Independence results for pattern sequences in distinct bases

by

YOHEI TACHIYA (Yokohama)

1. Introduction and results. Let $q \geq 2$ be an integer. Then any positive integer $n$ has a unique representation of the form

$$n = \sum_{i=0}^{k} a_i q^i, \quad a_i \in \Sigma_q := \{0, 1, \ldots, q-1\}, \quad a_k > 0.$$  

We denote by $\Sigma_q^*$ the set of all finite strings of elements in $\Sigma_q$,

$$\Sigma_q^* := \{b_l b_{l-2} \cdots b_0 \mid b_i \in \Sigma_q, \ l \geq 1\}.$$  

(Note that the set $\Sigma_q^*$ does not contain the empty string.) For an integer $n \geq 1$ having the expression (1), the string of digits

$$(n)_q := a_k a_{k-1} \cdots a_0 \in \Sigma_q^*$$

is called the $q$-ary expansion of $n$. Let $w \in \Sigma_q^*$. We put $w^l = w \cdots w$ ($l$ times). If $w = 0^l$ for some $l \geq 1$, we say that $w$ is a zero pattern; otherwise it is a nonzero pattern. We define $e_q(w; n)$ to be the number of (possibly overlapping) occurrences of $w$ in the $q$-ary expansion of an integer $n > 0$. Here if $w$ is a nonzero pattern, then in evaluating $e_q(w; n)$ we assume that the $q$-ary expansion of $n$ starts with an arbitrarily long string of zeros. On the other hand, if $w$ is a zero pattern, then $w = 0^l$ for some $l \geq 1$, and we just count the number of occurrences of $w$ in the $q$-ary expansion of $n$. We set $e_q(w; 0) = 0$ for any $w \in \Sigma_q^*$. The resulting sequence

$$\{e_q(w; n)\}_{n \geq 0}$$

is sometimes called the pattern sequence for the pattern $w \in \Sigma_q^*$ (cf. Allouche and Shallit [1]). We note that the value $e_2(1; n)$ coincides with the sum of the base-2 digits of $n$.

Uchida [11] gave necessary and sufficient conditions for algebraic independence over $\mathbb{C}(z)$ of generating functions of pattern sequences in one $q$-adic
number system. Recently, Shiokawa and the author [8] obtained similar results for pattern sequences in \((q,r)\)-number systems \((r = 0,1,\ldots,q - 2)\) with a fixed base \(q\). Generating functions and their values defined by digital properties of integers have also been studied in [3], [7], and [9]. In the case of different bases, only special pattern sequences have been discussed; for example Toshimitsu [10] proved that for a given integer \(b\) the generating functions of the pattern sequences \(\{e_q(b; n)\}_{n \geq 0} (q = b + 1, b + 2, \ldots)\) are algebraically independent over \(\mathbb{C}(z)\).

In this paper, for arbitrary given nonzero patterns \(w_q \in \Sigma_q^* \ (q = 2, 3, \ldots)\) we prove the algebraic independence of the values of the generating functions

\[
\sum_{n \geq 0} e_q(w_q; n)z^n, \quad q = 2, 3, \ldots
\]

which converge in \(|z| < 1\). Furthermore, we derive the algebraic independence over \(\mathbb{C}(z)\) of the above generating functions. In particular, the latter implies the linear independence of the pattern sequences in distinct bases (Corollary 1).

Theorem 1. Let \(w_q \in \Sigma_q^* \ (q \geq 2)\) be nonzero patterns and

\[
f_q(z) = \sum_{n \geq 0} e_q(w_q; n)z^n, \quad q = 2, 3, \ldots.
\]

Then for any algebraic number \(\alpha\) with \(0 < |\alpha| < 1\), their values \(\{f_q(\alpha)\}_{q \geq 2}\) are algebraically independent.

Theorem 2. The generating functions of the pattern sequences (2) are algebraically independent over \(\mathbb{C}(z)\).

