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Independence results for pattern sequences in distinct bases

by

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1. Introduction and results. Let $q \geq 2$ be an integer. Then any positive integer n has a unique representation of the form

$$(1) \qquad n = \sum_{i=0}^{k} a_i q^i, \quad a_i \in \Sigma_q := \{0, 1, \ldots, q-1\}, \quad a_k > 0.$$

We denote by Σ_q^* the set of all finite strings of elements in Σ_q,

$$\Sigma_q^* := \{b_{l-1} b_{l-2} \cdots b_0 \mid b_i \in \Sigma_q,\ l \geq 1\}.$$

(Note that the set Σ_q^* does not contain the empty string.) For an integer $n \geq 1$ having the expression (1), the string of digits

$$(n)_q := a_k a_{k-1} \cdots a_0 \in \Sigma_q^*$$

is called the *q-ary expansion* of n. Let $w \in \Sigma_q^*$. We put $w^l = w \cdots w$ (l times). If $w = 0^l$ for some $l \geq 1$, we say that w is a *zero pattern*; otherwise it is a *nonzero pattern*. We define $e_q(w; n)$ to be the number of (possibly overlapping) occurrences of w in the q-ary expansion of an integer $n > 0$. Here if w is a nonzero pattern, then in evaluating $e_q(w; n)$ we assume that the q-ary expansion of n starts with an arbitrarily long string of zeros. On the other hand, if w is a zero pattern, then $w = 0^l$ for some $l \geq 1$, and we just count the number of occurrences of w in the q-ary expansion of n. We set $e_q(w; 0) = 0$ for any $w \in \Sigma_q^*$. The resulting sequence

$$\{e_q(w; n)\}_{n \geq 0}$$

is sometimes called the *pattern sequence* for the pattern $w \in \Sigma_q^*$ (cf. Allouche and Shallit [1]). We note that the value $e_2(1; n)$ coincides with the sum of the base-2 digits of n.

Uchida [11] gave necessary and sufficient conditions for algebraic independence over $\mathbb{C}(z)$ of generating functions of pattern sequences in one q-adic

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number system. Recently, Shiokawa and the author [8] obtained similar results for pattern sequences in $\langle q, r\rangle$-number systems ($r = 0, 1, \ldots, q-2$) with a fixed base q. Generating functions and their values defined by digital properties of integers have also been studied in [3], [7], and [9]. In the case of different bases, only special pattern sequences have been discussed; for example Toshimitsu [10] proved that for a given integer b the generating functions of the pattern sequences $\{e_q(b; n)\}_{n\geq 0}$ ($q = b+1, b+2, \ldots$) are algebraically independent over $\mathbb{C}(z)$.

In this paper, for arbitrary given nonzero patterns $w_q \in \Sigma_q^*$ ($q = 2, 3, \ldots$) we prove the algebraic independence of the values of the generating functions

$$\sum_{n\geq 0} e_q(w_q; n) z^n, \qquad q = 2, 3, \ldots,$$

which converge in $|z| < 1$. Furthermore, we derive the algebraic independence over $\mathbb{C}(z)$ of the above generating functions. In particular, the latter implies the linear independence of the pattern sequences in distinct bases (Corollary 1).

Theorem 1. *Let $w_q \in \Sigma_q^*$ ($q \geq 2$) be nonzero patterns and*

$$f_q(z) = \sum_{n\geq 0} e_q(w_q; n) z^n, \qquad q = 2, 3, \ldots. \tag{2}$$

Then for any algebraic number α with $0 < |\alpha| < 1$, their values $\{f_q(\alpha)\}_{q\geq 2}$ are algebraically independent.

Theorem 2. *The generating functions of the pattern sequences* (2) *are algebraically independent over* $\mathbb{C}(z)$.

By Theorem 2, a nontrivial linear combination of the functions (2)

$$c_1 f_2(z) + c_2 f_3(z) + \cdots + c_{m-1} f_m(z)$$

over $\mathbb{C}$ is not a rational function for $|z| < 1$. Hence we obtain the following:

Corollary 1. *Let $w_q \in \Sigma_q^*$ ($q = 2, \ldots, m$) be $m-1$ nonzero patterns and $c_1, \ldots, c_{m-1} \in \mathbb{C}$ not all zero. Then the linear combination of the pattern sequences*

$$\{c_1 e_2(w_2; n) + c_2 e_3(w_3; n) + \cdots + c_{m-1} e_m(w_m; n)\}_{n\geq 0}$$

cannot be a linear recurrence sequence. In particular, the pattern sequences $\{e_q(w_q; n)\}_{n\geq 0}$ ($q = 2, 3, \ldots$) are linearly independent over $\mathbb{C}$.

