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On the generalization of the D. H. Lehmer problem II

by

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1. Statement of the main result. Let $q \geq 3$ be a positive odd integer. For an integer a with $1 \leq a \leq q$ and $(a,q)=1$, we write $\overline{a}$ for the integer satisfying $1 \leq \overline{a} \leq q$, $a\overline{a} \equiv 1 \pmod q$. The classical Lehmer problem is to study the nontrivial properties of

$$r(q) = \sideset{}{'}\sum_{\substack{1\leq a\leq q\\ 2\nmid a+\overline{a}}} 1,$$

where the dash means that the sum runs through the integers a which are coprime to q. W. P. Zhang ([4], [5]) gave an asymptotic formula for $r(q)$:

$$r(q) = \frac{1}{2}\,\phi(q) + O(q^{1/2}d^2(q)\log^2 q), \tag{1}$$

where $\phi(q)$ and $d(q)$ are the Euler function and divisor function, respectively. In 1994, Zhang [6] proved

$$\begin{aligned} M(q,k) &:= \sideset{}{'}\sum_{\substack{1\leq a\leq q\\ 2\mid a+\overline{a}+1}} (a-\overline{a})^{2k} \\ &= \frac{\phi(q)q^{2k}}{(2k+1)(2k+2)} + O(4^k q^{(4k+1)/2} d^2(q)\log^2 q), \end{aligned} \tag{2}$$

where k is a nonnegative integer.

The formula (1) was recently generalized by the present authors. Let $n \geq 2$ be a fixed positive integer and let $q \geq 3$ and c be two integers with $(n,q)=(c,q)=1$. Denote

$$r_n(\delta_1,\delta_2,c;q) = \sideset{}{'}\sum_{\substack{a\leq\delta_1 q\\ ab\equiv c \,(\mathrm{mod}\, q)\\ n\nmid a+b}} \sideset{}{'}\sum_{b\leq\delta_2 q} 1 \quad (0<\delta_1,\delta_2\leq 1).$$

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Then

$$r_n(\delta_1,\delta_2,c;q) = \left(1-\frac{1}{n}\right)\delta_1\delta_2\phi(q) + O(q^{1/2}d^6(q)\log^2 q). \tag{3}$$

In this paper, we deal with the Lehmer problem weighted by $|a-\overline{a}_c|^\alpha$ for $\alpha > 0$, where $\overline{a}_c$ is the unique integer satisfying $1 \le \overline{a}_c \le q$ and $a\overline{a}_c \equiv c \pmod q$. We will give an asymptotic formula for it:

Theorem. *Let $n \ge 2$ be a fixed positive integer, $q \ge 3$ and c be two integers with $(n,q) = (c,q) = 1$, and d be an integer with $1 \le d \le n$. For any integer M and a positive integer N, we write*

$$\mathcal{L} = \{a : M+1 \le a \le M+N,\ (a,q) = 1\},$$

and for $\alpha > 0$ define

$$S(\alpha,q) := \sum_{\substack{a,b\in\mathcal{L}\\ ab\equiv c\,(\mathrm{mod}\,q)\\ a+b\equiv d\,(\mathrm{mod}\,n)}} |a-b|^\alpha. \tag{4}$$

Then

$$S(\alpha,q) = \frac{2\phi(q)}{(\alpha+1)(\alpha+2)nq^2}N^{\alpha+2} + O(q^{1/2+\varepsilon}N^\alpha(Nq^{-1}+1)), \tag{5}$$

where the O constant depends on n, α and ε.

Note. In the special case of $n = 2$ and $d = 1$ (which amounts to the problem in (2)), we can find that $a \ne b$ in the sum in (4), and thus our result also holds for $\alpha = 0$.

Notation. We set $e(x) = e^{2\pi ix}$. For χ the Dirichlet character modulo q, we denote by $G(m,\chi) = \sum_{n\le q,\,(n,q)=1}\chi(n)e(mn/q)$ the Gauss sum. We write $\|x\|$ for the distance of x from the nearest integer. ε always denotes a sufficiently small positive real number which can be different at each occurrence. We will use $d(q) \ll q^\varepsilon$ throughout the paper without explicit statement.

