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UNIQUE GLOBAL SOLVABILITY OF 1D FRIED–GURTIN MODEL

Abstract. We investigate a 1-dimensional simple version of the Fried–Gurtin 3-dimensional model of isothermal phase transitions in solids. The model uses an order parameter to study solid-solid phase transitions. The free energy density has the Landau–Ginzburg form and depends on a strain, an order parameter and its gradient.

The problem considered here has the form of a coupled system of one-dimensional elasticity and a relaxation law for a scalar order parameter. Under some physically justified assumptions on the strain energy and data we prove the existence and uniqueness of a regular solution to the problem. The proof is based on the Leray–Schauder fixed point theorem.

1. Introduction. Fried and Gurtin [4] have proposed a theory for isothermal phase transitions in solids in which the material phase is characterized by an order parameter. This theory is based on balance laws of linear momentum and microforce, with underlying free energy depending on a strain, a multicomponent order parameter and its gradient.

The idea of an order parameter was applied in the well-known theories of solid-solid transitions developed by Falk and Frémond (see e.g. [1]). In these theories, the order parameter is identified with the strain tensor. In the nonisothermal case the free energy density is postulated to be a function of the strain, the strain gradient, and temperature.

In the Fried–Gurtin theory the order parameter represents a new quantity that can have varying physical interpretation. Originally, in the Fried–Gurtin theory the order parameter represents a characterization of the material phase in the solid. Another possible interpretation of the order parameter

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is the density of the chemical component. The order parameter can also be considered as an artifice that yields a regularization of mechanical equations. The free energy density is constitutively dependent on a strain, the order parameter and the gradient of the order parameter.

From the mathematical point of view it is important that in the Fried–Gurtin theory the stress tensor is a linear function of the strain. The non-linear effects are contained in the order parameter equation.

To the best of the author's knowledge the well-posedness of the Fried–Gurtin model in the one-dimensional case has not been examined so far. Only in Sikora *et al.* [8] a special 1D case of the model and its stationary (independent of time) solutions have been analyzed. A three-dimensional problem with homogeneous Dirichlet boundary conditions has been investigated in [5]. The existence of solutions was proved there with the help of the maximal regularity theory for parabolic equations. A homogeneous Dirichlet boundary condition for the order parameter was assumed in applications of this theory.

The 1D approach to the problem allows the application of a simplified method to prove the existence and uniqueness of the solution. It is important to mention that in this case, in contrast to [5], we adopt the homogeneous Neumann boundary condition for the order parameter, which is common in phase transitions theories.

Let us consider an elastic bar which occupies the interval $[0,1]$ in the reference configuration. The motion of the bar is described by the mapping

$$y(x,t) = x + u(x,t).$$

The order parameter is described by a function $\varphi : (0,1) \times (0,T) \to \mathbb{R}$.

We use the following notation:

$x \in (0,1) = \Omega$	position
$S = \{0\} \cup \{1\}$	boundary points
$t \in [0,T]$	time
$\Omega^T = \Omega \times (0,T)$	space-time cylinder
$S^T = S \times (0,T)$	lateral boundary
$u(x,t)$	displacement
$y(x,t)$	placement in space
$\varepsilon(x,t) = u_x(x,t)$	strain
$\varphi(x,t)$	order parameter
$\varphi_x(x,t)$	gradient of the order parameter
$b(x,t)$	distributed body force

In this article we study the Fried–Gurtin model in a special case of small strain approximation with the strain represented by the linearized strain $\varepsilon = \varepsilon(u)$ and an unconstrained scalar order parameter φ distinguishing between two phases, a and b, characterized by $\varphi = 0$ and $\varphi = 1$. The model under consideration has the form of a coupled system of partial differential equations. These equations represent the linear momentum balance for the displacement and the relaxation law for the order parameter with prescribed initial and boundary conditions:

$$\begin{aligned} & u_{tt} - [f_{,\varepsilon}(\varepsilon, \varphi, \varphi_x)]_x = b && \text{in } \Omega^T, \\ & u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1 && \text{in } \Omega, \\ & u = 0 && \text{on } S^T, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} & \beta\varphi_t + f_{,\varphi}(\varepsilon, \varphi, \varphi_x) - [f_{,\varphi_x}(\varepsilon, \varphi, \varphi_x)]_x = 0 && \text{in } \Omega^T, \\ & \varphi|_{t=0} = \varphi_0 && \text{in } \Omega, \\ & \varphi_x = 0 && \text{on } S^T. \end{aligned} \tag{1.2}$$

Here β is a positive constant, called the dumping modulus (in general β can depend on $\varepsilon, \varphi, \varphi_x, \varphi_t$), and f denotes the free energy density which is constitutively given as a function of the strain, the order parameter and its gradient,

$$f = f(\varepsilon, \varphi, \varphi_x).$$

The functions u_0, u_1, φ_0 represent initial conditions for the displacement, the velocity and the order parameter. For simplicity we consider the homogeneous Dirichlet boundary condition for the displacement and homogeneous Neumann boundary condition for the order parameter. The inhomogeneous boundary condition for u can always be reduced to the homogeneous one by an appropriate translation of the variable u.

We assume that the free energy density f has the typical Landau–Ginzburg form:

$$f(\varepsilon, \varphi, \varphi_x) = W(\varepsilon, \varphi) + \Psi(\varphi) + \frac{\gamma}{2}\,|\varphi_x|^2, \tag{1.3}$$

with the three terms representing the strain energy, the exchange energy and the gradient energy respectively, with constant coefficient $\gamma > 0$. The exchange energy $\Psi(\cdot)$ has the standard form of a double-well potential with equal minima at $\varphi = 0$ and $\varphi = 1$,

$$\Psi(\varphi) = \frac{1}{2}\,\varphi^2(1-\varphi)^2. \tag{1.4}$$

The sum of the last two terms in (1.3) represents the energy of the diffused phase interfaces. For this form of f the corresponding derivatives are:

$$\begin{aligned} f_{,\varepsilon}(\varepsilon,\varphi,\varphi_x) &= W_{,\varepsilon}(\varepsilon,\varphi), \\ f_{,\varphi}(\varepsilon,\varphi,\varphi_x) &= W_{,\varphi}(\varepsilon,\varphi) + \Psi'(\varphi), \\ f_{,\varphi_x}(\varepsilon,\varphi,\varphi_x) &= \gamma\varphi_x. \end{aligned}$$

The relevant expressions for the strain energy $W(\varepsilon,\varphi)$ are given by the following two examples (see e.g. [4], [3]):

Example 1.

$$W(\varepsilon,\varphi) = (1 - z(\varphi))W_a(\varepsilon) + z(\varphi)W_b(\varepsilon), \tag{1.5}$$

where

$$W_i(\varepsilon) = \frac{1}{2}\,A(\varepsilon - \overline{\varepsilon}_i)^2, \quad i = a, b,$$

is the strain energy of phase i, $\overline{\varepsilon}_i$ is the natural strain (eigenstrain), assumed to be constant, and $A > 0$ is the elasticity coefficient.

Furthermore, $z(\cdot)$ is a smooth, nondecreasing scalar interpolation function satisfying

$$z(0) = 0, \qquad z(1) = 1, \qquad 0 \le z(\varphi) \le 1 \quad \text{for all } \varphi \in \mathbb{R}. \tag{1.6}$$

The inequality constraint is imposed to ensure the physical meaning of (1.5).

Example 2.

$$W(\varepsilon,\varphi) = \frac{1}{2}\,A(\varepsilon - \overline{\varepsilon}(\varphi))^2, \tag{1.7}$$

where

$$\overline{\varepsilon}(\varphi) = z(\varphi)\overline{\varepsilon}$$

is the natural strain depending on the order parameter, $\overline{\varepsilon}$ is the constant misfit tensor, and $z(\cdot)$ is a smooth scalar interpolation function satisfying

$$z(0) = 0, \quad z(1) = 1, \tag{1.8}$$

but here, not necessarily constrained by the inequality of (1.6). Furthermore, $A > 0$ is the elasticity coefficient.

