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LOCAL WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE GENERALIZED CAMASSA–HOLM EQUATION IN BESOV SPACES

Abstract. We study local well-posedness of the Cauchy problem for the generalized Camassa–Holm equation $\partial_t u - \partial^3_{txx} u + 2\kappa\partial_x u + \partial_x[g(u)/2] = \gamma(2\partial_x u\partial^2_{xx} u + u\partial^3_{xxx} u)$ for the initial data $u_0(x)$ in the Besov space $B^s_{p,r}(\mathbb{R})$ with $\max(3/2, 1+1/p) < s \le m$ and $(p,r) \in [1,\infty]^2$, where $g : \mathbb{R} \to \mathbb{R}$ is a given C^m-function ($m \ge 4$) with $g(0) = g'(0) = 0$, and $\kappa \ge 0$ and $\gamma \in \mathbb{R}$ are fixed constants. Using estimates for the transport equation in the framework of Besov spaces, compactness arguments and Littlewood–Paley theory, we get a local well-posedness result.

1. Introduction. In this paper we study the Cauchy problem for the generalized Camassa–Holm equation

$$(1.1)\qquad \begin{cases} \partial_t u - \partial^3_{txx} u + 2\kappa\partial_x u + \partial_x[g(u)/2] \\ \qquad\qquad = \gamma(2\partial_x u\partial^2_{xx} u + u\partial^3_{xxx} u), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases}$$

for the initial data $u_0(x)$ in the Besov space $B^s_{p,r}(\mathbb{R})$ with $\max(3/2, 1+1/p) < s \le m$ and $(p,r) \in [1,\infty]^2$, where $g : \mathbb{R} \to \mathbb{R}$ is a given C^m-function ($m \ge 4$) with $g(0) = g'(0) = 0$, and $\kappa \ge 0$ and $\gamma \in \mathbb{R}$ are fixed constants.

If $g(u) = 3u^2$ and $\gamma = 1$, then (1.1) is the classical Camassa–Holm equation, derived independently by R. Camassa and D. Holm in [2], and by A. Fokas and B. Fuchssteiner in [16]. The classical Camassa–Holm equation models the unidirectional propagation of shallow water waves over a flat bottom; $u(t,x)$ stands for the fluid velocity at time $t \ge 0$ in the spatial x-direction and κ is a nonnegative parameter related to the critical

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shallow water speed. The classical Camassa–Holm equation possesses a bi-Hamiltonian structure and infinitely many conservation laws [2, 16], and is completely integrable [2, 4, 9]. Moreover, when $\kappa = 0$ it has an infinite number of solitary wave solutions, called peakons due to the discontinuity of their first derivatives at the wave peak, interacting like solitons:

$$u(t,x) = ce^{-|x-ct|}, \qquad c \in \mathbb{R}.$$

The Cauchy problem for the Camassa–Holm equation has been extensively studied (see [3, 5, 6, 8, 10, 14, 15, 19, 20]).

For $g(u) = 3u^2$, $\kappa = 0$ and $\gamma \in \mathbb{R}$, Dai [11–13] derived (1.1) as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyper-elastic rods, $u(t,x)$ representing the radial stretch relative to a prestressed state and γ being a parameter related to the material constants and the prestress of the rod. Moreover, if $\gamma = 0$, equation (1.1) becomes the regularized wave equation describing surface waves in a channel [1].

From the mathematical viewpoint equation (1.1) has been much less studied than the classical Camassa–Holm equation. Recently, Yin [21–23] (see also Constantin and Escher [7]) proved local well-posedness, and global well-posedness for a particular class of initial data; in particular, he showed that smooth solutions blow up in finite time for a large class of initial data.

In this paper, we study the generalized Camassa–Holm equation (1.1) in the framework of Besov spaces. Making use of some estimates for the transport equation in Besov spaces, compactness arguments and Littlewood—Paley theory, we get local well-posedness in the sense of Hadamard, i.e., (1.1) has a unique local solution in a suitable functional setting, and the solution is continuous with respect to the initial data.

Throughout this paper, we shall assume that $u_0 \in B^s_{p,r}$ with $\max(3/2, 1+1/p) < s \le m$ and $\gamma > 0$. Observe that the case $\gamma = 0$ is much simpler than the one we are considering. Moreover, if $\gamma < 0$, we can use a similar argument.

Applying the pseudo-differential operator $(1-\partial^2_{xx})^{-1}$ to (1.1), we can rewrite (1.1) as follows:

$$\begin{cases} \partial_t u + \gamma u \partial_x u = P(D)\Big(\dfrac{g(u)}{2} + \dfrac{\gamma}{2}(\partial_x u)^2 - \dfrac{\gamma}{2}u^2 + 2\kappa u\Big), \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

with $P(D) = -\partial_x(1-\partial^2_{xx})^{-1}$.

To state our results, we need the following function spaces:

$$E^s_{p,r}(T) \triangleq \begin{cases} \mathcal{C}([0,T]; B^s_{p,r}) \cap \mathcal{C}^1([0,T]; B^{s-1}_{p,r}) & \text{if } 1 \le r < \infty, \\ L^\infty(0,T; B^s_{p,\infty}) \cap \mathrm{Lip}([0,T]; B^{s-1}_{p,\infty}) & \text{if } r = \infty, \end{cases}$$

where $T > 0$, $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$, and Lip is the space of continuous bounded functions with bounded first derivatives.

Our main results are the following theorems.

THEOREM 1.1 (Local well-posedness). *Let* $(p,r) \in [1,\infty]^2$ *and* $\max(3/2, 1+1/p) < s \le m$. *Let* $u_0 \in B^s_{p,r}(\mathbb{R})$. *Then there exists a time* $T > 0$ *such that* (1.1) *or equivalently* (1.2) *has a unique solution in* $E^s_{p,r}(T)$.