Example 1. Let $w = b_{l-1} b_{l-2} \cdots b_0$ be a nonzero pattern with $b_i \in \{0, 1\}$. Then the pattern sequences

$$\{e_2(w; n)\}_{n\geq 0}, \quad \{e_3(w; n)\}_{n\geq 0}, \ \ldots, \ \{e_m(w; n)\}_{n\geq 0}, \ \ldots$$

are linearly independent over $\mathbb{C}$. For example, the sequences $\{e_2(10; n)\}_{n\geq 0}$, $\{e_3(10; n)\}_{n\geq 0}$, and $\{e_4(10; n)\}_{n\geq 0}$ which are defined by the number of 10's appearing in the dyadic, 3-ary, and 4-ary expansions of n, respectively, are linearly independent over $\mathbb{C}$.

On the other hand, within one fixed number system, the generating functions can be algebraically dependent over $\mathbb{C}(z)$.

EXAMPLE 2 (Shiokawa and Tachiya [8]). In the usual dyadic expansion, we consider the generating functions

$$f_1(z) = \sum_{n\geq 0} e_2(01;n)z^n, \qquad f_2(z) = \sum_{n\geq 0} e_2(10;n)z^n.$$

Then the sequence $\{e_2(01;n) - e_2(10;n)\}_{n\geq 0} = \{0,1,0,1,\dots\}$ is periodic, and so

$$f_1(z) - f_2(z) = \frac{z}{1-z^2}, \qquad |z| < 1.$$

EXAMPLE 3. Let $w \in \Sigma_q^*$. By the definition of $e_q(w;n)$, we have

$$e_q(w;n) = \sum_{b=0}^{q-1} e_q(bw;n).$$

Therefore the pattern sequences $\{e_q(w;n)\}_{n\geq 0}$, $\{e_q(bw;n)\}_{n\geq 0}$ $(b = 0,1,\dots$ $\dots, q-1)$ are linearly dependent over $\mathbb{C}$, and so are their generating functions.

2. Lemmas. In this section, we prepare some lemmas for proving Theorem 1. Fix an integer $q \geq 2$. For any nonzero pattern $w = b_{l-1}b_{l-2}\cdots b_0 \in \Sigma_q^*$ with $b_i \in \Sigma_q$, let $|w|$ denote the length l and put $\nu(w) = \sum_{k=0}^{l-1} b_k q^k$.

LEMMA 1. *Let $i \geq 1$ be an integer and $w \in \Sigma_{q^i}^*$ be a nonzero pattern. Then for any integer $d \geq 0$, we have*

$$e_{q^i}(w;\nu(w)q^d) = \begin{cases} 1, & i \mid d, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. We put

$$w = 0^l a_k \cdots a_0, \qquad a_j \in \Sigma_{q^i},\ a_k \neq 0,\ k,l \geq 0.$$

Then $\nu(w) = \sum_{j=0}^{k} a_j (q^i)^j$ and $(\nu(w))_{q^i} = a_k \cdots a_0$. Let h and r be integers with

$$d = ih + r, \qquad 0 \leq r < i. \tag{3}$$

First we consider the case that d is divisible by i. Since $r = 0$ in (3), the q^i-ary expansion of the integer $\nu(w)q^d$ is represented as

$$(\nu(w)q^d)_{q^i} = (\nu(w)q^{ih})_{q^i} = a_k \cdots a_0 0^h.$$

It is clear that $e_{q^i}(w;\nu(w)q^d) \geq 1$. If $e_{q^i}(w;\nu(w)q^d) > 1$, we get $w = a_{k-m}\cdots a_0 0^{l+m}$ for some integer m with $1 \leq m \leq k$. Then we have

$$\begin{aligned}\nu(w) &= a_{k-m}q^{i(k+l)} + a_{k-m-1}q^{i(k+l-1)} + \cdots + a_0 q^{i(k+l-(k-m))}\\ &= q^{i(l+m)}(a_{k-m}q^{i(k-m)} + a_{k-m-1}q^{i(k-m-1)} + \cdots + a_0)\\ &= q^{i(l+m)}(\nu(w) - a_k q^{ik} - \cdots - a_{k-m+1}q^{i(k-m+1)}),\end{aligned}$$