We point out that in the general case, the coefficient A in the above examples may depend on the phase, i.e. $A = A(\varphi)$. In this paper we restrict ourselves to the case of homogeneous elasticity assuming that A is constant and for simplicity we set

$$A = 1. \tag{1.9}$$

For the free energy (1.3) with $\Psi(\varphi)$ given by (1.4) and $W(\varepsilon,\varphi)$ as in Examples 1 and 2, the problem (1,1), (1,2) takes the form

$$
\begin{aligned}
&u_{tt} - Au_{xx} = z'(\varphi)B\varphi_x + b && \text{in } \Omega^T = (0,1)\times(0,T),\\
&u_{|t=0} = u_0, \quad u_{|t=0} = u_1 && \text{in } \Omega = (0,1),\\
&u = 0 && \text{on } S^T = S\times(0,T),
\end{aligned}
\tag{1.10}
$$

$$
\begin{aligned}
&\beta\varphi_t - \gamma\varphi_{xx} = -\Psi'(\varphi) - z'(\varphi)[Bu_x + h(\varphi)] && \text{in } \Omega^T,\\
&\varphi_{|t=0} = \varphi_0 && \text{in } \Omega,\\
&\varphi_x = 0 && \text{on } S^T.
\end{aligned}
\tag{1.11}
$$

where

- in Example 1: $B = A(\bar\varepsilon_a - \bar\varepsilon_b)$, $h(\varphi) = -\frac{1}{2}A(\bar\varepsilon_a^2 - \bar\varepsilon_b^2)$,
- in Example 2: $B = -A\bar\varepsilon$, $h(\varphi) = Az(\varphi)\bar\varepsilon^2$.

Assumptions. The problem (1.1), (1.2) is studied under the following assumptions:

(A1) $\Omega = [0,1] \subset \mathbb{R}$.
(A2) $A = \text{const}$; later we will set $A = 1$.
(A3) $f(\varepsilon,\varphi,\varphi_x)$ is defined by (1.3) with $\Psi(\cdot)$ given by (1.4) and $W(\varepsilon,\varphi)$ specified in Examples 1 and 2, or more generally, $h(\varphi) = Ez(\varphi)+F$, where E, F are constants.
(A4) The function $z(\cdot) \in C^2$ satisfies (1.6).
(A5) The data of the problem are such that:

$$b \in L_1(0,T;L_2(\Omega)), \quad u_0 \in H^1_0(\Omega), \quad u_1 \in L_2(\Omega), \quad \varphi_0 \in H^1(\Omega),$$

and satisfy the compatibility condition

$$\varphi_{0,x} = 0 \quad \text{on } S.$$

Remark. System (1.10), (1.11) with the nonhomogeneous Dirichlet boundary conditions for the displacement

$$u(0,t) = 0, \quad u(1,t) = d, \tag{1.12}$$

where d denotes a constant elongation, can be reduced to the homogeneous one by the translation of the variable u:

$$v(x,t) = u(x,t) - dx.$$

Then the pair (v,φ) satisfies

$$
\begin{aligned}
&v_{tt} - Av_{xx} = z'(\varphi)B\varphi_x + b && \text{in } \Omega^T,\\
&v_{|t=0} = u_0 - dx, \quad v_{|t=0} = u_1 && \text{in } \Omega,\\
&v = 0 && \text{on } S^T,
\end{aligned}
$$

$$\beta\varphi_t - \gamma\varphi_{xx} = -\Psi'(\varphi) - z'(\varphi)[Bv_x + h^*(\varphi)] \quad \text{in } \Omega^T,$$
$$\varphi_{|t=0} = \varphi_0 \quad \text{in } \Omega,$$
$$\varphi_x = 0 \quad \text{on } S^T,$$

where $h^*(\varphi) = h(\varphi) + Bd$.

The above system has the same form as (1.10), (1.11), so the results can be applied to this case as well.

The problem (1.10), (1.11) corresponding to the strain energy from Example 2 with $\overline{\varepsilon}(\varphi) = k\varphi$, $k = \text{const}$, and the boundary condition (1.12) has been investigated numerically in [8].

We now state the main results of this paper.

THEOREM 1 (Global existence). *Let assumptions* (A1)–(A5) *be satisfied. Then problem* (1.10), (1.11) *has a solution* $(u,\varphi) \in W^{1,1}_{2,\infty}(\Omega^T) \times W^{2,1}_2(\Omega^T)$. *Moreover*, *any solution* (u,φ) *satisfies the estimates*

$$\begin{gathered} \|u\|_{2,\infty} + \|u_t\|_{2,\infty} + \|\varphi_x\|_{2,\infty} + \|u_x\|_{2,\infty} + \|\varphi\|_{4,\infty} + \|\varphi_t\|_{2,2} \le C, \\ \|\varphi\|_{W^{2,1}_2(\Omega^T)} \le C(T), \end{gathered} \tag{1.13}$$

where the constant C *depends only on the data* u_0, u_1, φ_0 *and* b, *and* $C(T)$ *depends on the data and time* T.

THEOREM 2 (Uniqueness). *Let assumptions* (A1)–(A5) *be satisfied. Then the solution* $(u,\varphi) \in W^{1,1}_{2,\infty}(\Omega^T) \times W^{2,1}_2(\Omega^T)$ *of problem* (1.10), (1.11) *is unique.*

NOTATION. We use the following notation:

$L_p = L_p(\Omega)$, $\quad \|\cdot\|_p = \|\cdot\|_{L_p(\Omega)}$,
$L_{p,q} = L_q(0,T;L_p(\Omega))$, $\quad \|\cdot\|_{p,q} = \|\cdot\|_{L_q(0,T;L_p(\Omega))}$,
$W^{1,1}_2 = W^{1,1}_2(\Omega \times (0,T))$ Sobolev space,
$W^{2,1}_2 = W^{2,1}_2(\Omega \times (0,T))$ anisotropic Sobolev space,
$H^1 = H^1(\Omega) = W^1_2(\Omega)$.

Throughout this paper, C denotes a generic, positive constant different in various instances, even in the same expression, depending on the data of the problem, domain Ω and the time horizon T. Sometimes, when appropriate, this dependence will be additionally expressed. When using the Young or Cauchy inequalities we will use the symbol $C(1/\delta)$ to emphasize that the coefficients $C(1/\delta)$, $\delta > 0$, depend on δ and $C(1/\delta) \to \infty$ as $\delta \to 0$.

2. Auxiliary results. For the reader's convenience we now present the most important theorems used in the proof of the main results.

We will use the following theorems on solvability of linear partial differential equations.

Theorem 2.1 (Global existence for the hyperbolic problem; see e.g. [6, Chapter IV, Theorems 3.1, 3.2]). *Let* $f \in L_{2,1}, u_0 \in H^1(\Omega), u_1 \in L_2(\Omega)$. *Then the problem*

$$
\begin{aligned}
&u_{tt} - u_{xx} = f,\\
&u|_{t=0} = u_0,\\
&u_t|_{t=0} = u_1,\\
&u(0,t) = u(1,t) = 0
\end{aligned}
\tag{2.1}
$$

has a unique solution $u \in W_2^1(\Omega^T)$.

Theorem 2.2 (Global existence for the parabolic problem; see e.g. [6, Chapter III, Theorem 4.1]). *Let* $\beta, \gamma > 0$ *and* $f \in L_{2,2}$, $\varphi_0 \in H^1(\Omega)$. *Then the problem*

$$
\begin{aligned}
&\beta\varphi_t - \gamma\varphi_{xx} = f,\\
&\varphi|_{t=0} = \varphi_0,\\
&\varphi_x(0,t) = \varphi_x(1,t) = 0
\end{aligned}
\tag{2.2}
$$

has a unique solution $\varphi \in W_2^{2,1}(\Omega^T)$.

The proof of Theorem 1 will be based on the classical Leray–Schauder fixed point theorem. We recall it in one of its equivalent forms.