THEOREM 1.2 (Energy conservation). *Let* s,p,r *be as in Theorem* 1.1. *Let* $u \in E^s_{p,r}(T)$ *be a solution of* (1.1) *or* (1.2) *on* $[0,T] \times \mathbb{R}$ *with data* $u_0 \in B^s_{p,r} \cap H^1$. *Then*

$$\forall t \in [0,T], \qquad \|u(t)\|_{H^1} = \|u_0\|_{H^1}. \tag{1.3}$$

THEOREM 1.3 (Blow-up criterion). *Denote by* $T^\star_{u_0}$ *the maximal lifespan of the solution with data* u_0. *Under the assumptions of Theorem* 1.1, *if* $T^\star_{u_0} < \infty$, *then*

$$\int_0^{T^\star_{u_0}} \|\partial_x u(\tau)\|_{L^\infty}\, d\tau = \infty. \tag{1.4}$$

This paper is arranged as follows. In Section 2, we introduce some definitions and properties of nonhomogeneous Besov spaces and Littlewood–Paley decomposition. In Section 3, we introduce some estimates for transport equations in the framework of Besov spaces, and prove some estimates for the generalized Camassa–Holm equation. In Section 4, using the results derived in Section 3 and compactness arguments, we prove Theorem 1.1. In Section 5, we prove Theorems 1.2 and 1.3.

2. Besov spaces and Littlewood–Paley decomposition. The proofs of our results are based on a dyadic partition of unity in Fourier variables, the so-called *nonhomogeneous Littlewood–Paley decomposition.*

Let (χ,φ) be a couple of smooth functions valued in $[0,1]$ such that χ is supported in the ball $\{\xi \in \mathbb{R}^n \mid |\xi| \le 4/3\}$, φ is supported in the shell $\{\xi \in \mathbb{R}^n \mid 3/4 \le |\xi| \le 8/3\}$ and

$$\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \qquad \forall \xi \in \mathbb{R}^n.$$

Writing $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$ and $\widetilde{h} = \mathcal{F}^{-1}\chi$, we define the dyadic blocks as

$$\varDelta_q u \triangleq \begin{cases} 0 & \text{if } q \le -2,\\ \chi(D)u = \int_{\mathbb{R}^n} \widetilde{h}(y)u(x-y)\,dy & \text{if } q = -1,\\ \varphi(2^{-q}D)u = \int_{\mathbb{R}^n} h_q(y)u(x-y)\,dy & \text{if } q \ge 0.\end{cases}$$

We shall also use the following low-frequency cut-off:

$$S_q u \triangleq \sum_{q' \le q-1} \Delta_{q'} u = \chi(2^{-q} D) u.$$

The formal equality

$$u = \sum_{q \ge -1} \Delta_q u \tag{2.1}$$

holds in $\mathcal{S}'(\mathbb{R}^n)$ and is called the *nonhomogeneous Littlewood–Paley decomposition*. It has nice properties of quasi-orthogonality:

$$\Delta_{q'} \Delta_q u \equiv 0 \quad \text{if } |q'-q| \ge 2, \qquad \Delta_{q'}(S_{q-1} u \Delta_q v) \equiv 0 \quad \text{if } |q'-q| \ge 5. \tag{2.2}$$

Let us now define the nonhomogeneous Besov spaces:

Definition 2.1. For $s \in \mathbb{R}$, $(p,r) \in [1,\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we set

$$\|u\|_{B^s_{p,r}} \triangleq \Big(\sum_{q \ge -1} (2^{qs} \|\Delta_q u\|_{L^p})^r \Big)^{1/r} \qquad \text{if } 1 \le r < \infty$$

and

$$\|u\|_{B^s_{p,\infty}} \triangleq \sup_{q \ge -1} 2^{qs} \|\Delta_q u\|_{L^p}.$$

We then define the *nonhomogeneous Besov spaces* as

$$B^s_{p,r} \triangleq \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \|u\|_{B^s_{p,r}} < \infty \}.$$

The above definition does not depend on the choice of the couple (χ, φ).

For a more complete study of Besov spaces, we refer to [18, 17]. Let us just recall some basic properties.

Proposition 2.1 ([14]). *The following properties hold*:

1. Density: *the space $\mathcal{C}^\infty_c$ of smooth functions with compact support is dense in $B^s_{p,r}$ if and only if p and r are finite.*
2. $B^s_{2,2} = H^s$.
3. Generalized derivatives: *if f is a smooth function on $\mathbb{R}^n \backslash \{0\}$ which is homogeneous of degree m, then $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*
4. Sobolev embeddings: *if $p_1 \le p_2$ and $r_1 \le r_2$, then we have $B^s_{p_1,r_1} \hookrightarrow B^{s-n(1/p_1-1/p_2)}_{p_2,r_2}$; if $s_1 < s_2$ and $(p, r_1, r_2) \in [1,\infty]^3$, then the embedding $B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1}$ is locally compact.*
5. $B^0_{\infty,1} \hookrightarrow L^\infty \hookrightarrow B^0_{\infty,\infty}$.
6. $B^1_{\infty,1} \hookrightarrow \mathrm{Lip} \hookrightarrow B^1_{\infty,\infty}$.
7. Algebraic properties: *for $s > 0$ the intersection $B^s_{p,r} \cap L^\infty$ is an algebra with respect to pointwise multiplication. Moreover, ($B^s_{p,r}$ is an algebra) $\Leftrightarrow$ ($B^s_{p,r} \hookrightarrow L^\infty$) $\Leftrightarrow$ (either $s > n/p$, or $s = n/p$ and $r = 1$).*
8. Real interpolation: $\|u\|_{B^{\theta s_1 + (1-\theta) s_2}_{p,r}} \le \|u\|^\theta_{B^{s_1}_{p,r}} \|u\|^{1-\theta}_{B^{s_2}_{p,r}}$ *for $\theta \in [0,1]$.*