By Theorem 2, a nontrivial linear combination of the functions (2)

\[
c_1f_2(z) + c_2f_3(z) + \cdots + c_{m-1}f_m(z)
\]

over \(\mathbb{C}\) is not a rational function for \(|z| < 1\). Hence we obtain the following:

Corollary 1. Let \(w_q \in \Sigma_q^* \ (q = 2, \ldots, m)\) be \(m - 1\) nonzero patterns and \(c_1, \ldots, c_{m-1} \in \mathbb{C}\) not all zero. Then the linear combination of the pattern sequences

\[
\{c_1e_2(w_2; n) + c_2e_3(w_3; n) + \cdots + c_{m-1}e_m(w_m; n)\}_{n \geq 0}
\]

cannot be a linear recurrence sequence. In particular, the pattern sequences \(\{e_q(w_q; n)\}_{n \geq 0} (q = 2, 3, \ldots)\) are linearly independent over \(\mathbb{C}\).

Example 1. Let \(w = b_{l-1}b_{l-2}\cdots b_0\) be a nonzero pattern with \(b_i \in \{0, 1\}\). Then the pattern sequences

\[
\{e_2(w; n)\}_{n \geq 0}, \quad \{e_3(w; n)\}_{n \geq 0}, \ldots, \quad \{e_m(w; n)\}_{n \geq 0}, \ldots
\]

are linearly independent over \(\mathbb{C}\). For example, the sequences \(\{e_2(10; n)\}_{n \geq 0}, \{e_3(10; n)\}_{n \geq 0}, \text{and} \{e_4(10; n)\}_{n \geq 0}\) which are defined by the number of 10’s
appearing in the dyadic, 3-ary, and 4-ary expansions of \( n \), respectively, are linearly independent over \( \mathbb{C} \).

On the other hand, within one fixed number system, the generating functions can be algebraically dependent over \( \mathbb{C}(z) \).

**Example 2** (Shiokawa and Tachiya [8]). In the usual dyadic expansion, we consider the generating functions

\[
f_1(z) = \sum_{n \geq 0} e_2(01; n)z^n, \quad f_2(z) = \sum_{n \geq 0} e_2(10; n)z^n.
\]

Then the sequence \( \{e_2(01; n) - e_2(10; n)\}_{n \geq 0} = \{0, 1, 0, 1, \ldots\} \) is periodic, and so

\[
f_1(z) - f_2(z) = \frac{z}{1 - z^2}, \quad |z| < 1.
\]

**Example 3.** Let \( w \in \Sigma_q^* \). By the definition of \( e_q(w; n) \), we have

\[
e_q(w; n) = \sum_{b=0}^{q-1} e_q(bw; n).
\]

Therefore the pattern sequences \( \{e_q(w; n)\}_{n \geq 0}, \{e_q(bw; n)\}_{n \geq 0} (b = 0, 1, \ldots \ldots, q - 1) \) are linearly dependent over \( \mathbb{C} \), and so are their generating functions.

**2. Lemmas.** In this section, we prepare some lemmas for proving Theorem 1. Fix an integer \( q \geq 2 \). For any nonzero pattern \( w = b_{l-1}b_{l-2}\cdots b_0 \in \Sigma_q^* \) with \( b_i \in \Sigma_q \), let \( |w| \) denote the length \( l \) and put \( \nu(w) = \sum_{k=0}^{l-1} b_kq^k \).