Theorem 2.3 (Leray–Schauder; see e.g. [2]) *Let* $\mathcal{X}$ *be a Banach space. Assume that a map* $\mathcal{T} : [0,1] \times \mathcal{X} \to \mathcal{X}$ *has the following properties*:

(i) *for any fixed* $\tau \in [0,1]$ *the map* $\mathcal{T}(\tau,\cdot) : \mathcal{X} \to \mathcal{X}$ *is completely continuous* (*continuous and compact*),
(ii) *for every bounded subset* $\mathcal{X}_B$ *of* $\mathcal{X}$, *the family of maps* $\mathcal{T}(\cdot,\chi) : [0,1] \to \mathcal{X}$, $\chi \in \mathcal{X}_B$, *is uniformly equicontinuous*,
(iii) *there is a bounded subset* $\mathcal{X}_{\mathcal{F}}$ *of* $\mathcal{X}$ *such that any fixed point in* $\mathcal{X}$ *of the map* $\mathcal{T}(\tau,\cdot)$, $0 \le \tau \le 1$, *is contained in* $\mathcal{X}_{\mathcal{F}}$,
(iv) $\mathcal{T}(0,\cdot)$ *has precisely one fixed point.*

Then $\mathcal{T}(1,\cdot)$ *has at least one fixed point in* $\mathcal{X}$.

For later use, we prepare the following lemmas providing estimates of solutions to problems (2.1) and (2.2).

Lemma 2.4 (Estimate for the hyperbolic problem). *Let* u *be a solution for problem* (2.1). *Then*

$$
\|u\|_{2,\infty} + \|u_x\|_{2,\infty} + \|u_t\|_{2,\infty} \le C(\|f\|_{2,1} + \|u_1\|_2 + \|u_{0,x}\|_2).
\tag{2.3}
$$

Proof. Multiplying $(2.1)_1$ by u_t and integrating by parts with respect to x ($u_t(x,t) = 0$ for $x = 0$ and $x = 1$ because $u(x,t) = 0$ for $x = 0$ and $x = 1$), we get

$$
\int u_t u_{tt}\,dx + \int u_x u_{xt}\,dx = \int f u_t\,dx.
$$

By Schwarz's inequality,

$$\frac{1}{2}\frac{d}{dt}(\|u_t\|_2^2+\|u_x\|_2^2)\le\|f\|_2\|u_t\|_2,$$

thus we get

$$(\|u_t\|_2^2+\|u_x\|_2^2)^{1/2}\frac{d}{dt}(\|u_t\|_2^2+\|u_x\|_2^2)^{1/2}\le\|f\|_2(\|u_t\|_2^2+\|u_x\|_2^2)^{1/2}.$$

After dividing by $(\|u_t\|_2^2+\|u_x\|_2^2)^{1/2}$ and integrating with respect to t, we conclude that

$$(\|u_t(t)\|_2^2+\|u_x(t)\|_2^2)^{1/2}\le\int\|f\|_2\,dt+(\|u_t(0)\|_2^2+\|u_x(0)\|_2^2)^{1/2}.$$

Recalling the initial conditions $u(0)=u_0$, $u_t(0)=u_1$, it follows that for $t\le T$,

$$\|u_t(t)\|_2+\|u_x(t)\|_2\le C(\|f\|_{2,1}+\|u_1\|_2+\|u_{0,x}\|_2).$$

In view of the boundary condition $u(0,t)=0$,

$$u(x,t)=\int_0^x u_x(s,t)\,ds.$$

Hence,

$$\|u(t)\|_2^2\le\int_0^1\Big(\int_0^x u_x(s,t)\,ds\Big)^2dx\le\int_0^1\|u_x(t)\|_2^2\,dx=\|u_x(t)\|_2^2,$$

which concludes the proof. ∎

LEMMA 2.5 (Estimate for the parabolic problem). *Let φ be a solution for problem* (2.2). *Then*

$$\|\varphi\|_{2,\infty}+\|\varphi_x\|_{2,\infty}+\|\varphi_{xx}\|_{2,2}+\|\varphi_t\|_{2,2}\le c(T)(\|f\|_{2,2}+\|\varphi_0\|_2+\|\varphi_{0,x}\|_2). \tag{2.4}$$

Proof. Multiplying $(2.2)_1$ by φ and integrating by parts with respect to x ($\varphi_x\varphi|_0^1=0$, because $\varphi_x=0$ for $x=0,1$) we get

$$\beta\int\varphi_t\varphi\,dx+\gamma\int\varphi_x^2\,dx=\int f\varphi\,dx.$$

Omitting $\gamma\int\varphi_x^2\,dx$, we conclude that

$$\frac{\beta}{2}\frac{d}{dt}\|\varphi(t)\|_2^2=\beta\|\varphi(t)\|_2\frac{d}{dt}\|\varphi(t)\|_2\le\|f(t)\|_2\|\varphi(t)\|_2.$$

After dividing by $\|\varphi(t)\|_2$ and integrating with respect to t, it follows that

$$\beta\|\varphi(t)\|_2\le\int_0^t\|f(s)\|_2\,ds+\beta\|\varphi(0)\|_2,$$

so, for any $t\le T$,

$$\beta\|\varphi(t)\|_2\le\|f\|_{2,1}+\beta\|\varphi_0\|_2.$$

Thus

$$\|\varphi\|_{2,\infty} \le C(\|f\|_{2,1} + \|\varphi_0\|_2). \tag{2.5}$$

Further, multiplying $(2.2)_1$ by φ_t and integrating by parts with respect to x ($\varphi_x\varphi_t|_0^1 = 0$, because $\varphi_x = 0$ for $x = 0,1$), we conclude that

$$\beta\int \varphi_t^2\,dx + \gamma\int \varphi_x\varphi_{xt}\,dx = \int f\varphi_t\,dx.$$

By Young's inequality $fg \le \frac{1}{2\beta}f^2 + \frac{\beta}{2}g^2$, it follows that

$$\beta\|\varphi_t\|_2^2 + \frac{\gamma}{2}\frac{d}{dt}\|\varphi_x\|_2^2 \le \frac{1}{2\beta}\|f\|_2^2 + \frac{\beta}{2}\|\varphi_t\|_2^2.$$

Reducing $\frac{\beta}{2}\|\varphi_t\|_2^2$ and integrating with respect to t, we get

$$\frac{\beta}{2}\|\varphi_t\|_{2,2}^2 + \frac{\gamma}{2}\|\varphi_x(t)\|_2^2 \le \frac{1}{2\beta}\|f\|_{2,2}^2 + \frac{\gamma}{2}\|\varphi_x(0)\|_2^2,$$

which implies that for some $C > 0$,

$$\|\varphi_t\|_{2,2}^2 + \|\varphi_x(t)\|_2^2 \le C(\|f\|_{2,2}^2 + \|\varphi_{0,x}\|_2^2).$$

Thus for $t \le T$, we have

$$\|\varphi_t\|_{2,2} + \|\varphi_x(t)\|_2 \le C(\|f\|_{2,2} + \|\varphi_{0,x}\|_2).$$

As a result,

$$\|\varphi_t\|_{2,2} + \|\varphi_x\|_{2,\infty} \le C(\|f\|_{2,2} + \|\varphi_{0,x}\|_2).$$

This inequality and inequality (2.5) imply (2.4). ∎

We will also use

LEMMA 2.6 (Interpolation inequality). *Let* $\Omega = [a,b] \subset \mathbb{R}$ *and* $\varphi \in L_4(\Omega) \cap H^1(\Omega)$. *Then*

$$\begin{aligned} \|\varphi\|_4^2 &\le C\|\varphi\|_2(\|\varphi\|_2 + \|\varphi_x\|_2), \\ \|\varphi\|_4^2 &\le C(1/\delta)\|\varphi\|_2^2 + \delta\|\varphi_x\|_2^2, \end{aligned} \tag{2.6}$$

where:

- *the constant* C *depends only on* Ω,
- *for* $\delta > 0$, *the constant* $C(1/\delta)$ *depends only on* δ, Ω.