9. Fatou property: *if a sequence $(f_k)_{k\in\mathbb{N}}$ is uniformly bounded in $B^s_{p,r}$ and converges weakly in $\mathcal{S}'$ to f, then $f \in B^s_{p,r}$ and*
$$\|f\|_{B^s_{p,r}} \le \liminf_{k\to\infty} \|f_k\|_{B^s_{p,r}}.$$

We have the following continuity properties for the product of two functions:

PROPOSITION 2.2 ([14]). *If $(p,r) \in [1,\infty]^2$ and $s > 0$, then there exists a positive constant $C = C(n,s)$ such that*
$$\|uv\|_{B^s_{p,r}} \le C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}). \tag{2.3}$$
If $(p,r) \in [1,\infty]^2$, $s_1, s_2 < n/p$ if $r > 1$ ($s_1, s_2 \le n/p$ if $r = 1$) and $s_1 + s_2 > 0$, then there exists a positive constant $C = C(s_1,s_2,p,r,n)$ such that
$$\|uv\|_{B^{s_1+s_2-n/p}_{p,r}} \le C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}. \tag{2.4}$$
If $(p,r) \in [1,\infty]^2$, $s_1 \le n/p$, $s_2 > n/p$ ($s_2 \ge n/p$ if $r = 1$) and $s_1 + s_2 > 0$, then there exists a positive constant $C = C(s_1,s_2,p,r,n)$ such that
$$\|uv\|_{B^{s_1}_{p,r}} \le C\|u\|_{B^{s_1}_{p,r}}\|v\|_{B^{s_2}_{p,r}}. \tag{2.5}$$

We also have the following two important results (cf. [18]):

PROPOSITION 2.3. *Let I be an open interval of $\mathbb{R}$. Let $\sigma > 0$ and $\widetilde{\sigma}$ be the least integer such that $\widetilde{\sigma} \ge \sigma$. Let $g : I \to \mathbb{R}$ satisfy $g(0) = 0$ and $g' \in W^{\widetilde{\sigma},\infty}(I;\mathbb{R})$. Assume that $u \in B^\sigma_{p,r}$ has values in $J \subset\subset I$. Then $g(u) \in B^\sigma_{p,r}$ and there exists a constant C depending only on σ, I, J and n, such that*
$$\|g(u)\|_{B^\sigma_{p,r}} \le C(1 + \|u\|_{L^\infty})^{\widetilde{\sigma}}\|g'\|_{W^{\widetilde{\sigma},\infty}(I)}\|u\|_{B^\sigma_{p,r}}. \tag{2.6}$$

PROPOSITION 2.4. *Let I be an open interval of $\mathbb{R}$. Let $\sigma > 0$ and $\widetilde{\sigma}$ be the least integer such that $\widetilde{\sigma} \ge \sigma$. Let $g : I \to \mathbb{R}$ satisfy $g'(0) = 0$ and $g'' \in W^{\widetilde{\sigma},\infty}(I;\mathbb{R})$. Assume that $u, v \in B^\sigma_{p,r}$ have values in $J \subset\subset I$. Then there exists a constant C, depending only on σ, I, J and n, such that*
$$\begin{aligned}\|g(u) - g(v)\|_{B^\sigma_{p,r}} \le{}& C(1 + \|u\|_{L^\infty})^{\widetilde{\sigma}}\|g''\|_{W^{\widetilde{\sigma},\infty}(I)} \\ &\cdot \Big(\|u - v\|_{B^\sigma_{p,r}} \sup_{\theta\in[0,1]} \|v + \theta(u - v)\|_{L^\infty} \\ &+ \|u - v\|_{L^\infty} \sup_{\theta\in[0,1]} \|v + \theta(u - v)\|_{B^\sigma_{p,r}}\Big).\end{aligned} \tag{2.7}$$

3. Linear estimates. First, consider the following linear transport equation:
$$\begin{cases} \partial_t f + a \cdot \nabla f = F, \\ f|_{t=0} = f_0, \end{cases} \tag{3.1}$$
where $a : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ stands for a given time dependent vector field, $f_0 : \mathbb{R}^n \to \mathbb{R}^{n'}$ and $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n'}$ are known data.

For (3.1), we have the following a priori estimates (for the proof we refer to [14]):

LEMMA 3.1 (a priori estimates). *Let* $1 \le p \le p_1 \le \infty$, $1 \le r \le \infty$, $p' = (1-1/p)^{-1}$. *Assume that*

$$\sigma > -n\min\left(\frac{1}{p_1},\frac{1}{p'}\right), \quad \text{or} \quad \sigma > -1-n\min\left(\frac{1}{p_1},\frac{1}{p'}\right) \quad \text{if } \operatorname{div} a = 0. \tag{3.2}$$

There exists a constant $C_1 > 0$, *depending only on* n, p, p_1, r *and* σ, *such that*

$$\|f\|_{L^\infty(0,t;B^\sigma_{p,r})} \le \Big(\|f_0\|_{B^\sigma_{p,r}} + \int_0^t e^{-C_1\int_0^\tau Z(\tau')\,d\tau'}\|F(\tau)\|_{B^\sigma_{p,r}}\,d\tau\Big)e^{C_1\int_0^t Z(\tau)\,d\tau}, \tag{3.3}$$

with

$$Z(t) = \begin{cases} \|\nabla a(t)\|_{B^{n/p_1}_{p_1,\infty}\cap L^\infty} & \text{if } \sigma < 1+n/p_1, \\ \|\nabla a(t)\|_{B^{\sigma-1}_{p_1,r}} & \text{if } \sigma > 1+n/p_1 \\ & \quad \text{or } \{\sigma = 1+n/p_1 \text{ and } r = 1\}. \end{cases} \tag{3.4}$$

If $f = a$ *then for all* $\sigma > 0$, *or* $\sigma > -1$ *if* $\operatorname{div} a = 0$, *estimate* (3.3) *holds with* $Z(t) = \|\nabla a(t)\|_{L^\infty}$.