**Lemma 1.** Let \( i \geq 1 \) be an integer and \( w \in \Sigma_{q_i}^* \) be a nonzero pattern. Then for any integer \( d \geq 0 \), we have

\[
e_{q_i}(w; \nu(w)q^d) = \begin{cases} 1, & i \mid d, \\ 0, & \text{otherwise.} \end{cases}
\]

**Proof.** We put

\[
w = 0^l a_k \cdots a_0, \quad a_j \in \Sigma_{q_i}, \ a_k \neq 0, \ k, l \geq 0.
\]

Then \( \nu(w) = \sum_{j=0}^{k} a_j(q^i)^j \) and \( (\nu(w)q^i)_q = a_k \cdots a_0 \). Let \( h \) and \( r \) be integers with

\[
d = ih + r, \quad 0 \leq r < i.
\]

First we consider the case that \( d \) is divisible by \( i \). Since \( r = 0 \) in (3), the \( q^i \)-ary expansion of the integer \( \nu(w)q^d \) is represented as

\[
(\nu(w)q^d)_{q^i} = (\nu(w)q^{ih})_{q^i} = a_k \cdots a_00^h.
\]
It is clear that \( e_q^i(w; \nu(w)q^d) \geq 1 \). If \( e_q^i(w; \nu(w)q^d) > 1 \), we get \( w = a_{k-m} \cdots a_0 l^m \) for some integer \( m \) with \( 1 \leq m \leq k \). Then we have

\[
\nu(w) = a_{k-m} q^{i(k+l)} + a_{k-m-1} q^{i(k+l-1)} + \cdots + a_0 q^{i(k+l-(k-m))}
\]

\[
= q^{i(l+m)} (a_{k-m} q^{i(k-m)} + a_{k-m-1} q^{i(k-m-1)} + \cdots + a_0)
\]

so that \( (q^{i(l+m)} - 1) \nu(w) \equiv 0 \mod q^{i(k+l+1)} \). Noting that the integers \( q^{i(l+m)} - 1 \) and \( q^{i(k+l+1)} \) are coprime, we get \( \nu(w) \equiv 0 \mod q^{i(k+l+1)} \), that is, \( a_j = 0 \) for all \( j = 0, 1, \ldots, k \). This is a contradiction and hence we obtain \( e_q^i(w; \nu(w)q^d) = 1 \).

Next we consider the case that \( d \) is not divisible by \( i \). For the integer \( r \geq 1 \) defined in \[3\], we put

\[
(\nu(w)q^r)^{q^r} = b_s b_{s-1} \cdots b_0 \in \Sigma^*_q, \quad b_j \in \Sigma_q, \quad b_s \neq 0,
\]

where \( s = k, k+1, \) since

\[
k + 1 = |(\nu(w))_{q^r}| \leq |(\nu(w)q^r)_{q^r}| \leq |(\nu(w)q^r)| = |(\nu(w))_{q^r}| + 1 = k + 2.
\]

Suppose on the contrary that \( e_q^i(w; \nu(w)q^d) \neq 0 \), that is, the pattern \( w \) appears at least once in the \( q^i \)-ary expansion of \( \nu(w)q^d \):

\[
(\nu(w)q^d)^{q^i} = (\nu(w)q^{r+i})^{q^i} = b_s b_{s-1} \cdots b_0 0^h.
\]

Hence, as \( b_s \neq 0 \), the \( q^i \)-ary expansion of \( w \) must be of the form either

\[4\]
\[
w = 0^i b_s b_{s-1} \cdots b_{s-k},
\]

or

\[5\]
\[
w = b_{s-m} b_{s-m-1} \cdots b_{s-m-(k+l)}
\]

for some integer \( m \) with \( 1 \leq m \leq s \), where we define \( b_j = 0 \) for negative \( j \).

If the equality \[4\] is satisfied, we have

\[
\nu(w) = b_s q^{i-k} + b_{s-1} q^{i(k-1)} + \cdots + b_{s-k} = \begin{cases} \nu(w)q^r, & s = k, \\ q^{-i}(\nu(w)q^r - b_0), & s = k + 1. \end{cases}
\]