so that $(q^{i(l+m)} - 1)\nu(w) \equiv 0 \bmod q^{i(k+l+1)}$. Noting that the integers $q^{i(l+m)} - 1$ and $q^{i(k+l+1)}$ are coprime, we get $\nu(w) \equiv 0 \mod q^{i(k+l+1)}$, that is, $a_j = 0$ for all $j = 0, 1, \ldots, k$. This is a contradiction and hence we obtain $e_{q^i}(w;\nu(w)q^d) = 1$.

Next we consider the case that d is not divisible by i. For the integer $r \geq 1$ defined in (3), we put

$$(\nu(w)q^r)_{q^i} = b_s b_{s-1}\cdots b_0 \in \Sigma_{q^i}^*, \qquad b_j \in \Sigma_{q^i},\ b_s \neq 0,$$

where $s = k,\ k+1$, since

$$k + 1 = |(\nu(w))_{q^i}| \leq |(\nu(w)q^r)_{q^i}| \leq |(\nu(w)q^i)_{q^i}| = |(\nu(w))_{q^i}| + 1 = k + 2.$$

Suppose on the contrary that $e_{q^i}(w;\nu(w)q^d) \neq 0$, that is, the pattern w appears at least once in the q^i-ary expansion of $\nu(w)q^d$:

$$(\nu(w)q^d)_{q^i} = (\nu(w)q^{r+ih})_{q^i} = b_s b_{s-1}\cdots b_0 0^h.$$

Hence, as $b_s \neq 0$, the q^i-ary expansion of w must be of the form either

$$w = 0^l b_s b_{s-1}\cdots b_{s-k}, \tag{4}$$

or

$$w = b_{s-m}b_{s-m-1}\cdots b_{s-m-(k+l)} \tag{5}$$

for some integer m with $1 \leq m \leq s$, where we define $b_j = 0$ for negative j. If the equality (4) is satisfied, we have

$$\nu(w) = b_s q^{ik} + b_{s-1}q^{i(k-1)} + \cdots + b_{s-k} = \begin{cases} \nu(w)q^r, & s = k,\\ q^{-i}(\nu(w)q^r - b_0), & s = k+1.\end{cases}$$

Since $1 \leq r < i$, in any case we can deduce a contradiction. On the other hand, if the case (5) holds, we get

$$\begin{aligned}\nu(w) &= b_{s-m}q^{i(k+l)} + b_{s-m-1}q^{i(k+l-1)} + \cdots + b_{s-m-k-l}\\ &= q^{i(k+l-(s-m))}(b_{s-m}q^{i(s-m)} + \cdots + b_{s-m-k-l}q^{i(s-m-k-l)})\\ &= q^{i(k+l-(s-m))}(\nu(w)q^r - b_s q^{is} - \cdots - b_{s-m+1}q^{i(s-m+1)}),\end{aligned}$$

so that $(q^{r+i(k+l-(s-m))} - 1)\nu(w) \equiv 0 \bmod q^{i(k+l+1)}$. Since $r + i(k + l - (s - m)) \geq 1$, we obtain $\nu(w) \equiv 0 \mod q^{i(k+l+1)}$, which implies $a_j = 0$ for all $j = 0, 1, \ldots, k$. This is a contradiction and the lemma is proved. ∎

Let $m \geq 2$ be an integer. We set

$$S := \{(k_1, \ldots, k_{m-1}) \in \mathbb{Z}_{\geq 0}^{m-1} \mid 0 \leq k_j \leq j-1,\ j = 1, \ldots, m-1\}, \tag{6}$$

$$S_n := \{(k_1, \ldots, k_{m-1}) \in S \mid k_1 + \cdots + k_{m-1} = n\}. \tag{7}$$

LEMMA 2. *For every integer $m \geq 2$, there exist integers d_1 and d_2 with $0 \leq d_1 < d_2 \leq m-1$ such that*

$$\sum_{\substack{n \geq 0 \\ n \equiv d_1 \bmod m}} \sharp S_n \neq \sum_{\substack{n \geq 0 \\ n \equiv d_2 \bmod m}} \sharp S_n,$$

where $\sharp S_n$ is the number of elements of S_n.