Concerning the local well-posedness of the transport equation (3.1), we also have the following lemma (the proof is in [14]):

LEMMA 3.2. *Let* p, p_1, r *and* σ *be as in the statement of Lemma* 3.1. *Let* $f_0 \in B^\sigma_{p,r}$ *and* $F \in L^1(0,T;B^\sigma_{p,r})$. *Let* a *be a time dependent vector field with coefficients in* $L^\varrho(0,T;B^{-M}_{\infty,\infty})$ *for some* $\varrho > 1$ *and* $M > 0$, *and such that* $\nabla a \in L^1(0,T;B^{n/p_1}_{p_1,\infty}\cap L^\infty)$ *if* $\sigma < 1+n/p_1$, *and* $\nabla a \in L^1(0,T;B^{\sigma-1}_{p_1,r})$ *if* $\sigma > 1+n/p_1$ *or* $\sigma = 1+n/p_1$ *and* $r = 1$. *Then the Cauchy problem* (3.1) *has a unique solution* $f \in L^\infty(0,T;B^\sigma_{p,r}) \cap \bigcap_{\sigma'<\sigma}\mathcal{C}([0,T];B^{\sigma'}_{p,1})$ *and the inequality* (3.3) *holds true. If in addition* $r < \infty$ *then* $f \in \mathcal{C}([0,T];B^\sigma_{p,r})$.

Now, let us estimate the right-hand side of the generalized Camassa–Holm equation (1.2).

LEMMA 3.3. *Let* $(p,r) \in [1,\infty]^2$ *and* $\sigma > \max(1/2,1/p)$. *Then*:

1. *The function* $G : u \mapsto P(D)\frac{g(u)}{2}$ *is continuous from* $B^\sigma_{p,r}$ *to* $B^{\sigma+1}_{p,r}$.
2. *The function* $H : (u,v) \mapsto P(D)\big(\frac{\gamma}{2}\partial_x u\partial_x v - \frac{\gamma}{2}uv\big)$ *is continuous from* $B^\sigma_{p,r}\times B^{\sigma+1}_{p,r}$ *to* $B^\sigma_{p,r}$.
3. *The function* $K : u \mapsto P(D)(2\kappa u)$ *is continuous from* $B^\sigma_{p,r}$ *to* $B^{\sigma+1}_{p,r}$.

Proof. 1. Applying Proposition 2.3 and the fact that $P(D)$ is a multiplier of order -1, we immediately get the first conclusion.

2. When $\sigma > 1/p$, $B^{\sigma}_{p,r}$ is an algebra by Proposition 2.1. Therefore,

$$\begin{aligned} \Big\|-\frac{\gamma}{2}\,uv\Big\|_{B^{\sigma-1}_{p,r}} &\lesssim \Big\|-\frac{\gamma}{2}\,uv\Big\|_{B^{\sigma}_{p,r}} \lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma}_{p,r}} \\ &\lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma+1}_{p,r}}. \end{aligned} \tag{3.5}$$

If $\sigma > 1 + 1/p$, or $\sigma \ge 1 + 1/p$ and $r = 1$, then $B^{s-1}_{p,r}$ is also an algebra so that

$$\begin{aligned} \Big\|\frac{\gamma}{2}\,\partial_x u\partial_x v\Big\|_{B^{\sigma-1}_{p,r}} &\lesssim \gamma\|\partial_x u\|_{B^{\sigma-1}_{p,r}}\|\partial_x v\|_{B^{\sigma-1}_{p,r}} \lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma}_{p,r}} \\ &\lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma+1}_{p,r}}. \end{aligned} \tag{3.6}$$

Otherwise, we still have $\sigma > 1/p$ and $\sigma > 1/2$. According to (2.5), we have

$$\Big\|\frac{\gamma}{2}\,\partial_x u\partial_x v\Big\|_{B^{\sigma-1}_{p,r}} \lesssim \gamma\|\partial_x u\|_{B^{\sigma-1}_{p,r}}\|\partial_x v\|_{B^{\sigma}_{p,r}} \lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma+1}_{p,r}}. \tag{3.7}$$

Combining (3.5) with (3.6) or (3.7), we get

$$\Big\|-\frac{\gamma}{2}\,uv\Big\|_{B^{\sigma-1}_{p,r}} + \Big\|\frac{\gamma}{2}\,\partial_x u\partial_x v\Big\|_{B^{\sigma-1}_{p,r}} \lesssim \gamma\|u\|_{B^{\sigma}_{p,r}}\|v\|_{B^{\sigma+1}_{p,r}}.$$

Therefore the second conclusion is proved.

3. The third conclusion is obvious. ■

4. Local well-posedness. In this section we make use of the results derived in Section 3, compactness arguments and Littlewood–Paley theory to prove the local well-posedness of the Cauchy problem for the generalized Camassa–Holm equation (1.1) or (1.2) in Besov spaces.

Uniqueness and continuity with respect to the initial data are a corollary of the following:

PROPOSITION 4.1. *Let $(p,r) \in [1,\infty]^2$ and $\max(3/2, 1+1/p) < s \le m$. Suppose that $(u,v) \in (L^\infty(0,T; B^s_{p,r}) \cap \mathcal{C}([0,T]; B^{s-1}_{p,r}))^2$ are two solutions of (1.1) or (1.2) with initial data $u_0, v_0 \in B^s_{p,r}$. Then there exists a constant $C_2 > 0$ such that, for all $t \in [0,T]$,*

$$\begin{aligned} &\|u(t) - v(t)\|_{B^{s-1}_{p,r}} \\ &\quad\le \|u_0 - v_0\|_{B^{s-1}_{p,r}} \exp\Big(C_2(\gamma+1)\int_0^t (\kappa + 1 + \|u(\tau)\|_{B^s_{p,r}} + \|v(\tau)\|_{B^s_{p,r}})^m\,d\tau\Big). \end{aligned} \tag{4.1}$$