Since \( 1 \leq r < i \), in any case we can deduce a contradiction. On the other hand, if the case \[5\] holds, we get

\[
\nu(w) = b_{s-m} q^{i(k+l)} + b_{s-m-1} q^{i(k+l-1)} + \cdots + b_{s-m-k-l}
\]

\[
= q^{i(k+l-(s-m))} (b_{s-m} q^{i(s-m)} + \cdots + b_{s-m-k-l} q^{i(s-m-k-l)})
\]

\[
= q^{i(k+l-(s-m))} (\nu(w)q^r - b_s q^{i-k} - \cdots - b_{s-m+1} q^{i(s-m+1)}),
\]

so that \( (q^{r+i(k+l-(s-m))} - 1) \nu(w) \equiv 0 \mod q^{i(k+l+1)} \). Since \( r + i(k + l - (s - m)) \geq 1 \), we obtain \( \nu(w) \equiv 0 \mod q^{i(k+l+1)} \), which implies \( a_j = 0 \) for all \( j = 0, 1, \ldots, k \). This is a contradiction and the lemma is proved.
Let $m \geq 2$ be an integer. We set
\begin{equation}
S := \{(k_1, \ldots, k_{m-1}) \in \mathbb{Z}_{\geq 0}^{m-1} \mid 0 \leq k_j \leq j - 1, j = 1, \ldots, m - 1\},
\end{equation}
\begin{equation}
S_n := \{(k_1, \ldots, k_{m-1}) \in S \mid k_1 + \cdots + k_{m-1} = n\}.
\end{equation}

**Lemma 2.** For every integer $m \geq 2$, there exist integers $d_1$ and $d_2$ with $0 \leq d_1 < d_2 \leq m - 1$ such that
\[
\sum_{n \geq 0} \#S_n \neq \sum_{n \equiv d_1 \mod m} \#S_n,
\]
where $\#S_n$ is the number of elements of $S_n$.

**Proof.** We define
\[
f(x) = \frac{1}{(1 - x)^{m-1}} \prod_{k=1}^{m-1} (1 - x^k) \in \mathbb{Z}[x].
\]
Let $\xi$ be a primitive $m$th root of unity. Then it is clear that $f(\xi) \neq 0$. Since the polynomial $f(x)$ is expressed as
\[
f(x) = (1 + x)(1 + x + x^2) \cdots (1 + x + x^2 + \cdots + x^{m-2}) = \sum_{n \geq 0} (\#S_n) x^n,
\]
we have
\begin{equation}
f(\xi) = c_0 + c_1 \xi + \cdots + c_{m-1} \xi^{m-1},
\end{equation}
where
\[
c_i = \sum_{n \geq 0 \atop n \equiv i \mod m} \#S_n, \quad i = 0, 1, \ldots, m - 1.
\]
If $c_i = c_j$ for all $i, j$, then by (8) we get
\[
f(\xi) = c_0(1 + \xi + \cdots + \xi^{m-1}) = 0,
\]
a contradiction. ■

**Lemma 3 (Uchida [11]).** Let $d \geq 2$ and $l \geq 1$ be integers. If $c(z) \in \mathbb{C}(z)$ satisfies the functional equation
\[
c(z) = c(z^d) + \frac{(1 - z)a(z)}{1 - z^d}, \quad a(z) \in \mathbb{C}[z],
\]
then there exists $b(z) \in \mathbb{C}[z]$ such that
\[
c(z) = \frac{(1 - z)b(z)}{1 - z^{d-1}}.
\]

**Lemma 4 (Nishioka [5]).** Let $K$ be an algebraic number field and $d_1, \ldots, d_t \geq 2$ be integers with $\log d_i/\log d_j \notin \mathbb{Q}$ if $i \neq j$. Suppose that $f_{i,j}(z) \in K[[z]]$ (1 \leq i \leq t, 1 \leq j \leq m_i) satisfy the functional equations
\[
f_{i,j}(z^{d_i}) = a_{i,j}(z)f_{i,j}(z) + b_{i,j}(z) \quad (1 \leq i \leq t, 1 \leq j \leq m_i),
\]
where \( a_{i,j}(z), b_{i,j}(z) \in K(z) \), \( a_{i,j}(0) = 1 \), and \( f_{i,1}(z), \ldots, f_{i,m_i}(z) \) are algebraically independent over \( K(z) \) for each \( i = 1, \ldots, t \). If \( \alpha \) is an algebraic number with \( 0 < |\alpha| < 1 \), \( a_{i,j}(\alpha^{d_i}) \neq 0 \) \((k \geq 0)\) and all \( f_{i,j}(z) \) converge at \( z = \alpha \), then the values

\[
f_{i,j}(\alpha) \quad (1 \leq i \leq t, 1 \leq j \leq m_i)
\]

are algebraically independent.