Proof. We define

$$f(x) = \frac{1}{(1-x)^{m-1}} \prod_{k=1}^{m-1} (1 - x^k) \in \mathbb{Z}[x].$$

Let ξ be a primitive mth root of unity. Then it is clear that $f(\xi) \neq 0$. Since the polynomial $f(x)$ is expressed as

$$f(x) = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{m-2}) = \sum_{n \geq 0} (\sharp S_n) x^n,$$

we have

$$f(\xi) = c_0 + c_1\xi + \cdots + c_{m-1}\xi^{m-1}, \tag{8}$$

where

$$c_i = \sum_{\substack{n \geq 0 \\ n \equiv i \bmod m}} \sharp S_n, \qquad i = 0, 1, \ldots, m-1.$$

If $c_i = c_j$ for all i, j, then by (8) we get

$$f(\xi) = c_0(1 + \xi + \cdots + \xi^{m-1}) = 0,$$

a contradiction. ∎

LEMMA 3 (Uchida [11]). *Let $d \geq 2$ and $l \geq 1$ be integers. If $c(z) \in \mathbb{C}(z)$ satisfies the functional equation*

$$c(z) = c(z^d) + \frac{(1-z)a(z)}{1 - z^{d^l}}, \qquad a(z) \in \mathbb{C}[z],$$

then there exists $b(z) \in \mathbb{C}[z]$ such that

$$c(z) = \frac{(1-z)b(z)}{1 - z^{d^{l-1}}}.$$

LEMMA 4 (Nishioka [5]). *Let K be an algebraic number field and $d_1, \ldots, d_t \geq 2$ be integers with $\log d_i / \log d_j \notin \mathbb{Q}$ if $i \neq j$. Suppose that $f_{i,j}(z) \in K[[z]]$ $(1 \leq i \leq t, 1 \leq j \leq m_i)$ satisfy the functional equations*

$$f_{i,j}(z^{d_i}) = a_{i,j}(z) f_{i,j}(z) + b_{i,j}(z) \qquad (1 \leq i \leq t,\ 1 \leq j \leq m_i),$$

where $a_{i,j}(z), b_{i,j}(z) \in K(z)$, $a_{i,j}(0) = 1$, *and* $f_{i,1}(z), \dots, f_{i,m_i}(z)$ *are algebraically independent over* $K(z)$ *for each* $i = 1, \dots, t$. *If* α *is an algebraic number with* $0 < |\alpha| < 1$, $a_{i,j}(\alpha^{d_i^k}) \neq 0$ $(k \geq 0)$ *and all* $f_{i,j}(z)$ *converge at* $z = \alpha$, *then the values*

$$f_{i,j}(\alpha) \quad (1 \leq i \leq t, 1 \leq j \leq m_i)$$

are algebraically independent.

Lemma 5 (Kubota [2], Loxton and van der Poorten [4]; see Nishioka [6]). *Let* $d \geq 2$ *be an integer. Suppose that* $g_1(z), \dots, g_n(z) \in \mathbb{C}[[z]]$ *are algebraically dependent over* $\mathbb{C}(z)$ *and satisfy the functional equations*

$$g_i(z^d) = g_i(z) + a_i(z), \qquad a_i(z) \in \mathbb{C}(z),\ i = 1, \dots, n.$$

Then there exist constants $c_1, \dots, c_n \in \mathbb{C}$ *not all zero such that*

$$c_1 g_1(z) + \dots + c_n g_n(z) \in \mathbb{C}(z).$$

3. Proofs of Theorems 1 and 2. Define

$$M = \{q \in \mathbb{N} \mid q \neq a^n \text{ for any } a, n \in \mathbb{N},\ n \geq 2\}.$$

Then

$$\mathbb{N} \setminus \{1\} = \bigcup_{q \in M} \{q, q^2, \dots\} = \{q^j \in \mathbb{N} \mid q \in M, j \geq 1\}.$$

Let $q_1, \dots, q_t \in M$ be distinct integers, $w_{i,j} \in \Sigma^*_{q_i^j}$ $(j = 1, \dots, m_i)$ be nonzero patterns, and

$$f_{i,j}(z) = \sum_{n \geq 0} e_{q_i^j}(w_{i,j}; n) z^n \quad (1 \leq i \leq t,\ 1 \leq j \leq m_i).$$