Proof. Let $w \triangleq u - v$. Then w satisfies the following equation:

$$\partial_t w + \gamma u \partial_x w = -\gamma w \partial_x v + P(D)\Big(\frac{g(u)-g(v)}{2} + \frac{\gamma}{2}\,\partial_x w \partial_x(u+v) - \frac{\gamma}{2}\,w(u+v) + 2\kappa w\Big).$$

According to Lemma 3.1 and the Sobolev embedding $B^{s-1}_{p,r} \hookrightarrow B^{s-2}_{p,r}$, we have

$$\|w(t)\|_{B^{s-1}_{p,r}} \leq \|w_0\|_{B^{s-1}_{p,r}} e^{C_1\gamma \int_0^t \|\partial_x u\|_{B^{s-1}_{p,r}}\,d\tau'} + \int_0^t e^{C_1\gamma \int_\tau^t \|\partial_x u\|_{B^{s-1}_{p,r}}\,d\tau'} \Big(\gamma \|w\partial_x v\|_{B^{s-1}_{p,r}} + \Big\|P(D)\Big(\frac{g(u)-g(v)}{2} + \frac{\gamma}{2}\,\partial_x w \partial_x(u+v) - \frac{\gamma}{2}\,w(u+v) + 2\kappa w\Big)\Big\|_{B^{s-1}_{p,r}}\Big)\,d\tau. \tag{4.2}$$

Since $\max(3/2, 1+1/p) < s \leq m$, by Proposition 2.4 and Lemma 3.3 there exists a constant $C_3 > 0$ such that

$$\Big\|P(D)\Big(\frac{g(u)-g(v)}{2} + \frac{\gamma}{2}\,\partial_x w \partial_x(u+v\Big) - \frac{\gamma}{2}\,w(u+v) + 2\kappa w)\Big\|_{B^{s-1}_{p,r}} \leq C_3((1 + \|u\|_{B^{s-1}_{p,r}} + \|v\|_{B^{s-1}_{p,r}})^{\widetilde{s-1}} + \gamma(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}) + \kappa)\|w\|_{B^{s-1}_{p,r}}. \tag{4.3}$$

Plugging (4.3) in (4.2) and using the fact that $B^{s-1}_{p,r}$ is an algebra, we have

$$\begin{aligned}\|w(t)\|_{B^{s-1}_{p,r}} \leq \|w_0\|_{B^{s-1}_{p,r}} e^{C_1\gamma \int_0^t \|\partial_x u\|_{B^{s-1}_{p,r}}\,d\tau'} &+ \int_0^t e^{C_1\gamma \int_\tau^t \|\partial_x u\|_{B^{s-1}_{p,r}}\,d\tau'}\\ &\cdot (C_4\gamma\|\partial_x v\|_{B^{s-1}_{p,r}} + C_3((1 + \|u\|_{B^{s-1}_{p,r}} + \|v\|_{B^{s-1}_{p,r}})^{\widetilde{s-1}}\\ &+ \gamma(\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}}) + \kappa))\|w(\tau)\|_{B^{s-1}_{p,r}}\,d\tau,\end{aligned}$$

with a constant $C_4 > 0$. Therefore, there exists a constant $C_5 > 0$ such that

$$\begin{aligned}\|w(t)\|_{B^{s-1}_{p,r}} \leq \|w_0\|_{B^{s-1}_{p,r}} &e^{C_1(\gamma+1)\int_0^t (\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}\\ &+ C_5(\gamma+1)\int_0^t e^{C_1(\gamma+1)\int_\tau^t (\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}\\ &\cdot (\kappa + 1 + \|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^m \|w(\tau)\|_{B^{s-1}_{p,r}}\,d\tau.\end{aligned}$$

Thus, we have

$$e^{-C_1(\gamma+1)\int_0^t(\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}\|w(t)\|_{B^{s-1}_{p,r}}$$
$$\le \|w_0\|_{B^{s-1}_{p,r}} + C_5(\gamma+1)\int_0^t(\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m$$
$$\cdot\, e^{-C_1(\gamma+1)\int_0^\tau(\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}\|w(\tau)\|_{B^{s-1}_{p,r}}\,d\tau.$$

Applying the Gronwall lemma, we obtain

$$e^{-C_1(\gamma+1)\int_0^t(\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}\|w(t)\|_{B^{s-1}_{p,r}}$$
$$\le \|w_0\|_{B^{s-1}_{p,r}} e^{C_5(\gamma+1)\int_0^t(\kappa+1+\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}})^m\,d\tau'}.$$

Letting $C_2 \triangleq C_1 + C_5$, we get (4.1). ∎

Proof of Theorem 1.1. The uniqueness is an immediate consequence of Proposition 4.1.

Now, let us prove the existence. We will use a standard iterative process to construct a solution.

Step 1: *approximate solution.* Starting from $u^0 \triangleq 0$, we define recurrently a sequence $(u^i)_{i\in\mathbb{N}}$ of smooth functions by solving the following linear transport equation:

$$\begin{cases} (\partial_t + \gamma u^i\partial_x)u^{i+1} = P(D)\Big(\dfrac{g(u^i)}{2} + \dfrac{\gamma}{2}(\partial_x u^i)^2 - \dfrac{\gamma}{2}(u^i)^2 + 2\kappa u^i\Big), \\ u^{i+1}(0,x) = u_0^{i+1}(x) \triangleq S_{i+1}u_0. \end{cases} \tag{4.4}$$

Since all the data belong to $B^m_{p,r}$, Lemma 3.2 enables us to show by induction that for all $i \in \mathbb{N}$, (4.4) has a global solution which belongs to $\mathcal{C}(\mathbb{R}^+; B^m_{p,r})$.