Obviously, we also have

$$\|\varphi\|_4 \le C(1/\delta)\|\varphi\|_2 + \delta\|\varphi_x\|_2. \tag{2.7}$$

Estimates of nonlinearity for the parabolic equation, We point out some consequences of the assumptions on Ψ, z and h. The derivative of the double-well potential Ψ has the form

$$\Psi'(\varphi) = \varphi(1-\varphi)(1-2\varphi) = 2\varphi^3 - 3\varphi^2 + \varphi. \tag{2.8}$$

The function $z(\cdot) \in C^2(\mathbb{R})$, with $z'(\cdot)$ Lipschitz continuous, satisfies

$$\begin{gathered} 0 \le z(\varphi) \le 1, \\ |z(\varphi_1) - z(\varphi_2)| \le C|\varphi_1 - \varphi_2|, \quad |z'(\varphi_1) - z'(\varphi_2)| \le C|\varphi_1 - \varphi_2|. \end{gathered} \tag{2.9}$$

Let

$$H(\varphi) = \Psi(\varphi) + \frac{E}{2} z^2(\varphi) + Fz(\varphi). \tag{2.10}$$

Then

$$\begin{aligned} H'(\varphi) &= \Psi'(\varphi) + Ez'(\varphi)z(\varphi) + Fz'(\varphi), \\ \Psi'(\varphi) + z'(\varphi)[Bu_x + h(\varphi)] &= H'(\varphi) + z'(\varphi)Bu_x, \end{aligned}$$

where:

- $E = 0$, $F = \text{const}$ in Example 1,
- $E = \text{const}$, $F = 0$ in Example 2.

One can easily check the following inequalities:

$$\begin{gathered} C\varphi^4 - C \le H(\varphi) \le C\varphi^4 + C, \\ C\varphi^4 - C\varphi^2 \le H(\varphi) \le C\varphi^4 + C\varphi^2, \\ H'(\varphi_1) - H'(\varphi_2) \le [C(\varphi_1^2 + \varphi_2^2) + C]\,(\varphi_1 - \varphi_2). \end{gathered} \tag{2.11}$$

Using (2.10), the problem (1.10), (1.11) can be rewritten as

$$\begin{aligned} &u_{tt} - u_{xx} = Bz'(\varphi)\varphi_x + b, \\ &u|_{t=0} = u_0 \in H^1(\Omega), \quad u_t|_{t=0} = u_1 \in L_2(\Omega), \\ &u = 0 \quad \text{on } S^T, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\beta\varphi_t - \gamma\varphi_{xx} + H'(\varphi) = -Bz'(\varphi)u_x, \\ &\varphi_{|t=0} = \varphi_0 \in H^1(\Omega), \\ &\varphi_x = 0 \quad \text{on } S^T. \end{aligned} \tag{2.13}$$

3. Proof of Theorem 1. The proof is a direct application of the Leray–Schauder fixed point theorem. First, we choose a working space $\mathcal{X}$. Secondly, we construct a map $\mathcal{T} : [0,1] \times \mathcal{X} \to \mathcal{X}$. Thirdly, we check the assumptions of the Leray–Schauder theorem to show the existence of a solution to problem (1.10), (1.11) in simplified form (2.12), (2.13).

Working space. Let

$$\mathcal{X} = L_\infty(\Omega^T) \cap L_2(0,T; H^1(\Omega)) = \{\varphi : \varphi \in L_\infty(\Omega^T),\ \varphi_x \in L_2(\Omega^T)\}$$

with the norm

$$\|\varphi\|_{\mathcal{X}} = \|\varphi\|_{L_\infty(\Omega^T)} + \|\varphi_x\|_{L_2(\Omega^T)}. \tag{3.1}$$

$\mathcal{X}$ is a Banach space and the imbedding $W_2^{2,1}(\Omega^T) \hookrightarrow \mathcal{X}$ is compact.

Construction of the map. Let $\mathcal{T} : [0,1] \times \mathcal{X} \to \mathcal{X}$ be defined by the following procedure: for any $\tau \in [0,1]$, $\overline{\varphi} \in \mathcal{X}$, $b \in L_1(0,T;L_2(\Omega))$, we define u as a solution of the linear equation

$$u_{tt} - u_{xx} = -\tau B z'(\overline{\varphi})\overline{\varphi}_x + b \tag{3.2}$$

with initial and boundary conditions

$$\begin{aligned} &u|_{t=0} = u_0 \in H^1(\Omega), \qquad u_t|_{t=0} = u_1 \in L_2(\Omega), \\ &u = 0 \quad \text{ on } S^T. \end{aligned} \tag{3.3}$$

By Theorem 2.1, the solution u exists, is unique and $u \in W_2^1(\Omega^T)$, in particular $u_x \in L_2(\Omega^T)$. Given u_x, we define $\mathcal{T}(\tau,\overline{\varphi})$ as a solution of the linear equation

$$\beta\varphi_t - \gamma\varphi_{xx} = -\tau H'(\overline{\varphi}) - \tau B z'(\overline{\varphi}) u_x \tag{3.4}$$

with initial and boundary conditions

$$\begin{aligned} &\varphi_{|t=0} = \varphi_0 \in H^1(\Omega), \\ &\varphi_x = 0 \quad \text{ on } S^T. \end{aligned} \tag{3.5}$$

By Theorem 2.2, the solution φ exists, is unique and $\varphi \in W_2^{2,1}(\Omega^T) \subset \mathcal{X}$. This shows that $\mathcal{T}(\tau,\overline{\varphi})$ is well-defined.

Continuity and compactness of the map $\mathcal{T}(\tau,\cdot)$. To show the continuity we estimate the difference $\mathcal{T}(\tau,\overline{\varphi}_1) - \mathcal{T}(\tau,\overline{\varphi}_2)$ in the norm of the space $\mathcal{X}$. First we estimate $\|(u_1 - u_2)_x\|_{L_2(\Omega^T)}$, and then $\|\varphi_1 - \varphi_2\|_{W_2^{2,1}(\Omega^T)}$.

LEMMA 3.1. *Let $\overline{\varphi}_1, \overline{\varphi}_2 \in \mathcal{X}$, and u_1, u_2 be solutions of*

$$\begin{aligned} u_{1,tt} - u_{1,xx} &= \tau B z'(\overline{\varphi}_1)\overline{\varphi}_{1,x} + b, \\ u_{2,tt} - u_{2,xx} &= \tau B z'(\overline{\varphi}_2)\overline{\varphi}_{2,x} + b, \end{aligned}$$

with the same initial conditions $(3.3)_1$ *and with boundary conditions*

$$u_1 = 0, \qquad u_2 = 0 \quad \text{on } S^T.$$

Then the difference $v = u_1 - u_2$ satisfies the estimate

$$\|v_x\|_{2,\infty} \le \tau C \|\overline{\varphi}\|_{\mathcal{X}} \tag{3.6}$$

where $\overline{\varphi} = \overline{\varphi}_1 - \overline{\varphi}_2$ and C depends on $\overline{\varphi}_{1,x}$.

Proof. The difference v satisfies

$$\begin{aligned} v_{tt} - v_{xx} &= \tau B z'(\overline{\varphi}_1)\overline{\varphi}_{1,x} - \tau B z'(\overline{\varphi}_2)\overline{\varphi}_{2,x} \\ &= \tau B(z'(\overline{\varphi}_1) - z'(\overline{\varphi}_2))\overline{\varphi}_{1,x} + \tau B z'(\overline{\varphi}_2)\overline{\varphi}_x. \end{aligned}$$

In view of the boundary conditions and estimate (2.3) we deduce that

$$\|v_x\|_{2,\infty} \le \tau \|B z'(\overline{\varphi}_1)\overline{\varphi}_{1,x} - B z'(\overline{\varphi}_2)\overline{\varphi}_{2,x}\|_{2,1},$$

Using the estimate

$$|B(z'(\overline{\varphi}_1) - z'(\overline{\varphi}_2))\overline{\varphi}_{1,x}| \le |Bz''(\xi)|\,|\overline{\varphi}|\,|\overline{\varphi}_{1,x}| \le C|\overline{\varphi}|\,|\overline{\varphi}_{1,x}|$$

where ξ depends on $\overline{\varphi}_1$, $\overline{\varphi}_2$, we get

$$\|B(z'(\overline{\varphi}_1) - z'(\overline{\varphi}_2))\overline{\varphi}_{1,x}\|_{2,1} \le C\|\overline{\varphi}\|_{L_\infty(\Omega^T)}\|\overline{\varphi}_{1,x}\|_{2,1}.$$