It is easily seen that $\log q_i / \log q_j \notin \mathbb{Q}$ if $i \neq j$. Then by Theorem 1 in [11] the functional equations

$$f_{i,j}(z) = \frac{1 - z^{q_i^j}}{1 - z} f_{i,j}(z^{q_i^j}) + \frac{z^{\nu(w_{i,j})}}{1 - z^{q_i^{j|w_{i,j}|}}} \quad (1 \leq i \leq t,\ 1 \leq j \leq m_i)$$

are satisfied. Here, putting $F_{i,j}(z) = (1 - z) f_{i,j}(z)$, we have

$$F_{i,j}(z) = F_{i,j}(z^{q_i^j}) + z^{\nu(w_{i,j})} \frac{1 - z}{1 - z^{q_i^{j|w_{i,j}|}}} \quad (1 \leq i \leq t,\ 1 \leq j \leq m_i),$$

and so

$$F_{i,j}(z) = F_{i,j}(z^{q_i^{D_i}}) + \sum_{k=0}^{D_i/j - 1} z^{q^{kj}\nu(w_{i,j})} \frac{1 - z^{q_i^{kj}}}{1 - z^{q_i^{j|w_{i,j}| + kj}}} \quad (1 \leq i \leq t,\ 1 \leq j \leq m_i),$$

where $D_i = \operatorname{lcm}(1, \dots, m_i)$. Hence, if the functions $F_{i,1}(z), \dots, F_{i,m_i}(z)$ are algebraically independent over $\mathbb{C}(z)$ for each $i = 1, \dots, t$, then by Lemma 4 the values $F_{i,j}(\alpha) = (1 - \alpha) f_{i,j}(\alpha)$ $(1 \leq i \leq t,\ 1 \leq j \leq m_i)$ are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$. Therefore, to prove Theorem 1, it is enough to show the algebraic independence over $\mathbb{C}(z)$ of the functions

$$F_i(z) := (1-z)\sum_{n\geq 0} e_{q^i}(w_i;n)z^n, \qquad i = 1,\ldots,m, \tag{9}$$

for any fixed integer $q \geq 2$ and for nonzero patterns $w_i \in \Sigma_{q^i}^*$.

Proof of Theorem 1. Let $q \geq 2$ be a fixed integer and $w_i \in \Sigma_{q^i}^*$ ($i = 1,\ldots,m$) be nonzero patterns. In what follows, we prove the algebraic independence over $\mathbb{C}(z)$ of the functions $F_1(z),\ldots,F_m(z)$ given in (9). We use induction on m. By Theorem 1 in [11] the function $F_1(z)$ is transcendental over $\mathbb{C}(z)$, and hence the claim is satisfied in the case of $m = 1$. Let $m \geq 2$ and assume the claim is true for $m-1$. Towards a contradiction, suppose that the functions $F_1(z),\ldots,F_m(z)$ are algebraically dependent over $\mathbb{C}(z)$. Since they satisfy the functional equations

$$F_i(z) = F_i(z^{q^D}) + \sum_{k=0}^{D/i-1} z^{q^{ki}\nu(w_i)} \frac{1-z^{q^{ki}}}{1-z^{q^{i|w_i|+ki}}}, \qquad i = 1,\ldots,m, \tag{10}$$

with $D = \operatorname{lcm}(1,\ldots,m)$, applying Lemma 5 we see that there exist constants $c_1,\ldots,c_m \in \mathbb{C}$ not all zero such that

$$R(z) := c_1F_1(z) + \cdots + c_mF_m(z) \in \mathbb{C}(z).$$

We may suppose $c_m \neq 0$ from the assumption of induction. Substituting z^{q^D} for z in the above identity and using the functional equation (10), we have

$$R(z) = R(z^{q^D}) + \frac{1-z}{1-z^{q^{Dl}}} \sum_{i=1}^{m}\sum_{k=0}^{D/i-1} c_i z^{q^{ki}\nu(w_i)} \frac{1-z^{q^{ki}}}{1-z} \frac{1-z^{q^{Dl}}}{1-z^{q^{i|w_i|+ki}}}, \tag{11}$$

where $l := \max_{1\leq i\leq m}|w_i|$ and

$$i|w_i| + ki \leq i(l+k) \leq i(l + D/i - 1) \leq Dl.$$

Thus the functional equation (11) can be written as

$$R(z) = R(z^{q^D}) + \frac{(1-z)a(z)}{1-z^{q^{Dl}}} \tag{12}$$

with

$$a(z) = \sum_{i=1}^{m}\sum_{k=0}^{D/i-1} c_i z^{q^{ki}\nu(w_i)} \frac{1-z^{q^{ki}}}{1-z} \frac{1-z^{q^{Dl}}}{1-z^{q^{i|w_i|+ki}}} \in \mathbb{C}[z].$$