**Lemma 5** (Kubota [2], Loxton and van der Poorten [4]; see Nishioka [6]). Let \( d \geq 2 \) be an integer. Suppose that \( g_1(z), \ldots, g_n(z) \in \mathbb{C}[[z]] \) are algebraically dependent over \( \mathbb{C}(z) \) and satisfy the functional equations

\[
g_i(z^d) = g_i(z) + a_i(z), \quad a_i(z) \in \mathbb{C}(z), \quad i = 1, \ldots, n.
\]

Then there exist constants \( c_1, \ldots, c_n \in \mathbb{C} \) not all zero such that

\[
c_1 g_1(z) + \cdots + c_n g_n(z) \in \mathbb{C}(z).
\]

**3. Proofs of Theorems 1 and 2.** Define

\[
M = \{ q \in \mathbb{N} \mid q \neq a^n \text{ for any } a, n \in \mathbb{N}, n \geq 2 \}.
\]

Then

\[
\mathbb{N} \setminus \{1\} = \bigcup_{q \in M} \{ q, q^2, \ldots \} = \{ q^j \in \mathbb{N} \mid q \in M, j \geq 1 \}.
\]

Let \( q_1, \ldots, q_t \in M \) be distinct integers, \( w_{i,j} \in \Sigma_{q_i}^* \) \((j = 1, \ldots, m_i)\) be nonzero patterns, and

\[
f_{i,j}(z) = \sum_{n \geq 0} e_{q_i}(w_{i,j}; n)z^n \quad (1 \leq i \leq t, 1 \leq j \leq m_i).
\]

It is easily seen that \( \log q_i / \log q_j \notin \mathbb{Q} \) if \( i \neq j \). Then by Theorem 1 in [11] the functional equations

\[
f_{i,j}(z) = \frac{1 - z^{q_i^j}}{1 - z} f_{i,j}(z^{q_i^j}) + \frac{z^{\nu(w_{i,j})}}{1 - z^{q_i^{\nu(w_{i,j})}}} \quad (1 \leq i \leq t, 1 \leq j \leq m_i)
\]

are satisfied. Here, putting \( F_{i,j}(z) = (1 - z)f_{i,j}(z) \), we have

\[
F_{i,j}(z) = F_{i,j}(z^{q_i^j}) + z^{\nu(w_{i,j})} \frac{1 - z}{1 - z^{q_i^{\nu(w_{i,j})}}} \quad (1 \leq i \leq t, 1 \leq j \leq m_i),
\]

and so

\[
F_{i,j}(z) = F_{i,j}(z^{q_i^{D_i}}) + \sum_{k=0}^{D_i/j - 1} z^{q_i^{D_i kj}} \frac{1 - z^{q_i^{kj}}}{1 - z^{q_i^{\nu(w_{i,j}) + kj}}} \quad (1 \leq i \leq t, 1 \leq j \leq m_i),
\]

where \( D_i = \text{lcm}(1, \ldots, m_i) \). Hence, if the functions \( F_{i,1}(z), \ldots, F_{i,m_i}(z) \) are algebraically independent over \( \mathbb{C}(z) \) for each \( i = 1, \ldots, t \), then by Lemma 4 the values \( F_{i,j}(\alpha) = (1 - \alpha) f_{i,j}(\alpha) \) \((1 \leq i \leq t, 1 \leq j \leq m_i)\) are algebraically independent.
independent for any algebraic number \( \alpha \) with \( 0 < |\alpha| < 1 \). Therefore, to prove Theorem 1 it is enough to show the algebraic independence over \( \mathbb{C}(z) \) of the functions

\[
F_i(z) := (1 - z) \sum_{n \geq 0} e_{q_i}(w_i; n) z^n, \quad i = 1, \ldots, m,
\]

for any fixed integer \( q \geq 2 \) and for nonzero patterns \( w_i \in \Sigma_q^* \).