The estimate $|Bz'(\overline{\varphi}_2)\overline{\varphi}_x| \le C|\overline{\varphi}_x|$ implies that

$$\|Bz'(\overline{\varphi}_2)\overline{\varphi}_x\|_{2,1} \le C\|\overline{\varphi}_x\|_{2,1}.$$

Hence,

$$\begin{aligned}\|Bz'(\overline{\varphi}_1)\overline{\varphi}_{1,x} - Bz'(\overline{\varphi}_2)\overline{\varphi}_{2,x}\|_{2,1} &\le C\|\overline{\varphi}\|_{L_\infty(\Omega^T)}\|\overline{\varphi}_{1,x}\|_{2,1} + C\|\overline{\varphi}_x\|_{2,1}\\ &\le C\|\overline{\varphi}_{1,x}\|_{2,1}\|\overline{\varphi}\|_{L_\infty(\Omega^T)} + C(T)\|\overline{\varphi}_x\|_{2,2}\\ &\le (C\|\overline{\varphi}_{1,x}\|_{2,1} + C(T))(\|\overline{\varphi}\|_{L_\infty(\Omega^T)} + \|\overline{\varphi}_x\|_{2,2}).\end{aligned}$$

This yields inequality (3.6). ■

LEMMA 3.2. *Let φ_1, φ_2 be solutions of the equations*

$$\begin{aligned}\beta\varphi_{1,t} - \gamma\varphi_{1,xx} &= -\tau H'(\overline{\varphi}_1) - \tau Bz'(\overline{\varphi}_1)u_{1,x},\\ \beta\varphi_{2,t} - \gamma\varphi_{2,xx} &= -\tau H'(\overline{\varphi}_2) - \tau Bz'(\overline{\varphi}_2)u_{2,x}\end{aligned}$$

with initial and boundary conditions

$$\begin{aligned}&\varphi_1|_{t=0} = \varphi_0, \quad \varphi_2|_{t=0} = \varphi_0 \quad && \text{in } \Omega,\\ &\varphi_{1,x} = 0, \quad \varphi_{2,x} = 0 && \text{on } S^T,\end{aligned}$$

where u_1, u_2 are constructed in Lemma 3.1. *Then*

$$\|\varphi_1 - \varphi_2\|_{W_2^{2,1}} \le \tau C\|\overline{\varphi}\|_{\mathcal{X}}$$

where $\overline{\varphi} = \overline{\varphi}_1 - \overline{\varphi}_2$, and C depends on $\overline{\varphi}_{1,x}$.

Proof. The difference $\varphi = \varphi_1 - \varphi_2$ satisfies

$$\begin{aligned}&\beta\varphi_t - \gamma\varphi_{xx} = -\tau[H'(\overline{\varphi}_1) - H'(\overline{\varphi}_2)] + \tau B[z'(\overline{\varphi}_1)u_{1,x} - z'(\overline{\varphi}_2)u_{2,x}],\\ &\varphi_{|t=0} = 0, \quad \varphi_x = 0 \quad \text{on } S^T.\end{aligned}$$

In view of (2.4) and the vanishing initial conditions, it is enough to estimate

$$\begin{aligned}(3.7)\qquad &\|H'(\overline{\varphi}_1) - H'(\overline{\varphi}_2) + Bz'(\overline{\varphi}_1)u_{1,x} - Bz'(\overline{\varphi}_2)u_{2,x}\|_{2,2}\\ &\quad \le \|H'(\overline{\varphi}_1) - H'(\overline{\varphi}_2)\|_{2,2} + \|Bz'(\overline{\varphi}_1)u_{1,x} - Bz'(\overline{\varphi}_2)u_{2,x}\|_{2,2}.\end{aligned}$$

Let $v = u_1 - u_2$. Then

$$\begin{aligned}|H'(\overline{\varphi}_1) - H'(\overline{\varphi}_2)| &\le C(\overline{\varphi}_1^2 + \overline{\varphi}_2^2 + C)|\overline{\varphi}_1 - \overline{\varphi}_2|,\\ |Bz'(\overline{\varphi}_1)u_{1,x} - Bz'(\overline{\varphi}_2)u_{2,x}| &\le |B(z'(\overline{\varphi}_1) - z'(\overline{\varphi}_2))u_{1,x}| + |Bz'(\overline{\varphi}_2)v_x|\\ &\le |Bz''(\xi)u_{1,x}\overline{\varphi}| + |Bz'(\overline{\varphi}_2)v_x|\\ &\le C_1|u_{1,x}|\,|\overline{\varphi}| + C|v_x|\end{aligned}$$

where ξ depends on $\overline{\varphi}_1, \overline{\varphi}_2$. The above implies

$$\begin{aligned}\|H'(\overline{\varphi}_1) - H'(\overline{\varphi}_2)\|_{2,2} &\le C\|(\overline{\varphi}_1^2 + \overline{\varphi}_2^2 + C)(\overline{\varphi}_1 - \overline{\varphi}_2)\|_{2,2} \\ &\le C\|\overline{\varphi}_1^2 + \overline{\varphi}_2^2 + C\|_{L_4(\Omega^T)}\|\overline{\varphi}_1 - \overline{\varphi}_2\|_{L_4(\Omega^T)} \\ &\le C\|\overline{\varphi}_1^2 + \overline{\varphi}_2^2 + C\|_{L_\infty(\Omega^T)}\|\overline{\varphi}_1 - \overline{\varphi}_2\|_{L_\infty(\Omega^T)} \\ &\le C(\|\overline{\varphi}_1\|^2_{L_\infty(\Omega^T)} + \|\overline{\varphi}_2\|^2_{L_\infty(\Omega^T)} + 1)\|\overline{\varphi}\|_{L_\infty(\Omega^T)}.\end{aligned}$$

The estimate (3.6) yields $\|v_x\|_{2,2} \le C\|\overline{\varphi}\|_{\mathcal{X}}$. By the above estimates and (3.7),

$$\begin{aligned}&\|\varphi_1 - \varphi_2\|_{W_2^{2,1}(\Omega^T)} \\ &\quad\le \tau C((\|\overline{\varphi}_1\|^2_{L_\infty(\Omega^T)} + \|\overline{\varphi}_2\|^2_{L_\infty(\Omega^T)} + 1)\|\overline{\varphi}\|_{L_\infty(\Omega^T)} + \|u_{1,x}\overline{\varphi}\|_{2,2} + \|v_x\|_{2,2}) \\ &\quad\le \tau C((\|\overline{\varphi}_1\|^2_{L_\infty(\Omega^T)} + + \|\overline{\varphi}_2\|^2_{L_\infty(\Omega^T)} + 1)\|\overline{\varphi}\|_{L_\infty(\Omega^T)} \\ &\qquad + \|u_{1,x}\|_{2,2}\|\overline{\varphi}\|_{L_\infty(\Omega^T)} + \|\overline{\varphi}\|_{\mathcal{X}}) \\ &\quad\le \tau C\|\overline{\varphi}\|_{\mathcal{X}} = \tau C\|\overline{\varphi}_1 - \overline{\varphi}_2\|_{\mathcal{X}}.\end{aligned}$$

where the constant C depends on $\overline{\varphi}_1$ (for small $\|\overline{\varphi}_1 - \overline{\varphi}_2\|_{L_\infty(\Omega^T)}$). ∎

Lemmas 3.1 and 3.2 show that the map $\mathcal{T}(\tau, \cdot) : \mathcal{X} \to W_2^{2,1}(\Omega^T)$ is continuous for any $\tau \in [0, 1]$. Its compactness follows from the compactness of the imbedding $W_2^{2,1}(\Omega^T) \hookrightarrow \mathcal{X}$.

Uniform equicontinuity. In the next step of the proof we prove that if $\mathcal{X}_0$ is a bounded subset of $\mathcal{X}$ then the family of maps $\{\mathcal{T}(\cdot, \varphi)\}_{\varphi\in\mathcal{X}_0} : [0, 1] \to \mathcal{X}$ is uniformly equicontinuous.