Step 2: *uniform bounds.* According to Lemma 3.2, Proposition 2.3 and Lemma 3.3, we have the following inequality for all $i \in \mathbb{N}$:

$$\|u^{i+1}(t)\|_{B^s_{p,r}} \le C_6 e^{C_6(\gamma+1)U^i(t)}\Big(\|u_0\|_{B^s_{p,r}} + (\gamma+1)\int_0^t e^{-C_6(\gamma+1)U^i(\tau)} \cdot (\kappa+1+\|u^i(\tau)\|_{B^s_{p,r}})^m\|u^i(\tau)\|_{B^s_{p,r}}\,d\tau\Big), \tag{4.5}$$

with $U^i(t) \triangleq \int_0^t \|u^i(\tau)\|_{B^s_{p,r}}\,d\tau$ and a constant $C_6 > 1$.

Fix a $T > 0$ such that $2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m T < 1$ and suppose that

$$(\kappa+1+\|u^i(t)\|_{B^s_{p,r}})^m \le \frac{C_6^m(\kappa+1+\|u_0\|_{B^s_{p,r}})^m}{1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t}. \tag{4.6}$$

Plugging (4.6) in (4.5), we have

$$\begin{aligned}
&\|u^{i+1}(t)\|_{B^s_{p,r}}\\
&\quad\le C_6(\kappa+1+\|u_0\|_{B^s_{p,r}})[1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t]^{-1/m}\\
&\qquad - C_6(\kappa+1)[1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t]^{-1/2m}\\
&\quad\le \frac{C_6(\kappa+1+\|u_0\|_{B^s_{p,r}})}{[1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t]^{1/m}}-(\kappa+1).
\end{aligned}$$

Thus we get

$$\kappa+1+\|u^{i+1}(t)\|_{B^s_{p,r}}\le \frac{C_6(\kappa+1+\|u_0\|_{B^s_{p,r}})}{[1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t]^{1/m}}.$$

Therefore,

$$(\kappa+1+\|u^{i+1}(t)\|_{B^s_{p,r}})^m\le \frac{C_6^m(\kappa+1+\|u_0\|_{B^s_{p,r}})^m}{1-2mC_6^{m+1}(\gamma+1)(\kappa+1+\|u_0\|_{B^s_{p,r}})^m t}.$$

Therefore, $(u^i)_{i\in\mathbb{N}}$ is uniformly bounded in $\mathcal{C}([0,T];B^s_{p,r})$. This clearly entails that $u^i\partial_x u^{i+1}$ is uniformly bounded in $\mathcal{C}([0,T];B^{s-1}_{p,r})$. As the right-hand side of (4.4) has been shown to be uniformly bounded in $\mathcal{C}([0,T];B^s_{p,r})$, one can conclude that the sequence $(u^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$.

Step 3: *convergence.* We will prove that $(u^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];B^{s-1}_{p,r})$.

For all $(i,j)\in\mathbb{N}^2$, we have

$$\begin{aligned}
&(\partial_t+\gamma u^{i+j}\partial_x)(u^{i+j+1}-u^{i+1})\\
&\qquad = -\gamma(u^{i+j}-u^i)\partial_x u^{i+1}+P(D)\Big(\frac{g(u^{i+j})-g(u^i)}{2}\\
&\qquad\quad +\frac{\gamma}{2}\,\partial_x(u^{i+j}-u^i)\partial_x(u^{i+j}+u^i)\\
&\qquad\quad -\frac{\gamma}{2}\,(u^{i+j}-u^i)(u^{i+j}+u^i)+2\kappa(u^{i+j}-u^i)\Big).
\end{aligned}$$

Applying Lemma 3.2, Lemma 3.3 and Proposition 2.4, and using the fact that $B^{s-1}_{p,r}$ is an algebra, we show that for all $t\in[0,T]$,

$$\|(u^{i+j+1}-u^{i+1})(t)\|_{B^{s-1}_{p,r}}$$
$$\le e^{C_7\gamma U^{i+j}(t)}\Big(\|u_0^{i+j+1}-u_0^{i+1}\|_{B^{s-1}_{p,r}}+C_7\int_0^t e^{-C_7\gamma U^{i+j}(\tau)}$$
$$\cdot((1+\|u^{i+j}\|_{B^s_{p,r}}+\|u^i\|_{B^s_{p,r}})^m+\gamma(\|u^{i+1}\|_{B^s_{p,r}}+\|u^{i+j}\|_{B^s_{p,r}}+\|u^i\|_{B^s_{p,r}})+\kappa)$$
$$\cdot\|(u^{i+j}-u^i)(\tau)\|_{B^{s-1}_{p,r}}\,d\tau\Big),$$

for a constant $C_7>0$.

Since $(u^i)_{i\in\mathbb{N}}$ is uniformly bounded in $E^s_{p,r}(T)$ and

$$u_0^{i+j+1}-u_0^{i+1}=\sum_{q=i+1}^{i+j}\Delta_q u_0,$$

we finally get a constant C_T independent of i,j and such that for all t in $[0,T]$,

$$\|(u^{i+j+1}-u^{i+1})(t)\|_{B^{s-1}_{p,r}}\le C_T\Big(2^{-i}+\int_0^t\|(u^{i+j}-u^i)(\tau)\|_{B^{s-1}_{p,r}}\,d\tau\Big).$$

Arguing by induction, one can easily prove that

$$\|u^{i+j+1}-u^{i+1}\|_{L^\infty(0,T;B^{s-1}_{p,r})}$$
$$\le\frac{(TC_T)^{i+1}}{(i+1)!}\|u^j\|_{L^\infty(0,T;B^s_{p,r})}+C_T\sum_{l=0}^{i}2^{-(i-l)}\frac{(TC_T)^l}{l!}.$$

As $\|u^j\|_{L^\infty(0,T;B^s_{p,r})}$ may be bounded independently of j, we deduce the existence of some new constant C'_T such that

$$\|u^{i+j+1}-u^{i+1}\|_{L^\infty(0,T;B^{s-1}_{p,r})}\le C'_T2^{-i}.$$

Hence $(u^i)_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T];B^{s-1}_{p,r})$, whence it converges to some limit function $u\in\mathcal{C}([0,T];B^{s-1}_{p,r})$.