2. Auxiliary lemmas. All the lemmas introduced here will be used in estimating the error term of $S(\alpha,q)$.

Lemma 1 ([3, §5.1, Lemma 3]). *Assume that U is a positive real number, K_0 is an integer, K is a positive integer, and that α and β are real numbers. If α can be written in the form*

$$\alpha = \frac{s}{r} + \frac{\theta}{r^2}, \qquad (r,s) = 1,\ r \ge 1,\ |\theta| \le 1,$$

we have

$$\sum_{k=K_0+1}^{K_0+K} \min\left(U, \frac{1}{\|\alpha k+\beta\|}\right) \ll \left(\frac{K}{r}+1\right)(U + r\log r).$$

Lemma 2 ([2, Lemma 4]). *Suppose that q and c are integers satisfying $q \ge 3$ and $(c,q)=1$. Then for any integers k_1, k_2, we have*

$$\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}} \overline{\chi}(c)G(k_1,\chi)G(k_2,\chi) \ll \phi(q)q^{1/2}d(q)\min\{(k_1,q),(k_2,q)\}.$$

3. Proof of the Theorem. It is obvious that

$$S(\alpha,q) = 2\sum_{\substack{a,b\in\mathcal{L},\, a<b\\ ab\equiv c \,(\mathrm{mod}\, q)\\ a+b\equiv d\,(\mathrm{mod}\, n)}} |a-b|^\alpha = 2\sum_{a\in\mathcal{L}}\sum_{\substack{l>0,\, a+l\in\mathcal{L}\\ a(a+l)\equiv c\,(\mathrm{mod}\, q)\\ 2a+l\equiv d\,(\mathrm{mod}\, n)}} l^\alpha \tag{6}$$

$$= \frac{2}{\phi(q)}\sum_{a\in\mathcal{L}}\sum_{\substack{l>0,\, a+l\in\mathcal{L}\\ 2a+l\equiv d\,(\mathrm{mod}\, n)}} l^\alpha \sum_{\chi \bmod q}\overline{\chi}(c)\chi(a(a+l))$$

$$= \frac{2}{\phi(q)}\sum_{a\in\mathcal{L}}\sum_{\substack{l>0,\, a+l\in\mathcal{L}\\ 2a+l\equiv d\,(\mathrm{mod}\, n)}} l^\alpha$$

$$+ \frac{2}{\phi(q)}\sum_{\substack{\chi \bmod q\\ \chi\ne\chi^0}}\overline{\chi}(c)\sum_{a\in\mathcal{L}}\sum_{\substack{l>0,\, a+l\in\mathcal{L}\\ 2a+l\equiv d\,(\mathrm{mod}\, n)}} l^\alpha\chi(a(a+l))$$

$$= S+E$$

say. Here

$$S = \frac{2}{\phi(q)}\sum_{a\in\mathcal{L}}\sum_{\substack{l>0,\, a+l\in\mathcal{L}\\ 2a+l\equiv d\,(\mathrm{mod}\, n)}} l^\alpha$$

$$= \frac{2}{\phi(q)}\sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}}\ \sum_{\substack{l\le M+N-a\\ l\equiv d-2a\,(\mathrm{mod}\, n)\\ (a+l,q)=1}} l^\alpha$$

$$= \frac{2}{\phi(q)}\sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}}\ \sum_{m\mid q}\mu(m)\sum_{\substack{l\le M+N-a\\ l\equiv d-2a\,(\mathrm{mod}\, n)\\ l\equiv -a\,(\mathrm{mod}\, m)}} l^\alpha.$$

From $(n,q)=1$ we know that there is a d' such that

$$S = \frac{2}{\phi(q)}\sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}}\ \sum_{m\mid q}\mu(m)\sum_{\substack{l\le M+N-a\\ l\equiv d'\,(\mathrm{mod}\, mn)}} l^\alpha. \tag{7}$$

Partial summation gives

$$\sum_{\substack{l\le M+N-a\\ l\equiv d' \,(\mathrm{mod}\, mn)}} l^{\alpha} = \int_{0}^{M+N-a} u^{\alpha}\, d\Big(\sum_{\substack{l\le u\\ l\equiv d' \,(\mathrm{mod}\, mn)}} 1\Big) = (M+N-a)^{\alpha}\bigg(\frac{M+N-a}{(\alpha+1)mn}+O(1)\bigg), \tag{8}$$

thus

$$\begin{aligned} S &= \frac{2}{\phi(q)} \sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}} \sum_{m|q} \mu(m)(M+N-a)^{\alpha}\bigg(\frac{M+N-a}{(\alpha+1)mn}+O(1)\bigg) \\ &= \frac{2}{(\alpha+1)nq} \sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}} (M+N-a)^{\alpha+1} \\ &\quad + O\bigg(\frac{d(q)}{\phi(q)} \sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}} (M+N-a)^{\alpha}\bigg). \end{aligned} \tag{9}$$