Let $\overline{\varphi} \in \mathcal{X}_0$, $\|\overline{\varphi}\|_{\mathcal{X}} \le M$, and u_1, u_2 be solutions of (3.2), (3.3) corresponding to τ_1, τ_2 respectively. Let φ_1, φ_2 be solutions of (3.4) corresponding to τ_1, u_1 and τ_2, u_2 respectively.

The difference $v = u_1 - u_2$ satisfies

$$v_{tt} - v_{xx} = (\tau_1 - \tau_2)z'(\overline{\varphi})\overline{\varphi}_x,$$

with vanishing initial and boundary conditions. In view of (2.3) we have

$$\|v_x\|_{2,2} \le \|(\tau_1 - \tau_2)z'(\overline{\varphi})\overline{\varphi}_x\|_{2,1} \le |\tau_1 - \tau_2|C\|\overline{\varphi}_x\|_{2,2}. \tag{3.8}$$

The difference $\varphi = \varphi_1 - \varphi_2$ satisfies

$$\beta\varphi_t - \gamma\varphi_{xx} = -(\tau_1 - \tau_2)H'(\overline{\varphi}) - B(\tau_1 z'(\overline{\varphi})u_{1,x} - \tau_2 z'(\overline{\varphi})u_{2,x}),$$

with vanishing initial and boundary conditions. The estimate (2.4) implies

$$\|\varphi\|_{W_2^{2,1}(\Omega^T)} \le \|(\tau_1 - \tau_2)H'(\overline{\varphi}) + B(\tau_1 z'(\overline{\varphi})u_{1,x} - \tau_2 z'(\overline{\varphi})u_{2,x})\|_{2,2}.$$

Using the inequalities

$$\begin{gathered}|H'(\overline{\varphi})| \le C|\overline{\varphi}|^3 + C, \\ |\tau_1 z'(\overline{\varphi})u_{1,x} - \tau_2 z'(\overline{\varphi})u_{2,x}| \le |(\tau_1 - \tau_2)u_{1,x}| + |\tau_2 z'(\overline{\varphi})(u_{1,x} - u_{2,x})|,\end{gathered}$$

taking into account that u_1 satisfies (3.2) for $\tau = \tau_1$, and using Lemma 2.4 we get

$$\begin{aligned}\|u_{1,x}\|_{2,\infty} &\le C(\|\tau_1 z'(\overline{\varphi})\overline{\varphi}_x\|_{2,1}) + C_2 \le \tau_1 C\|\overline{\varphi}_x\|_{2,1} + C_2\\ &\le \tau_1 C\|\overline{\varphi}_x\|_{2,2} + C_2\end{aligned}$$

where C_2 depends on the initial conditions (3.3). Hence (3.8) and $\|\overline{\varphi}\|_{\mathcal{X}} \le M$ yield

$$\begin{aligned}\|\varphi\|_{W_2^{2,1}(\Omega^T)} &\le |\tau_1 - \tau_2|(\|H'(\overline{\varphi})\|_{2,2} + \|u_{1,x}\|_{2,2}) + C\tau_2\|v_x\|_{2,2}\\ &\le |\tau_1 - \tau_2|(C\|\overline{\varphi})\|^3_{L_\infty(\Omega^T)} + C + C\|\overline{\varphi}_x\|_{2,2} + C\|\overline{\varphi}_x\|_{2,2})\\ &\le C|\tau_1 - \tau_2|.\end{aligned}$$

By continuity of the imbedding $W_2^{2,1}(\Omega^T) \hookrightarrow \mathcal{X}$, the above estimate shows the desired uniform equicontinuity. ∎

Boundedness of the fixed point. In the next step we prove that there exists a constant M such that every solution of the equation $\mathcal{T}(\tau,\varphi) = \varphi$ for any $\tau \in [0,1]$ satisfies $\|\varphi\|_{\mathcal{X}} \le M$.

If φ is a fixed point of $\mathcal{T}(\tau,\cdot)$ for some τ, then

$$\begin{aligned}&u_{tt} - u_{xx} = -\tau Bz'(\varphi)\varphi_x + b,\\ &\beta\varphi_t - \gamma\varphi_{xx} + \tau H'(\varphi) = -\tau Bz'(\varphi)u_x,\\ &u|_{t=0} = u_0 \in H^1(\Omega), \quad u_t|_{t=0} = u_1 \in L_2(\Omega),\\ &u = 0 \quad \text{on } S^T.\\ &\varphi_{|t=0} = \varphi_0 \in H^1(\Omega),\\ &\varphi_x = 0 \quad \text{on } S^T.\end{aligned} \tag{3.9}$$

In virtue of the inequality (2.3),

$$\begin{aligned}\|u_x\|_{2,2} &\le C(\|\tau Bz'(\varphi)\varphi_x + b\|_{2,1} + \|u_1\|_2 + \|u_{0,x}\|_2)\\ &\le C\|\varphi_x\|_{2,2} + C(\|b\|_{2,1} + \|u_1\|_2 + \|u_{0,x}\|_2) = C\|\varphi_x\|_{2,2} + D\end{aligned}$$

where $D = C(\|b\|_{2,1} + \|u_1\|_2 + \|u_{0,x}\|_2)$. This shows that

$$\|u_x\|^2_{2,2} \le C\|\varphi_x\|^2_{2,2} + D^2. \tag{3.10}$$

Multiplying $(3.9)_2$ by φ and integrating by parts with respect to x yields

$$\beta\int \varphi\varphi_t\,dx + \gamma\int \varphi_x\varphi_x\,dx + \tau\int H'(\varphi)\varphi\,dx = -\tau\int Bz'(\varphi)\varphi u_x\,dx.$$

Taking into account that $H'(\varphi)\varphi \ge C\varphi^4 - C\varphi^2$, $|Bz'(\varphi)| \le C$, we get

$$\beta\int \varphi\varphi_t\,dx + \gamma\int \varphi_x\varphi_x\,dx + \tau C\int \varphi^4\,dx \le \tau C\int \varphi^2\,dx + \tau C\int |\varphi u_x|\,dx.$$

If we omit $\tau C \int \varphi^4\,dx$, the Young inequality yields

$$\begin{aligned}\frac{\beta}{2}\frac{d}{dt}\|\varphi(t)\|_2^2 + \gamma\|\varphi_x(t)\|_2^2 &\le C\|\varphi(t)\|_2^2 + C\|\varphi(t)\|_2\|u_x(t)\|_2 \\ &\le C\|\varphi(t)\|_2^2 + C(1/\delta)\|\varphi(t)\|_2^2 + \delta\|u_x(t)\|_2^2.\end{aligned}$$

Integrating with respect to $t \in (0,T]$ and applying (3.10) we get

$$\begin{aligned}&\frac{\beta}{2}\,\|\varphi(t)\|_2^2 + \gamma\|\varphi_x\|_{2,2}^2 \\ &\quad\le (C + C(1/\delta))\int_0^t \|\varphi(s)\|_2^2\,ds + \delta\|u_x(t)\|_{2,2}^2 + \frac{\beta}{2}\,\|\varphi(0)\|_2^2 \\ &\quad\le (C + C(1/\delta))\int_0^t \|\varphi(s)\|_2^2\,ds + \frac{\beta}{2}\,\|\varphi(0)\|_2^2 + \delta C\|\varphi_x\|_{2,2}^2 + \delta D^2.\end{aligned}$$

Choosing $\delta = \gamma/2C$, after reducing the term $\delta C\|\varphi_x\|_{2,2}^2$, we get

$$\frac{\beta}{2}\,\|\varphi(t)\|_2^2 + \frac{\gamma}{2}\|\varphi_x\|_{2,2}^2 \le C\int_0^t \|\varphi(s)\|_2^2\,ds + \frac{\beta}{2}\,\|\varphi(0)\|_2^2 + \delta D^2.$$