**Proof of Theorem 1.** Let \( q \geq 2 \) be a fixed integer and \( w_i \in \Sigma_q^* \) \((i = 1, \ldots, m)\) be nonzero patterns. In what follows, we prove the algebraic independence over \( \mathbb{C}(z) \) of the functions \( F_1(z), \ldots, F_m(z) \) given in (9). We use induction on \( m \). By Theorem 1 in [11] the function \( F_1(z) \) is transcendental over \( \mathbb{C}(z) \), and hence the claim is satisfied in the case of \( m = 1 \). Let \( m \geq 2 \) and assume the claim is true for \( m - 1 \). Towards a contradiction, suppose that the functions \( F_1(z), \ldots, F_m(z) \) are algebraically dependent over \( \mathbb{C}(z) \).

Since they satisfy the functional equations

\[
F_i(z) = F_i(z^{qD}) + \sum_{k=0}^{D/i-1} z^{q^k i \nu(w_i)} \frac{1 - z^{q^k i}}{1 - z^{q^k i |w_i| + ki}}, \quad i = 1, \ldots, m,
\]

with \( D = \text{lcm}(1, \ldots, m) \), applying Lemma 5 we see that there exist constants \( c_1, \ldots, c_m \in \mathbb{C} \) not all zero such that

\[
R(z) := c_1 F_1(z) + \cdots + c_m F_m(z) \in \mathbb{C}(z).
\]

We may suppose \( c_m \neq 0 \) from the assumption of induction. Substituting \( z^{qD} \) for \( z \) in the above identity and using the functional equation (10), we have

\[
R(z) = R(z^{qD}) + \frac{1 - z}{1 - z^{qDl}} \sum_{i=1}^{m} \sum_{k=0}^{D/i-1} c_i z^{q^k i \nu(w_i)} \frac{1 - z^{q^k i}}{1 - z} \frac{1 - z^{qDl}}{1 - z^{q^k i |w_i| + ki}},
\]

where \( l := \max_{1 \leq i \leq m} |w_i| \) and

\[
i |w_i| + ki \leq i(l + k) \leq i(l + D/i - 1) \leq Dl.
\]

Thus the functional equation (11) can be written as

\[
R(z) = R(z^{qD}) + \frac{(1 - z)a(z)}{1 - z^{qDl}}
\]

with

\[
a(z) = \sum_{i=1}^{m} \sum_{k=0}^{D/i-1} c_i z^{q^k i \nu(w_i)} \frac{1 - z^{q^k i}}{1 - z} \frac{1 - z^{qDl}}{1 - z^{q^k i |w_i| + ki}} \in \mathbb{C}[z].
\]

Using Lemma 3 we see that there exists \( b(z) \in \mathbb{C}[z] \) such that

\[
R(z) = \frac{(1 - z)b(z)}{1 - z^{qD(i-1)}}.
\]
Substituting the expression (13) into (12) and multiplying both sides by 
\( (1 - z^{q^D l})/(1 - z) \), we have

\[
\frac{1 - z^{q^D l}}{1 - z^{q^D (l-1)}} b(z) = \frac{1 - z^{q^D}}{1 - z} b(z^{q^D}) + a(z),
\]

where \( \deg a(z) \leq q^D l - 1 \). If the degree of the first term of the right-hand side is not greater than that of the left-hand side, we get \( \deg b(z) \leq q^{D(l-1)} - 1 \). Otherwise, the degree of the first term coincides with \( \deg a(z) \); then we can deduce \( \deg b(z) \leq q^{D(l-1)} - 1 \). In any case, we have

(14) \( \deg b(z) \leq q^{D(l-1)} - 1 \).