Analogously to (8), we may get

$$\begin{aligned} \sum_{\substack{M+1\le a\le M+N\\ (a,q)=1}} (M+N-a)^{\alpha} &= \sum_{m|q} \mu(m) \sum_{\substack{M+1\le a\le M+N\\ m|a}} (M+N-a)^{\alpha} \\ &= \frac{\phi(q)}{(\alpha+1)q} N^{\alpha+1} + O(N^{\alpha} d(q)). \end{aligned}$$

Combining with (9), this implies that

$$S = \frac{2\phi(q)}{(\alpha+1)(\alpha+2)nq^2} N^{\alpha+2} + O(N^{\alpha+1} q^{-1+\varepsilon}). \tag{10}$$

The remaining task is to estimate E:

$$\begin{aligned} E &= \frac{2}{\phi(q)} \sum_{\substack{\chi \,\mathrm{mod}\, q\\ \chi\ne\chi^0}} \overline{\chi}(c) \sum_{\substack{M+1\le a\le M+N \quad l\le M+N-a\\ 2a+l\equiv d \,(\mathrm{mod}\, n)}} l^{\alpha}\chi(a(a+l)) \\ &= \frac{2}{\phi(q)} \sum_{\substack{\chi \,\mathrm{mod}\, q\\ \chi\ne\chi^0}} \overline{\chi}(c) \sum_{M+1\le a\le M+N} \sum_{l\le M+N-a} l^{\alpha}\chi(a(a+l)) \\ &\quad \times \sum_{j=1}^{n} e\bigg(\Big(\frac{2a+l-d}{n}\Big) j\bigg). \end{aligned}$$

Making use of the identity

$$\chi(a)=\frac{1}{q}\sum_{k=1}^{q-1}G(k,\chi)e\Big(-\frac{ak}{q}\Big)\qquad(\chi\neq\chi^0),$$

we obtain

$$\begin{aligned}E&=\frac{2}{\phi(q)q^2}\sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}}\overline{\chi}(c)\sum_{M+1\le a\le M+N}\ \sum_{l\le M+N-a}l^{\alpha}\sum_{j=1}^{n}e\Big(\Big(\frac{2a+l-d}{n}\Big)j\Big)\\&\quad\times\sum_{k_1\le q-1}G(k_1,\chi)e\Big(-\frac{ak_1}{q}\Big)\sum_{k_2\le q-1}G(k_2,\chi)e\Big(-\frac{a+l}{q}\,k_2\Big)\\&=\frac{2}{\phi(q)q^2}\sum_{j\le n}e\Big(-\frac{d}{n}\,j\Big)\sum_{k_1\le q-1}\ \sum_{k_2\le q-1}\ \sum_{l\le N-1}l^{\alpha}e\Big(\Big(\frac{j}{n}-\frac{k_2}{q}\Big)l\Big)\\&\quad\times\sum_{M+1\le a\le M+N-l}e\Big(\Big(\frac{2j}{n}-\frac{k_1+k_2}{q}\Big)a\Big)\sum_{\substack{\chi \bmod q\\ \chi\neq\chi^0}}\overline{\chi}(c)G(k_1,\chi)G(k_2,\chi).\end{aligned}$$

Lemma 2 yields

$$\begin{aligned}E&\ll q^{-3/2}d(q)\sum_{j\le n}\ \sum_{k_1\le q-1}\ \sum_{k_2\le q-1}\min\{(k_1,q),(k_2,q)\}\\&\quad\times\Big|\sum_{l\le N-1}l^{\alpha}e\Big(\Big(\frac{j}{n}-\frac{k_2}{q}\Big)l\Big)\sum_{M+1\le a\le M+N-l}e\Big(\Big(\frac{2j}{n}-\frac{k_1+k_2}{q}\Big)a\Big)\Big|\\&=q^{-3/2}d(q)\sum_{j\le n}\Big(\sum_{\substack{k_1\le q-1\\ k_1+k_2=q}}\ \sum_{k_2\le q-1}+\sum_{\substack{k_1\le q-1\\ k_1+k_2\neq q}}\ \sum_{k_2\le q-1}\Big)=E_1+E_2\end{aligned}\tag{11}$$

say.

