ anisotropic. In the latter case we let

$$\theta_i(\boldsymbol{\delta}) = \sum_{s=0}^{\infty} \sigma_{p_i}(\boldsymbol{\delta}, s). \tag{13}$$

These series are convergent by Theorem 3.4. They are also positive since $F(\mathbf{x}) = \mathbf{p}^{\boldsymbol{\delta}}$ has a solution over all p-adic fields. So if $p = p_i$, then multiplying such a solution by $p_i^{s_i}$ for a sufficiently large s_i gives rise to an element of the set $M_{p_i}(\boldsymbol{\delta}, s_i, 4(s_i+1))$, from which one deduces that $\sigma_{p_i}(\boldsymbol{\delta}, s_i) > 0$.

Let $\mathcal{N}^*(P)$ be the counting function obtained by omitting the requirement that $\mathbf{x}$ is primitive from the definition of $\mathcal{N}(P)$. Since

$$\mathcal{N}(P) = \sum_{Y} \mu(Y) \mathcal{N}^*(P/Y),$$

an instance of Möbius inversion shows that it suffices to consider $\mathcal{N}^*(P)$. In analogy with (4) we have

$$\mathcal{N}^*(P) = \sum_{\boldsymbol{\delta} \in \Delta} \sum_{Y \ll P^2}{}' I_B(P; Y) \mathfrak{S}(Y) + O(P^{n-2} K^{-1}),$$

where the prime indicates that Y is of type $\boldsymbol{\delta}$, and $K = \log P$.

By initially considering only that part of the summation which has $|Y| \leq P^2K^{-5m}$, we will be able to take advantage of Lemma 2.1. This lemma is fruitful only in combination with Lemma 2.2, whose requirements may not be satisfied by B. It is clearly possible though to choose some n-dimensional box $B_0 \subset B$ that does satisfy the conditions of the lemma. One can then partition B into a finite number of equally oriented n-dimensional boxes $B_0, B_1, \dots, B_g$ which may or may not contain some or all of their faces. (Note that the absence of faces from the boxes B_i is irrelevant with regard to the results proved in Section 2, as the faces are nullsets in $\mathbb{R}^n$.) We obtain

$$\sum_{Y \leq PK^{-5m}}' I_B(P;Y)\mathfrak{S}(Y) = \sum_{Y \leq PK^{-5m}}' \sum_{i=0}^{g} I_{B_i}(P;0)\mathfrak{S}(Y) + O(P^{n-2}K^{-1})$$

since $\mathfrak{S}(Y) \ll 1$ by (10). The definition of the singular integral in (3) shows that

$$I_{B_i}(P;0) = H_{B_i}P^{n-2}$$

for some constant H_{B_i}. In addition, H_{B_i} must be real and non-negative. For otherwise Theorem 1.2 applied to $N = \mathbf{p}^{\boldsymbol{\delta}} p_1^{2k_1} \cdots p_l^{2k_l} p_{l+1}^{2s_{l+1}} \cdots p_m^{2s_m}$, where the s_i are fixed and defined as above and the k_i tend to infinity with P, would, in combination with the resulting positivity of the singular series and Lemma 2.1, imply that there is a complex or negative number of zeros $\mathbf{x} \in PB_i$ of $Q(\mathbf{x}, N)$. Furthermore, $H_0 > 0$ since B_0 fulfils the conditions of Lemma 2.2. Therefore our summation becomes

$$HP^{n-2} \sum_{|Y| \leq P^2K^{-5m}}' \mathfrak{S}(Y) + O(P^{n-2}K^{-1}), \tag{14}$$

where $H = H_0 + \cdots + H_g$ is a positive constant. Now let us write

$$\sigma_1(Y) = \hat{\sigma}_{p_1}(Y) \cdots \hat{\sigma}_{p_l}(Y) \quad \text{and} \quad \sigma_2(Y) = \hat{\sigma}_{p_{l+1}}(Y) \cdots \hat{\sigma}_{p_m}(Y).$$

Then the summation in (14) is

$$L_{\boldsymbol{\delta}}\rho_{p_1} \cdots \rho_{p_l} \sum_{|Y| \leq P^2K^{-5m}}' \sigma_2(Y) + O\Big(\sum_{Y \leq P^2K^{-5m}}' |\sigma_1(Y) - \rho_{p_1} \cdots \rho_{p_l}|\sigma_2(Y)\Big),$$

where the error term is due to the convergence of the series (13) bounded by

$$\sum_{k_1,\dots,k_l \ll K} |\sigma_{p_1}(\boldsymbol{\delta}, k_1) \cdots \sigma_{p_l}(\boldsymbol{\delta}, k_l) - \rho_{p_1} \cdots \rho_{p_l}| \ll K^{l-1} \log K.$$