Since $\varphi_{1,x} = 0$, $\varphi_{2,x} = 0$ on S^T, the integral Gronwall inequality shows that for $t \le T$,

$$\|\varphi(t)\|_2^2 \le \left(\|\varphi(0)\|_2^2 + \frac{2\delta}{\beta}\,D^2\right)(1 + cte^{ct})$$

where $c = 2C/\beta$. Consequently, $\|\varphi\|_{2,2} \le \|\varphi\|_{2,\infty} = \operatorname{ess\,sup}\|\varphi(t)\|_2 \le C(T) < \infty$, so

$$\begin{aligned}\frac{\gamma}{2}\,\|\varphi_x\|_{2,2}^2 &\le C\int_0^t \|\varphi(s)\|_2^2\,ds + \frac{\beta}{2}\,\|\varphi(0)\|_2^2 + \delta D^2 \\ &\le C\|\varphi\|_{2,2}^2 + \frac{\beta}{2}\,\|\varphi(0)\|_2^2 + \delta D^2,\end{aligned}$$

thus

$$\|\varphi_x\|_{2,2} \le C(T).$$

Estimates (2.4) and (3.10) imply that

$$\begin{aligned}\|\varphi\|_{W_2^{2,1}} &\le C(T)(\|-\tau Bz'(\varphi)u_x\|_{2,2} + \|\varphi_0\|_2 + \|\varphi_{0,x}\|_2) \\ &\le C(T)(C\|u_x\|_{2,2} + \|\varphi_0\|_2 + \|\varphi_{0,x}\|_2) \\ &\le C(T)(C\|\varphi_x\|_{2,2} + C + \|\varphi_0\|_2 + \|\varphi_{0,x}\|_2) \\ &\le C(T)(C + \|\varphi_0\|_2 + \|\varphi_{0,x}\|_2) = M. \ \blacksquare\end{aligned}$$

Proof of the energy estimate. In this section, we prove the estimate $(1.13)_1$ from Theorem 1.

Multiplying $(2.12)_1$ by u_t and integrating by parts with respect to x ($\frac{d}{dx}(z(\varphi)u_t) = z'(\varphi)\varphi_x u_t + z(\varphi)u_{xt}$) we get

$$\int u_t u_{tt}\,dx + \int u_x u_{xt}\,dx = -B\int z(\varphi)u_{xt}\,dx + \int bu_t\,dx.$$

Multiplying $(2.13)_1$ by φ_t and integrating by parts with respect to x, we deduce that

$$\beta\int \varphi_t^2\,dx + \gamma\int \varphi_x\varphi_{xt}\,dx + \int H'(\varphi)\varphi_t\,dx = -B\int z'(\varphi)\varphi_t u_x\,dx.$$

In view of $\frac{d}{dt}(z(\varphi)u_x) = z'(\varphi)\varphi_t u_x + z(\varphi)u_{xt}$, adding the above equalities, we arrive at the energy identity

$$\beta\int \varphi_t^2\,dx + \frac{d}{dt}\biggl(\frac{1}{2}\int u_t^2\,dx + \frac{\gamma}{2}\int \varphi_x^2\,dx + \frac{1}{2}\int u_x^2\,dx + \int H(\varphi) + Bz(\varphi)u_x\,dx\biggr) = \int bu_t\,dx. \tag{3.11}$$

We use the estimates $H(\varphi) \ge C\varphi^4 - C\varphi^2$ (see (2.11)), $z(\varphi) \le C|\varphi| + C$ and the inequality $\varphi u_x \ge -(\varphi^2 + u_x^2)/2$ to conclude that

$$\frac{1}{2}\,u_x^2 + H(\varphi) + Bz(\varphi)u_x \ge Cu_x^2 + C\varphi^4 - C\varphi^2 \ge Cu_x^2 + C\varphi^4 - C. \tag{3.12}$$

Furthermore, integrating (3.11) with respect to t yields

$$\frac{1}{2}\,(\|u_t(t)\|_2^2 - \|u_t(0)\|_2^2) + \beta\|\varphi_t\|_{2,2}^2 + \frac{\gamma}{2}\,(\|\varphi_x(t)\|_2^2 - \|\varphi_x(0)\|_2^2) + \biggl(\frac{1}{2}\int u_x^2\,dx + \int H(\varphi)\,dx + B\int z(\varphi)u_x\,dx\biggr)\bigg|_0^t = \iint bu_t\,dx\,dt \tag{3.13}$$

for any $t \le T$. In virtue of the Hölder and Young inequalities we get

$$\iint bu_t\,dx\,dt \le \int \|b(s)\|_2\|u_t(s)\|_2\,ds \le \|b\|_{2,1}\|u_t\|_{2,\infty} \le \|b\|_{2,1}^2 + \frac{1}{4}\,\|u_t\|_{2,\infty}^2.$$

Hence, (3.13) yields, for any $t \le T$,

$$\begin{aligned}\frac{1}{2}\,\|u_t(t)\|_2^2 + \beta\|\varphi_t\|_{2,2}^2 + \frac{\gamma}{2}\,\|\varphi_x(t)\|_2^2 + \int (Cu_x^2(t) + C\varphi^4(t) - C)\,dx \\ \le \frac{1}{4}\,\|u_t\|_{2,\infty}^2 + \frac{1}{2}\,\|u_t(0)\|_2^2 + \frac{\gamma}{2}\,\|\varphi_x(0)\|_2^2 \\ + \int\biggl(\frac{1}{2}\,u_x^2(0) + H(\varphi(0)) + Bz(\varphi(0))u_x(0)\biggr)\,dx + \|b\|_{2,1}^2,\end{aligned}$$

so that

$$\begin{aligned}\frac{1}{4}\,\|u_t(t)\|_2^2 + \frac{\gamma}{2}\,\|\varphi_x(t)\|_{2,\infty}^2 + C\|u_x\|_{2,\infty}^2 + C\|\varphi\|_{4,\infty}^4 + \beta\|\varphi_t\|_{2,2}^2 \\ \le \frac{1}{2}\,\|u_t(0)\|_2^2 + \frac{\gamma}{2}\,\|\varphi_x(0)\|_2^2 + \|b\|_{2,1}^2 + C\end{aligned}$$

where $C = C + \int (\frac{1}{2} u_x^2(0) + H(\varphi(0)) + Bz(\varphi(0))u_x(0))\, dx$. The above inequality and the inequality $\|u\|_{2,\infty} \le \|u_x\|_{2,\infty}$ yield the estimate $(1.13)_1$. ∎

Existence of the solution. For $\tau = 0$, system (3.2), (3.3) has exactly one solution, because in this case both independent linear equations have unique solutions. Thus the assumptions of the Leray–Schauder's fixed point theorem are proven. Hence $\mathcal{T}(1, \cdot)$ has at least one fixed point $\varphi \in \mathcal{X}$. Then the pair (u, φ) is the solution of system (3.2), (3.3) for $\tau = 1$, thus also for (2.12), (2.13). Theorem 1 is proved. ∎

4. Proof of Theorem 2. Let (u_1, φ_1) and (u_2, φ_2) be two solutions of the system (2.12), (2.13). Let $v = u_1 - u_2$ and $\varphi = \varphi_1 - \varphi_2$. The pair (v, φ) satisfies

$$
\begin{aligned}
&v_{tt} - v_{xx} = Bz'(\varphi_1)\varphi_{1,x} - Bz'(\varphi_2)\varphi_{2,x},\\
&\beta\varphi_t - \gamma\varphi_{xx} = -[H'(\varphi_1) - H'(\varphi_2)] - B[z'(\varphi_1)u_{1,x} - z'(\varphi_2)u_{2,x}],\\
&v|_{t=0} = 0, \quad v_t|_{t=0} = 0, \quad \varphi_{|t=0} = 0,\\
&v = 0, \quad \varphi_x = 0 \quad \text{on } S^T.
\end{aligned}
\tag{4.1}
$$