We estimate E_1 first:

$$\begin{aligned}&E_1\ll\\&q^{-3/2}d(q)\sum_{j\le n}\sum_{k_1\le q-1}(k_1,q)\Big|\sum_{l\le N-1}l^{\alpha}e\Big(\Big(\frac{j}{n}+\frac{k_1}{q}\Big)l\Big)\sum_{M+1\le a\le M+N-l}e\Big(\frac{2j}{n}\,a\Big)\Big|\\&=q^{-3/2}d(q)\sum_{\substack{j\le n\\ n\mid 2j}}\sum_{k_1\le q-1}(k_1,q)\Big|\sum_{l\le N-1}(N-l)l^{\alpha}e\Big(\Big(\frac{j}{n}+\frac{k_1}{q}\Big)l\Big)\Big|\\&+q^{-3/2}d(q)\sum_{\substack{j\le n\\ n\nmid 2j}}\sum_{k_1\le q-1}(k_1,q)\Big|\sum_{l\le N-1}l^{\alpha}e\Big(\Big(\frac{j}{n}+\frac{k_1}{q}\Big)l\Big)\frac{e(2j(N-l)/n)-1}{1-e(2j/n)}\Big|.\end{aligned}\tag{12}$$

Notice that $k_1/q \pm l/n \neq 0$ for $j \leq n$ and $k_1 \leq q-1$ since $(n,q)=1$, we obtain by partial summation

$$\sum_{l\leq N-1}(N-l)l^{\alpha}e\biggl(\Bigl(\frac{j}{n}+\frac{k_1}{q}\Bigr)l\biggr) \ll N^{\alpha+1}\frac{1}{\|\frac{k_1}{q}+\frac{j}{n}\|},$$

and also

$$\sum_{l\leq N-1}l^{\alpha}e\biggl(\Bigl(\frac{j}{n}+\frac{k_1}{q}\Bigr)l\biggr)\biggl(e\Bigl(\frac{2j(N-l)}{n}\Bigr)-1\biggr) \ll N^{\alpha}\biggl(\frac{1}{\|\frac{k_1}{q}+\frac{j}{n}\|}+\frac{1}{\|\frac{k_1}{q}-\frac{j}{n}\|}\biggr).$$

Combining these with (12) implies

$$\begin{aligned}(13)\qquad E_1 &\ll q^{-3/2}d(q)N^{\alpha+1}\sum_{\substack{j\leq n\\ n|2j}}\sum_{k_1\leq q-1}(k_1,q)\frac{1}{\|\frac{k_1}{q}+\frac{j}{n}\|}\\ &\quad+q^{-3/2}d(q)N^{\alpha}\sum_{\substack{j\leq n\\ n\nmid 2j}}\frac{1}{\|2j/n\|}\sum_{k_1\leq q-1}(k_1,q)\biggl(\frac{1}{\|\frac{k_1}{q}+\frac{j}{n}\|}+\frac{1}{\|\frac{k_1}{q}-\frac{j}{n}\|}\biggr).\end{aligned}$$

Here

$$\begin{aligned}\sum_{k_1\leq q-1}\frac{(k_1,q)}{\|\frac{k_1}{q}\pm\frac{j}{n}\|} &= \sum_{\substack{m|q\\ m<q}}\sum_{\substack{k_1\leq q-1\\ (k_1,q)=m}}\frac{(k_1,q)}{\|\frac{k_1}{q}\pm\frac{j}{n}\|}\\ &= \sum_{\substack{m|q\\ m<q}}m\sum_{\substack{k\leq (q-1)/m\\ (k,q)=1}}\frac{1}{\|\frac{mk}{q}\pm\frac{j}{n}\|}\\ &= \sum_{\substack{m|q\\ m<q}}m\sum_{h|q}\mu(h)\sum_{k\leq (q-1)/(mh)}\frac{1}{\|\frac{mhk}{q}\pm\frac{j}{n}\|}.\end{aligned}$$

Since $\|mhk/q \pm j/n\| \gg m/q$ with the $\gg$ constant depending on n, we have

$$(14)\qquad \sum_{k_1\leq q-1}\frac{(k_1,q)}{\|\frac{k_1}{q}\pm\frac{j}{n}\|} \ll \sum_{\substack{m|q\\ m<q}}m\sum_{h|q}\sum_{k\leq (q-1)/(mh)}\min\biggl(\frac{q}{m},\frac{1}{\|\frac{mhk}{q}\pm\frac{j}{n}\|}\biggr).$$