The final estimate above follows from splitting the summation according to whether or not $k_i \geq \log K$ for all $i = 1, \dots, l$, and then using the approximation (12). We have shown that (14) is equal to

$$HL_{\boldsymbol{\delta}}\rho_{p_1} \cdots \rho_{p_l} P^{n-2} \sum_{|Y| \leq P^2K^{-5m}}' \sigma_2(Y) + O(P^{n-2}K^{l-1} \log K).$$

In what follows, we use the decomposition $Y = Y_1Y_2$ into factors $Y_1 > 0$ and Y_2 which contain the isotropic and anisotropic prime factors of Y respectively. With this notation we can write the summation over the Y as

$$\sum_{Y_1\le P^2K^{-5m}}{}' \sum_{Y_2}{}' \sigma_2(Y) - \sum_{Y_1\le P^2K^{-5m}}{}' \sum_{|Y_2|>P^2K^{-5m}Y_1^{-1}}{}' \sigma_2(Y), \tag{15}$$

where the primes indicate that Y_1 is of type $(\delta_1,\dots,\delta_l)$ and similarly for Y_2. Since the final summation above is bounded by

$$\ll \sum_{|Y_2|>P^2K^{-5m}Y_1^{-1}} |Y_2|^{-1} \ll \left(\frac{P^2}{K^{5m}Y_1}\right)^{-1/2} \sum_{Y_2} |Y_2|^{-1/2} \ll \left(\frac{P^2}{K^{5m}Y_1}\right)^{-1/2},$$

we have

$$\sum_{Y_1\le P^2K^{-5m}}{}' \sum_{|Y_2|>P^2K^{-5m}Y_1^{-1}}{}' \sigma_2(Y) \ll P^{-1}K^{5m/2} \sum_{Y_1\le P^2K^{-5m}} Y_1^{1/2}.$$

Now the sum above is $\ll PK^{-5m/2}$ if $l > 0$ and equal to 1 otherwise. Therefore

$$\sum_{|Y|\le P^2K^{-5m}}{}' \sigma_2(Y) = \theta_{l+1}(\boldsymbol{\delta})\cdots\theta_m(\boldsymbol{\delta}) \sum_{Y_1\le P^2K^{-5m}}{}' 1 + O(K^{l-1}),$$

and it only remains to note that the l-dimensional pyramid given by the equations

$$k_i \ge 0, \quad i = 1,\dots,l, \quad \text{and} \quad p_1^{\delta_1+2k_1}\cdots p_l^{\delta_l+2k_l} \le P^2K^{-5m}$$

has volume

$$\frac{1}{l!}\prod_{i=1}^{l}\frac{K}{\log p_i} + O(K^{l-1}\log K).$$

Via (15) and (14) we conclude that the contribution from $|Y| \le P^2K^{-5m}$ to $\mathcal{N}^*(P)$ is given by

$$\frac{HL\rho_{p_1}\cdots\rho_{p_l}}{l!\log p_1\cdots\log p_l} P^{n-2}K^l + O(P^{n-2}K^{l-1}\log K),$$

where we set $L = \sum_{\boldsymbol{\delta}\in\Delta} L_{\boldsymbol{\delta}}\theta_{l+1}(\boldsymbol{\delta})\cdots\theta_m(\boldsymbol{\delta})$.

Lastly, we need to consider the contribution from all S-units whose modulus is greater than P^2K^{-5m}. By the results of the previous two sections this contribution clearly is

$$\ll P^{n-2} \sum_{Y_1\le PK^{-6m}} \sum_{Y_2\ge K^m} \sigma_2(Y) + O(P^{n-2}K^{l-1}\log K) \ll P^{n-2}K^{l-1}\log K$$

since the sum over all $Y_2 \ge K^m$ is of size $O(K^{-1})$ by Theorem 3.4.

If for all $\boldsymbol{\delta} \in \Delta$ there exists a prime $p \in \Lambda$ such that (1) has no solution, then obviously $\mathcal{N}(P) = 0$. Otherwise $L > 0$, and Theorem 1.1 holds with

$$c = \frac{HL\rho_{p_1} \cdots \rho_{p_l}}{l! \log p_1 \cdots \log p_l} \prod_{p \in S} (1 - p^{2-n}),$$

where the product over the $p \in S$ stems from a Möbius inversion.

Acknowledgements. The author of this paper is studying for a D.Phil. at the University of Oxford. He would like to thank Prof. Heath-Brown for the excellent supervision that has accompanied this research.

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Received on 13.1.2009
and in revised form on 11.5.2009 (5912)