In view of the the zero initial conditions, Lemma 2.4 yields

$$\|v\|_{2,\infty} + \|v_x\|_{2,\infty} + \|v_t\|_{2,\infty} \le C\|Bz'(\varphi_1)\varphi_{1,x} - Bz'(\varphi_2)\varphi_{2,x}\|_{2,1}.$$

for any $t < T$. Taking into account the regularity of $z(\cdot)$ we have

$$
\begin{aligned}
|z'(\varphi_1)\varphi_{1,x} &- z'(\varphi_2)\varphi_{2,x}|\\
&\le |z'(\varphi_1)\varphi_{1,x} - z'(\varphi_2)\varphi_{1,x}| + |z'(\varphi_2)\varphi_{1,x} - z'(\varphi_2)\varphi_{2,x}|\\
&= |(z'(\varphi_1) - z'(\varphi_2))\varphi_{1,x}| + |z'(\varphi_2)(\varphi_{1,x} - \varphi_{2,x})|\\
&\le |z''(\xi)(\varphi_1 - \varphi_2)\varphi_{1,x}| + |z'(\varphi_2)(\varphi_{1,x} - \varphi_{2,x})|\\
&\le C(|\varphi\varphi_{1,x}| + |\varphi_x|)
\end{aligned}
$$

where ξ depends on φ_1, φ_2. Thus, for $t < T$,

$$\|v\|_{2,\infty} + \|v_x\|_{2,\infty} + \|v_t\|_{2,\infty} \le C\int_0^t (\|\varphi\varphi_{1,x}\|_2 + \|\varphi_x\|_2)\, ds.$$

Applying the Hölder inequality, we deduce that

$$\int_0^t \|\varphi\varphi_{1,x}\|_2\, ds \le \int_0^t \|\varphi\|_4\|\varphi_{1,x}\|_4\, ds \le \|\varphi\|_{4,2}\|\varphi_{1,x}\|_{4,2}.$$

The solution φ_1 of problem (2.13) belongs to $W_2^{2,1}$ and $\|\varphi_1\|_{W_2^{2,1}} \le C(T)$. The continuity of the imbedding $D_x : W_2^{2,1} \to L_{4,2}$ yields

$$\|\varphi_{1,x}\|_{4,2} \le C\|\varphi_1\|_{W_2^{2,1}} \le C.$$

Applying the inequality $\|\varphi_x\|_{2,1} \le C\|\varphi_x\|_{2,2}$ we deduce that

$$\|v\|_{2,\infty} + \|v_x\|_{2,\infty} + \|v_t\|_{2,\infty} \le C\|\varphi\|_{4,2} + C\|\varphi_x\|_{2,2}. \tag{4.2}$$

Multiplying $(4.1)_2$ by φ and integrating by parts with respect to x, we get

$$\beta\int \varphi\varphi_t\,dx + \gamma\int \varphi_x^2\,dx + \int[H'(\varphi_1) - H'(\varphi_2)]\varphi\,dx = -B\int(z'(\varphi_1)u_{1,x} - z(\varphi_2)u_{2,x})\varphi\,dx.$$

The boundedness of $z''(\cdot)$ yields

$$|B(z'(\varphi_1)u_{1,x} - z'(\varphi_2)u_{2,x})| = |B(z'(\varphi_1) - z'(\varphi_2))u_{1,x}| + |Bz'(\varphi_2)(u_{1,x} - u_{2,x})| \le C(|\varphi u_{1,x}| + |v_x|).$$

Taking into account the inequality $[H'(\varphi_1) - H'(\varphi_2)]\varphi \ge C[\varphi_1^2 + \varphi_2^2]\varphi^2 - C\varphi^2$, we conclude that

$$\begin{aligned}\frac{\beta}{2}\frac{d}{dt}\int \varphi^2\,dx + \gamma\int\varphi_x^2\,dx + C\int[\varphi_1^2 + \varphi_2^2]\varphi^2\,dx &\le C\int\varphi^2\,dx + \int|B(z'(\varphi_1)u_{1,x} - z(\varphi_2)u_{2,x})\varphi|\,dx \\ &\le C\int\varphi^2\,dx + C\int |\varphi^2 u_{1,x}| + |v_x\varphi|\,dx.\end{aligned}$$

Hence, omitting $C\int[\varphi_1^2 + \varphi_2^2]\varphi^2\,dx$, we get

$$\frac{\beta}{2}\frac{d}{dt}\|\varphi(t)\|_2^2 + \gamma\|\varphi_x(t)\|_2^2 \le C\|\varphi(t)\|_2^2 + C\int(|\varphi^2 u_{1,x}| + |v_x\varphi|)\,dx. \tag{4.3}$$

By virtue of the Hölder inequality,

$$C\int|\varphi^2 u_{1,x}|\,dx \le C\|\varphi\|_4^2\|u_{1,x}\|_2.$$

Applying the estimate $\|u_{1,x}\|_{2,\infty} \le C$, which is true by (1.13), we deduce that

$$C\int|\varphi^2 u_{1,x}|\,dx \le C\|\varphi\|_4^2.$$

In view of the interpolation inequality $(2.6)_2$, the above inequality yields

$$C\int|\varphi^2 u_{1,x}|\,dx \le C(1/\delta)\|\varphi\|_2^2 + \delta\|\varphi_x\|_2^2.$$

Thus, if we set $\delta = \gamma/4$ inequality (4.3) takes the form

$$\frac{\beta}{2}\frac{d}{dt}\|\varphi(t)\|_2^2 + \frac{3}{4}\gamma\|\varphi_x(t)\|_2^2 \le C\|\varphi(t)\|_2^2 + C\int |v_x\varphi|\,dx. \tag{4.4}$$

The inequality (4.2) implies that for any $t < T$,

$$\|v_x(t)\|_2 \le \|v_x\|_{2,\infty} \le C\|\varphi\|_{4,2} + C\|\varphi_x\|_{2,2}.$$

Hence, applying the Hölder inequality we can estimate the last term in (4.4) as

$$C\int |v_x\varphi|\,dx \le C\|\varphi(t)\|_2\|\varphi\|_{4,2} + C\|\varphi(t)\|_2\|\varphi_x\|_{2,2}.$$

In view of the vanishing initial conditions and the above inequality, integrating (4.4) with respect to t yields

$$\begin{aligned}\|\varphi(t)\|_2^2 &+ \frac{3}{4}\gamma\|\varphi_x\|_{2,2}^2 \\ &\le C\|\varphi\|_{2,2}^2 + C\int_0^t (\|\varphi(t)\|_2\|\varphi\|_{4,2} + \|\varphi(t)\|_2\|\varphi_x\|_{2,2})\,dt' \\ &\le C(\|\varphi\|_{2,2}^2 + \|\varphi\|_{2,2}\|\varphi\|_{4,2} + \|\varphi\|_{2,2}\|\varphi_x\|_{2,2}).\end{aligned}$$

Applying the interpolation inequality (2.7) again, we get

$$\begin{aligned}\|\varphi(t)\|_2^2 + \frac{3}{4}\gamma\|\varphi_x\|_{2,2}^2 &\le C(\|\varphi\|_{2,2}^2 + C\|\varphi\|_{2,2}^2 + 2\|\varphi\|_{2,2}\|\varphi_x\|_{2,2}) \\ &\le C(\|\varphi\|_{2,2}^2 + \|\varphi\|_{2,2}\|\varphi_x\|_{2,2}).\end{aligned}$$

Using the Young inequality we conclude that

$$\begin{aligned}\|\varphi(t)\|_2^2 + \frac{3}{4}\gamma\|\varphi_x\|_{2,2}^2 &\le C(\|\varphi\|_{2,2}^2 + C(1/\delta)\|\varphi\|_{2,2}^2 + \delta\|\varphi_x\|_{2,2}^2) \\ &\le C\|\varphi\|_{2,2}^2 + \frac{\gamma}{4}\|\varphi_x\|_{2,2}^2,\end{aligned}$$

where we have set $\delta = \gamma/4C$.

In view of the vanishing initial conditions, the integral Gronwall inequality yields, for $t \le T$,

$$\|\varphi(t)\|_2^2 = 0.$$

Hence, $\varphi(t) = 0$ for any $t < T$. Using (4.2) and the zero initial condition in (4.1) we conclude that $v = 0$. This shows that $\varphi_1 = \varphi_2$, $u_1 = u_2$. The uniqueness is proved. ∎

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