We write $mhk/q = k'/q'$ with k'/q' a reduced fraction and $q' \geq 1$. Then obviously, $q/(mh) \leq q' \leq q/m$. With Lemma 1 and (14), this gives

$$\begin{aligned}(15)\qquad \sum_{k_1\leq q-1}\frac{(k_1,q)}{\|\frac{k_1}{q}\pm\frac{j}{n}\|} &\ll \sum_{\substack{m|q\\ m<q}}m\sum_{h|q}\biggl(\frac{(q-1)/(mh)}{q'}+1\biggr)\biggl(\frac{q}{m}+q'\log q'\biggr)\\ &\ll q^{1+\varepsilon}.\end{aligned}$$

Combining this with (13), we have

$$E_1 \ll q^{-1/2+\varepsilon} N^{\alpha+1}. \tag{16}$$

Estimating E_2 proceeds almost in the same way:

$$\begin{aligned} E_2 &= q^{-3/2} d(q) \sum_{j\le n} \sum_{\substack{k_1\le q-1 \\ k_1+k_2\neq q}} \sum_{k_2\le q-1} \min\{(k_1,q),(k_2,q)\} \\ &\quad \times \Big| \sum_{l\le N-1} l^{\alpha} e\Big(\Big(\frac{j}{n}-\frac{k_2}{q}\Big)l\Big) \sum_{M+1\le a\le M+N-l} e\Big(\Big(\frac{2j}{n}-\frac{k_1+k_2}{q}\Big)a\Big)\Big| \\ &= q^{-3/2} d(q) \sum_{j\le n} \sum_{\substack{k_1\le q-1 \\ k_1+k_2\neq q}} \sum_{k_2\le q-1} \min\{(k_1,q),(k_2,q)\} \\ &\quad \times \Big| \sum_{l\le N-1} l^{\alpha} e\Big(\Big(\frac{j}{n}-\frac{k_2}{q}\Big)l\Big) \frac{e\big(\big(\frac{2j}{n}-\frac{k_1+k_2}{q}\big)(N-l)\big)-1}{1-e\big(\frac{2j}{n}-\frac{k_1+k_2}{q}\big)}\Big| \\ &\ll q^{-3/2} d(q) \sum_{j\le n} \sum_{\substack{k_1\le q-1 \\ k_1+k_2\neq q}} \sum_{k_2\le q-1} \frac{\min\{(k_1,q),(k_2,q)\}}{\big\|\frac{k_1+k_2}{q}-\frac{2j}{n}\big\|} \\ &\quad \times \Big\{ \Big| \sum_{l\le N-1} l^{\alpha} e\Big(\Big(\frac{k_1}{q}-\frac{j}{n}\Big)l\Big)\Big| + \Big| \sum_{l\le N-1} l^{\alpha} e\Big(\Big(\frac{j}{n}-\frac{k_2}{q}\Big)l\Big)\Big| \Big\} \\ &\ll q^{-3/2} d(q) N^{\alpha} \sum_{j\le n} \sum_{\substack{k_1\le q-1 \\ k_1+k_2\neq q}} \sum_{k_2\le q-1} \frac{\min\{(k_1,q),(k_2,q)\}}{\big\|\frac{k_1+k_2}{q}-\frac{2j}{n}\big\|} \\ &\quad \times \Bigg(\frac{1}{\big\|\frac{k_1}{q}-\frac{j}{n}\big\|} + \frac{1}{\big\|\frac{k_2}{q}-\frac{j}{n}\big\|} \Bigg) \\ &\ll q^{-3/2} d(q) N^{\alpha} \sum_{j\le n} \sum_{k_1\le q-1} \frac{(k_1,q)}{\big\|\frac{k_1}{q}-\frac{j}{n}\big\|} \sum_{\substack{k_2\le q-1 \\ k_1+k_2\neq q}} \frac{1}{\big\|\frac{k_1+k_2}{q}-\frac{2j}{n}\big\|}. \end{aligned} \tag{17}$$

Making use of (15), we have

$$E_2 \ll q^{1/2+\varepsilon} N^{\alpha}. \tag{18}$$

By (11), (16) and (18), we obtain $E \ll q^{1/2+\varepsilon} N^{\alpha}(Nq^{-1}+1)$. With (6) and (10), this establishes the theorem.

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