Using Lemma 3, we see that there exists $b(z) \in \mathbb{C}[z]$ such that

$$R(z) = \frac{(1-z)b(z)}{1-z^{q^{D(l-1)}}}. \tag{13}$$

Substituting the expression (13) into (12) and multiplying both sides by $(1-z^{q^{Dl}})/(1-z)$, we have

$$\frac{1-z^{q^{Dl}}}{1-z^{q^{D(l-1)}}}\,b(z)=\frac{1-z^{q^{D}}}{1-z}\,b(z^{q^{D}})+a(z),$$

where $\deg a(z)\le q^{Dl}-1$. If the degree of the first term of the right-hand side is not greater than that of the left-hand side, we get $\deg b(z)\le q^{D(l-1)}-1$. Otherwise, the degree of the first term coincides with $\deg a(z)$; then we can deduce $\deg b(z)\le q^{D(l-1)}-1$. In any case, we have

$$\deg b(z)\le q^{D(l-1)}-1. \tag{14}$$

By the expression (13), we have

$$\begin{aligned} b(z)&=\frac{1-z^{q^{D(l-1)}}}{1-z}\,R(z)=(1-z^{q^{D(l-1)}})\sum_{n\ge 0}\sum_{i=1}^{m}c_i e_{q^i}(w_i;n)z^n\\ &=\sum_{n=0}^{q^{D(l-1)}-1}a_n z^n+\sum_{n\ge 0}(a_{n+q^{D(l-1)}}-a_n)z^{n+q^{D(l-1)}}, \end{aligned}$$

where $a_n=\sum_{i=1}^{m}c_i e_{q^i}(w_i;n)$. Therefore by (14) we obtain $a_n=a_{n+q^{D(l-1)}}$ $(n\ge 0)$, so that the sequence

$$\Big\{\sum_{i=1}^{m}c_i e_{q^i}(w_i;n)\Big\}_{n\ge 0} \tag{15}$$

is periodic with period $q^{D(l-1)}$.

Now we prove $c_m=0$ and deduce a contradiction. We choose integers d_1,d_2 $(0\le d_1<d_2\le m-1)$ as in Lemma 2. Define the positive integers

$$N_j=\nu(w_m)\sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_{m-1}=0}^{m-2}q^{k_1+\cdots+k_{m-1}+m-d_j+DL(1+k_1+mk_2+\cdots+m^{m-2}k_{m-1})}$$

for $j=1,2$, where $L>0$ is a sufficiently large integer. Noting that $w_m\in\Sigma_{q^m}^*$ is a nonzero pattern and

$$k_1+mk_2+\cdots+m^{m-2}k_{m-1}\ne k_1'+mk_2'+\cdots+m^{m-2}k_{m-1}'$$

if $(k_1,\ldots,k_{m-1})\ne(k_1',\ldots,k_{m-1}')$, we have

$$e_{q^i}(w_i;N_j)=\sum_{k_1=0}^{0}\cdots\sum_{k_{m-1}=0}^{m-2}e_{q^i}(w_i;\nu(w_m)q^{k_1+\cdots+k_{m-1}+m-d_j+DL}),\qquad j=1,2,$$

for every $i = 1, \ldots, m$. For a fixed integer $i \geq 1$, if s_1 and s_2 are nonnegative integers with $s_1 \equiv s_2 \bmod i$, then the identity

$$e_{q^i}(w_i; \nu(w_m) q^{s_1+DL}) = e_{q^i}(w_i; \nu(w_m) q^{s_2+DL})$$

holds. Hence for each $i = 1, \ldots, m-1$ we get

$$\sum_{k_i=m-d_1}^{i-1+m-d_1} e_{q^i}(w_i; \nu(w_m) q^{k_1+\cdots+k_{m-1}+DL}) = \sum_{k_i=m-d_2}^{i-1+m-d_2} e_{q^i}(w_i; \nu(w_m) q^{k_1+\cdots+k_{m-1}+DL}),$$