STEP 4: *conclusion.* Finally, let us check that u belongs to $E^s_{p,r}(T)$ and satisfies (1.1) or equivalently (1.2).

Since $(u^i)_{i\in\mathbb{N}}$ is uniformly bounded in $L^\infty(0,T;B^s_{p,r})$, the Fatou property for Besov spaces guarantees that u also belongs to $L^\infty(0,T;B^s_{p,r})$.

On the other hand, as $(u^i)_{i\in\mathbb{N}}$ converges to u in $\mathcal{C}([0,T];B^{s-1}_{p,r})$, an interpolation argument ensures that convergence actually holds in $\mathcal{C}([0,T];B^{s'}_{p,r})$ for any $s'<s$. It is then easy to pass to the limit in (4.4) to conclude that u is indeed a solution to (1.1) or (1.2).

Now, because u belongs to $L^\infty(0,T;B^s_{p,r})$, the right-hand side of (1.2) is also in $L^\infty(0,T;B^s_{p,r})$. In the case $r<\infty$, Lemma 3.2 enables us to conclude that $u\in\mathcal{C}([0,T];B^s_{p,r})$. Finally, using again (1.2), we see that $\partial_t u$

is in $\mathcal{C}([0,T];B^{s-1}_{p,r})$ if r is finite, and in $L^\infty(0,T;B^{s-1}_{p,r})$ otherwise. Thus, $u\in E^s_{p,r}(T)$. ∎

5. Energy conservation and blow-up criterion. This section is devoted to the proofs of Theorems 1.2 and 1.3. Both theorems are based on the following lemma:

Lemma 5.1. *Let $(p,r)\in[1,\infty]^2$ and $1<s\le m$. Let $u\in L^\infty(0,T;B^s_{p,r})$ solve (1.2) on $[0,T)\times\mathbb{R}$ with $u_0\in B^s_{p,r}$ as an initial datum. Then there exist a constant $C_8>0$, depending only on s and p, and a constant $C_9>0$ such that for all $t\in[0,T)$,*

$$\|u(t)\|_{B^s_{p,r}}\le\|u_0\|_{B^s_{p,r}}e^{C_8(\gamma+1)\int_0^t(\kappa+1+\|u(\tau)\|_{\mathrm{Lip}})^m\,d\tau},\tag{5.1}$$

$$\|u(t)\|_{\mathrm{Lip}}\le\|u_0\|_{\mathrm{Lip}}e^{C_9\int_0^t(\kappa+(\gamma+1)\|\partial_x u(\tau)\|_{L^\infty})\,d\tau}.\tag{5.2}$$

Proof. Step 1. Lemma 3.1 and the fact that $P(D)$ is a multiplier of order -1 yield

$$\begin{aligned}&e^{-C_1\gamma\int_0^t\|\partial_x u\|_{L^\infty}\,d\tau'}\|u(t)\|_{B^s_{p,r}}\\&\le\|u_0\|_{B^s_{p,r}}+\int_0^t e^{-C_1\gamma\int_0^\tau\|\partial_x u\|_{L^\infty}\,d\tau'}\\&\qquad\cdot\bigg(\Big\|\frac{g(u)}{2}\Big\|_{B^{s-1}_{p,r}}+\Big\|\frac{\gamma}{2}(\partial_x u)^2\Big\|_{B^{s-1}_{p,r}}+\Big\|\frac{\gamma}{2}u^2\Big\|_{B^{s-1}_{p,r}}+\|2\kappa u\|_{B^{s-1}_{p,r}}\bigg)\,d\tau.\end{aligned}$$

As $s-1>0$, we have, according to Proposition 2.2,

$$\Big\|\frac{\gamma}{2}(\partial_x u)^2\Big\|_{B^{s-1}_{p,r}}+\Big\|\frac{\gamma}{2}u^2\Big\|_{B^{s-1}_{p,r}}\le C\gamma\|u\|_{\mathrm{Lip}}\|u\|_{B^s_{p,r}}.$$

By Proposition 2.3, we have

$$\|g(u)/2\|_{B^{s-1}_{p,r}}\le C(1+\|u\|_{L^\infty})^{\widetilde{s-1}}\|u\|_{B^s_{p,r}}\le C(1+\|u\|_{\mathrm{Lip}})^{\widetilde{s-1}}\|u\|_{B^s_{p,r}}.$$

Therefore,

$$\begin{aligned}&e^{-C_1(\gamma+1)\int_0^t(\kappa+1+\|u\|_{\mathrm{Lip}})^m\,d\tau'}\|u(t)\|_{B^s_{p,r}}\\&\qquad\le\|u_0\|_{B^s_{p,r}}+C_{10}(\gamma+1)\int_0^t e^{-C_1(\gamma+1)\int_0^\tau(\kappa+1+\|u\|_{\mathrm{Lip}})^m\,d\tau'}\\&\qquad\qquad\cdot(\kappa+1+\|u\|_{\mathrm{Lip}})^m\|u\|_{B^s_{p,r}}\,d\tau,\end{aligned}$$

with a constant $C_{10}>0$. Applying the Gronwall lemma, we obtain

$$\begin{aligned}&e^{-C_1(\gamma+1)\int_0^t(\kappa+1+\|u\|_{\mathrm{Lip}})^m\,d\tau'}\|u(t)\|_{B^s_{p,r}}\\&\qquad\le\|u_0\|_{B^s_{p,r}}e^{C_{10}(\gamma+1)\int_0^t(\kappa+1+\|u\|_{\mathrm{Lip}})^m\,d\tau'}.\end{aligned}$$

Letting $C_8=C_1+C_{10}$, we get (5.1).