By the expression (13), we have

\[
b(z) = \frac{1 - z^{q^{D(l-1)}}}{1 - z} R(z) = (1 - z^{q^{D(l-1)}}) \sum_{n=0}^{m} \sum_{i=1}^{d} c_i e_{q^i}(w_i; n) z^n
\]

\[
= \sum_{n=0}^{q^{D(l-1)}-1} a_n z^n + \sum_{n=0}^{q^{D(l-1)}} (a_{n+q^{D(l-1)}} - a_n) z^{n+q^{D(l-1)}},
\]

where \( a_n = \sum_{i=1}^{m} c_i e_{q^i}(w_i; n) \). Therefore by (14) we obtain \( a_n = a_{n+q^{D(l-1)}} \) \((n \geq 0)\), so that the sequence

(15) \( \left\{ \sum_{i=1}^{m} c_i e_{q^i}(w_i; n) \right\}_{n \geq 0} \)

is periodic with period \( q^{D(l-1)} \).

Now we prove \( c_m = 0 \) and deduce a contradiction. We choose integers

\( d_1, d_2 \) \((0 \leq d_1 < d_2 \leq m - 1)\) as in Lemma 2. Define the positive integers

\[
N_j = \nu(w_m) \sum_{k_1=0}^{0} \sum_{k_2=0}^{1} \cdots \sum_{k_{m-1}=0}^{m-2} q^{k_1 + \cdots + k_{m-1} + m-d_j+DL(1+k_1+mk_2+\cdots+m^{m-2}k_{m-1})}
\]

for \( j = 1, 2 \), where \( L > 0 \) is a sufficiently large integer. Noting that \( w_m \in \Sigma_{q^m}^* \) is a nonzero pattern and

\[
k_1 + mk_2 + \cdots + m^{m-2}k_{m-1} \neq k'_1 + mk'_2 + \cdots + m^{m-2}k'_{m-1}
\]

if \((k_1, \ldots, k_{m-1}) \neq (k'_1, \ldots, k'_{m-1})\), we have

\[
e_{q^i}(w_i; N_j) = \sum_{k_1=0}^{0} \cdots \sum_{k_{m-1}=0}^{m-2} e_{q^i}(w_i; \nu(w_m)q^{k_1 + \cdots + k_{m-1} + m-d_j+DL}), \quad j = 1, 2,
\]
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for every \(i = 1, \ldots, m\). For a fixed integer \(i \geq 1\), if \(s_1\) and \(s_2\) are nonnegative integers with \(s_1 \equiv s_2 \mod i\), then the identity

\[
e_{q^i}(w_i; \nu(w_m)q^{s_1+DL}) = e_{q^i}(w_i; \nu(w_m)q^{s_2+DL})
\]

holds. Hence for each \(i = 1, \ldots, m - 1\) we get

\[
\sum_{k_i=m-d_i}^{i-1+m-d_1} e_{q^i}(w_i; \nu(w_m)q^{k_1+\cdots+k_m-1+DL}) = \sum_{k_i=m-d_2}^{i-1+m-d_2} e_{q^i}(w_i; \nu(w_m)q^{k_1+\cdots+k_m-1+DL}),
\]

so that

\[
e_{q^i}(w_i; N_1) = \sum_{k_1=0}^{0} \cdots \sum_{k_m-1=0}^{m-2} \sum_{k_i=m-d_1}^{i-1+m-d_1} \cdots \sum_{k_m-1=0}^{m-2} e_{q^i}(w_i; \nu(w_m)q^{k_1+\cdots+k_m-1+DL})
\]

\[
= \sum_{k_1=0}^{0} \cdots \sum_{k_m-1=0}^{m-2} \sum_{k_i=m-d_2}^{i-1+m-d_2} \cdots \sum_{k_m-1=0}^{m-2} e_{q^i}(w_i; \nu(w_m)q^{k_1+\cdots+k_m-1+DL})
\]

\[
= e_{q^i}(w_i; N_2), \quad i = 1, \ldots, m - 1.
\]