so that

$$\begin{aligned} (16) \qquad e_{q^i}(w_i; N_1) &= \sum_{k_1=0}^{0} \cdots \sum_{k_i=m-d_1}^{i-1+m-d_1} \cdots \sum_{k_{m-1}=0}^{m-2} e_{q^i}(w_i; \nu(w_m) q^{k_1+\cdots+k_{m-1}+DL}) \\ &= \sum_{k_1=0}^{0} \cdots \sum_{k_i=m-d_2}^{i-1+m-d_2} \cdots \sum_{k_{m-1}=0}^{m-2} e_{q^i}(w_i; \nu(w_m) q^{k_1+\cdots+k_{m-1}+DL}) \\ &= e_{q^i}(w_i; N_2), \qquad i = 1, \ldots, m-1. \end{aligned}$$

On the other hand, by Lemma 1 we have

$$\begin{aligned} e_{q^m}(w_m; N_j) &= \sum_{k_1=0}^{0} \cdots \sum_{k_{m-1}=0}^{m-2} e_{q^m}(w_m; \nu(w_m) q^{k_1+\cdots+k_{m-1}+m-d_j+DL}) \\ &= \sharp\{(k_1, \ldots, k_{m-1}) \in S \mid k_1 + \cdots + k_{m-1} \equiv d_j \bmod m\} \\ &= \sum_{\substack{n \geq 0 \\ n \equiv d_j \bmod m}} \sharp S_n, \qquad j = 1, 2, \end{aligned}$$

where S and S_n are the sets defined by (6) and (7), respectively. Hence it follows from Lemma 2 that

$$(17) \qquad e_{q^m}(w_m; N_1) \neq e_{q^m}(w_m; N_2).$$

Since the sequence (15) is periodic with period $q^{D(l-1)}$ and $N_1 \equiv N_2 \bmod q^{D(l-1)}$ if L is large, we have

$$\sum_{i=1}^{m} c_i e_{q^i}(w_i; N_1) = \sum_{i=1}^{m} c_i e_{q^i}(w_i; N_2).$$

Combining (16), (17), and the above identity, we obtain $c_m = 0$. This is a contradiction and the proof of Theorem 1 is complete. ∎

Proof of Theorem 2. Suppose that the functions $f_q(z)$ $(q = 2, 3, \dots)$ are algebraically dependent over $\mathbb{C}(z)$, so that

$$\sum_{0 \le i_1, \dots, i_m \le N} a_{i_1,\dots,i_m}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} = 0 \tag{18}$$

with $a_{i_1,\dots,i_m}(z) \in \mathbb{C}[z]$ not all zero. Let $\{\beta_1, \dots, \beta_s\}$ be a maximal subset of the set of all the coefficients of $a_{i_1,\dots,i_m}(z)$ which is linearly independent over $\mathbb{Q}$. Then the polynomials $a_{i_1,\dots,i_m}(z)$ can be written as

$$a_{i_1,\dots,i_m}(z) = \sum_{j=1}^{s} b_{i_1,\dots,i_m,j}(z)\beta_j, \qquad b_{i_1,\dots,i_m,j}(z) \in \mathbb{Q}[z],$$

and so by (18) we have

$$\sum_{j=1}^{s} \Big(\sum_{0 \le i_1, \dots, i_m \le N} b_{i_1,\dots,i_m,j}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} \Big) \beta_j = 0.$$

Since $\beta_1, \dots, \beta_s$ are linearly independent over $\mathbb{Q}$, we get

$$\sum_{0 \le i_1, \dots, i_m \le N} b_{i_1,\dots,i_m,j}(z) f_{q_1}(z)^{i_1} \cdots f_{q_m}(z)^{i_m} = 0$$

for all $j = 1, \dots, s$. Noting that at least one of $b_{i_1,\dots,i_m,j}(z)$ is not zero, we obtain the algebraic dependence over $\mathbb{Q}(z)$ of the functions $f_{q_1}(z), \dots, f_{q_m}(z)$. Hence $f_{q_1}(\alpha), \dots, f_{q_m}(\alpha)$ are algebraically dependent for some algebraic number α with $0 < |\alpha| < 1$. This is a contradiction by Theorem 1. ∎

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