STEP 2. Differentiating equation (1.2) with respect to x, we easily prove that

$$(5.3)\qquad \|u(t)\|_{\mathrm{Lip}}\le \|u_0\|_{\mathrm{Lip}}+\int_0^t \left\|P(D)\left(\frac{g(u)}{2}+\frac{\gamma}{2}(\partial_x u)^2-\frac{\gamma}{2}u^2+2\kappa u\right)\right\|_{\mathrm{Lip}} d\tau.$$

Noting that $(1-\partial_{xx}^2)^{-1}u=\frac12 e^{-|\cdot|}*u$, we get

$$\left\|P(D)\left(\frac{g(u)}{2}+\frac{\gamma}{2}(\partial_x u)^2-\frac{\gamma}{2}u^2+2\kappa u\right)\right\|_{\mathrm{Lip}} \le C_9(\kappa+(\gamma+1)\|\partial_x u(\tau)\|_{L^\infty})\|u\|_{\mathrm{Lip}},$$

for a constant $C_9>0$. Plugging this into (5.3), we obtain

$$\|u(t)\|_{\mathrm{Lip}}\le \|u_0\|_{\mathrm{Lip}}+C_9\int_0^t (\kappa+(\gamma+1)\|\partial_x u(\tau)\|_{L^\infty})\|u(\tau)\|_{\mathrm{Lip}}\, d\tau.$$

By the Gronwall lemma, we get (5.2). ∎

Proof of Theorem 1.3. Let $u\in\bigcap_{T<T^\star}E^s_{p,r}(T)$ with $\int_0^{T^\star}\|\partial_x u(\tau)\|_{L^\infty}\,d\tau$ finite. Then by (5.2), $\int_0^{T^\star}\|u(\tau)\|_{\mathrm{Lip}}\,d\tau$ is also finite. According to (5.1), we have

$$\forall t\in[0,T^\star),\ \|u(t)\|_{B^s_{p,r}}\le M_{T^\star}\triangleq \|u_0\|_{B^s_{p,r}} e^{C_8(\gamma+1)\int_0^{T^\star}(\kappa+1+\|u\|_{\mathrm{Lip}})^m\,d\tau}<\infty.$$

Let $\varepsilon>0$ be such that $2mC_6^{m+1}(\gamma+1)(\kappa+1+M_{T^\star})^m\varepsilon<1$. We then have a solution $\widetilde{u}\in E^s_{p,r}(\varepsilon)$ to (1.2) with the initial datum $u(T^\star-\varepsilon/2)$. By uniqueness, $\widetilde{u}(t)=u(t+T^\star-\varepsilon/2)$ on $[0,\varepsilon/2)$ so that $\widetilde{u}$ extends the solution u beyond $T^\star$. We conclude that $T^\star<T^\star_{u_0}$ and Theorem 1.3 is proved. ∎

Proof of Theorem 1.2. Introduce a nonnegative mollifier $\phi\in\mathcal{C}_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}}\phi=1$, and write $\phi^j(x)=j\phi(jx)$. We then set $u_0^j\triangleq\phi^j*u_0$ and define u^j as the maximal solution of (1.2) corresponding to u_0^j.

As $\Delta_q(\phi^j*u_0)=\phi^j*\Delta_q u_0$ and $\|\phi^j\|_{L^1}=1$, we have

$$\|u_0^j\|_{B^s_{p,r}}\le\|u_0\|_{B^s_{p,r}}\quad\text{and}\quad \|u_0^j\|_{H^1}\le\|u_0\|_{H^1}.$$

Following the second step of the proof of Theorem 1.1, we discover that there exists a constant $C_{11}>0$ such that u^j is a solution of (1.2) on $[0,\overline{T}]\times\mathbb{R}$ with

$$\overline{T}\triangleq\frac{C_{11}}{\|u\|_{L^\infty(0,T;B^s_{p,r})}}$$

and $u^j\in E^s_{p,r}(\overline{T})$ uniformly.

On the other hand, we also have $u_0^j\in H^4$, so that there exists some $T^j>0$ such that (1.2) with data u_0^j has a solution $\widetilde{u}^j\in\mathcal{C}([0,T^j];H^4)$. Thanks to the uniqueness property, we actually have $\widetilde{u}^j\equiv u^j$ on $[0,\min(\overline{T},T^j))$. According to Lemma 5.1 and Proposition 2.1, there exists a constant $C_{12} > 0$ such that $\forall t \in [0, \min(\overline{T}, T^j)]$,

$$\|u^j(t)\|_{H^4} \le \|u_0^j\|_{H^4} \exp\Big(C_{12}(\gamma+1)\int_0^t (\kappa + 1 + \|u^j(\tau)\|_{B^s_{p,r}})^m \, d\tau\Big).$$

Note that the right-hand side may be bounded independently of j and of $t \in [0, \overline{T}]$. Therefore, arguing as in the proof of Theorem 1.3, one can conclude that T^j may be chosen greater than $\overline{T}$.

Now, the smoothness of u^j enables us to derive directly from (1.1) that

$$\forall t \in [0, \overline{T}], \qquad \|u^j(t)\|_{H^1} = \|u_0^j\|_{H^1} \le \|u_0\|_{H^1}.$$

Therefore, passing to the limit and using the Fatou property for H^1, one eventually gets $\|u(t)\|_{H^1} \le \|u_0\|_{H^1}$ for $t \in [0, \overline{T}]$.

To prove the reverse inequality, one can solve the equation backward, starting from $u(\overline{T})$. Then arguing as above and using uniqueness, one can assert that

$$\|u(\overline{T} - t)\|_{H^1} \le \|u(\overline{T})\|_{H^1} \qquad \text{for } t \le \frac{C_{11}}{\|u\|_{L^\infty(0,T;B^s_{p,r})}}.$$

Repeating the argument several times, we finally get $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$ for all $t \in [0, \overline{T}]$.

It is now easy to get equality on $[0, T]$. Indeed, as before, the above yields equality on $[\overline{T}, 2\overline{T}]$, $[2\overline{T}, 3\overline{T}]$ etc., until the whole interval $[0, T]$ is exhausted. ∎

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