On the other hand, by Lemma 1 we have

\[
e_{q^m}(w_m; N_j) = \sum_{k_1=0}^{0} \cdots \sum_{k_m-1=0}^{m-2} e_{q^m}(w_m; \nu(w_m)q^{k_1+\cdots+k_m-1+m-d_j+DL})
\]

\[
= \#\{(k_1, \ldots, k_m-1) \in S \mid k_1 + \cdots + k_m-1 \equiv d_j \mod m\}
\]

\[
= \sum_{n \geq 0} \#S_n, \quad j = 1, 2,
\]

where \(S\) and \(S_n\) are the sets defined by (6) and (7), respectively. Hence it follows from Lemma 2 that

\[
e_{q^m}(w_m; N_1) \neq e_{q^m}(w_m; N_2).
\]

Since the sequence (15) is periodic with period \(q^{D(l-1)}\) and \(N_1 \equiv N_2 \mod q^{D(l-1)}\) if \(L\) is large, we have

\[
\sum_{i=1}^{m} c_i e_{q^i}(w_i; N_1) = \sum_{i=1}^{m} c_i e_{q^i}(w_i; N_2).
\]

Combining (16), (17), and the above identity, we obtain \(c_m = 0\). This is a contradiction and the proof of Theorem 1 is complete. \(\blacksquare\)
Proof of Theorem 2. Suppose that the functions \( f_q(z) \) \((q = 2, 3, \ldots)\) are algebraically dependent over \( \mathbb{C}(z) \), so that

\[
\sum_{0 \leq i_1, \ldots, i_m \leq N} a_{i_1, \ldots, i_m}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} = 0
\]

with \( a_{i_1, \ldots, i_m}(z) \in \mathbb{C}[z] \) not all zero. Let \( \{\beta_1, \ldots, \beta_s\} \) be a maximal subset of the set of all the coefficients of \( a_{i_1, \ldots, i_m}(z) \) which is linearly independent over \( \mathbb{Q} \). Then the polynomials \( a_{i_1, \ldots, i_m}(z) \) can be written as

\[
a_{i_1, \ldots, i_m}(z) = \sum_{j=1}^{s} b_{i_1, \ldots, i_m,j}(z) \beta_j, \quad b_{i_1, \ldots, i_m,j}(z) \in \mathbb{Q}[z],
\]

and so by (18) we have

\[
\sum_{j=1}^{s} \left( \sum_{0 \leq i_1, \ldots, i_m \leq N} b_{i_1, \ldots, i_m,j}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} \right) \beta_j = 0.
\]

Since \( \beta_1, \ldots, \beta_s \) are linearly independent over \( \mathbb{Q} \), we get

\[
\sum_{0 \leq i_1, \ldots, i_m \leq N} b_{i_1, \ldots, i_m,j}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} = 0
\]

for all \( j = 1, \ldots, s \). Noting that at least one of \( b_{i_1, \ldots, i_m,j}(z) \) is not zero, we obtain the algebraic dependence over \( \mathbb{Q}(z) \) of the functions \( f_{q_1}(z), \ldots, f_{q_m}(z) \). Hence \( f_{q_1}(\alpha), \ldots, f_{q_m}(\alpha) \) are algebraically dependent for some algebraic number \( \alpha \) with \( 0 < |\alpha| < 1 \). This is a contradiction by Theorem 1. \( \blacksquare \)

References


Yohei Tachiya
Department of Mathematics
Keio University
Hiyoshi, Kohoku-ku, Yokohama 223–8522
Japan
E-mail: bof@math.keio.ac.jp

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