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COMPARISON OF EXPLICIT AND IMPLICIT DIFFERENCE METHODS FOR QUASILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. We give a theorem on error estimates of approximate solutions for explicit and implicit difference functional equations with unknown functions of several variables. We apply this general result to investigate the stability of difference methods for quasilinear functional differential equations with initial boundary condition of Dirichlet type. We consider first order partial functional differential equations and parabolic functional differential problems. We compare the properties of explicit and implicit difference methods.

We use a comparison technique with nonlinear estimates of Perron type for given functions with respect to the functional variables.

1. Introduction. We are interested in numerical approximation of classical solutions to quasilinear functional differential equations with initial boundary conditions of Dirichlet type. First order partial functional differential equations and parabolic functional differential problems are considered.

Difference schemes for evolution functional differential equations consist in replacing partial derivatives with difference operators. Moreover, because differential equations contain functional variables, some interpolating operators are needed. This leads to difference functional problems which satisfy consistency conditions on classical solutions of original equations. The main task in these considerations is to find difference approximations of functional differential equations which are stable. A comparison technique is used to investigate the stability of functional difference problems.

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It is not our aim here to give a full review of papers concerning explicit difference methods for quasilinear functional differential equations. The papers [3], [4], [18] and the monograph [7] contain such reviews. In recent years, a number of papers on implicit difference methods for functional partial differential equations have been published. Difference approximations of classical solutions to first order partial functional differential equations were considered in [9]. Implicit difference schemes for parabolic equations with initial boundary condition of Dirichlet type were studied in [5], [10]. From the abundant literature concerning the convergence of difference schemes for nonlinear functional differential equations we mention the papers [1], [8], [13], [18]. Monotone iterative methods and implicit difference schemes for computing approximate solutions to parabolic equations with time delays were investigated in [11], [12], [14], [15], [20].

The aim of the present paper is to compare explicit and implicit difference methods for quasilinear functional differential equations.

Two types of assumptions are needed in theorems on convergence of difference methods for evolution functional differential equations. The first type conditions concern regularity of the given functions. The second type conditions concern the mesh. We show that the equations considered here have the following properties. Assumptions on the regularity of the given functions are the same in theorems on convergence of explicit and implicit difference schemes. We prove that assumptions on the mesh are needed for explicit difference methods, but are not necessary for implicit schemes. We show that there are implicit methods which are convergent, while the corresponding explicit schemes are not.

Our results are based on the following idea. Normally, difference inequalities and theorems on recurrent inequalities are used to investigate the stability of explicit and implicit difference schemes. As a rule, these considerations require a lot of calculations so the main properties of the corresponding operators cannot be easily seen. The aim of the present paper is to show that results on explicit and implicit difference schemes are consequences of results on abstract difference equations with an unknown function of several variables.

The authors of [1]–[5], [8]–[15], [18], [20] have assumed that the given functions satisfy the Lipschitz condition or nonlinear estimates of Perron type with respect to the functional variables, and these conditions are global with respect to the functional variables. Our assumptions are more general. We assume that the nonlinear estimates of Perron type are local with respect to the functional variables. It is clear that there are differential equations with deviated variables and differential integral equations such that estimates of Perron type only hold locally.

We formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let $M_{n\times n}$ be the set of all $n\times n$ matrices with real entries. Write

$$E_0=[-b_0,0]\times[-b,b],\quad E=[0,a]\times[-b,b],\quad \partial_0E=[0,a]\times([-b,b]\setminus(-b,b)),$$

where $a>0$, $b_0\in\mathbb{R}_+$, $\mathbb{R}_+=[0,\infty)$, and $b=(b_1,\dots b_n)$, $b_i>0$ for $i=1,\dots,n$. For $(t,x)\in E$ we define

$$D[t,x]=\{(\tau,y)\in\mathbb{R}^{1+n}:\tau\le 0,\,(t+\tau,x+y)\in E_0\cup E\}.$$

It is clear that $D[t,x]=[-b_0-t,0]\times[-b-x,b-x]$ For a function $z:E_0\cup E\to\mathbb{R}$ and a point $(t,x)\in E$ we define a function $z_{(t,x)}:D[t,x]\to\mathbb{R}$ by $z_{(t,x)}(\tau,y)=z(t+\tau,x+y)$, $(\tau,y)\in D[t,x]$; that is, we restrict z to the set $(E_0\cup E)\cap([-b_0,t]\times\mathbb{R}^n)$ and then shift this restriction to the set $D[t,x]$. Write $B=[-b_0-a,0]\times[-2b,2b]$. Then $D[t,x]\subset B$ for $(t,x)\in E$. Set $\varXi=E\times C(B,\mathbb{R})$ and suppose that

$$\mathbf{f}:\varXi\to\mathbb{R}^n,\quad \mathbf{f}=(f_1,\dots,f_n),\quad g:\varXi\to\mathbb{R},\quad \varphi:E_0\cup\partial_0E\to\mathbb{R}$$

are given functions. Let z be an unknown function of the variables (t,x), $x=(x_1,\dots,x_n)$. We consider the functional differential equation

$$\partial_t z(t,x)=\sum_{i=1}^{n} f_i(t,x,z_{(t,x)})\,\partial_{x_i}z(t,x)+g(t,x,z_{(t,x)}) \tag{1}$$

with the initial boundary condition

$$z(t,x)=\varphi(t,x)\quad \text{on } E_0\cup\partial_0E. \tag{2}$$

We will say that $\mathbf{f}$ and g satisfy the *condition* (V) if for each $(t,x)\in E$ and for $w,\tilde w\in C(B,\mathbb{R})$ such that $w(\tau,y)=\tilde w(\tau,y)$ for $(\tau,y)\in D[t,x]$ we have $\mathbf{f}(t,x,w)=\mathbf{f}(t,x,\tilde w)$ and $g(t,x,w)=g(t,x,\tilde w)$. The condition (V) means that the values of $\mathbf{f}$ and g at the point $(t,x,w)\in\varXi$ depend on (t,x) and on the restriction of w to the set $D[t,x]$ only. We assume that $\mathbf{f}$ and g satisfy the condition (V) and we consider classical solutions to (1), (2).

Now we formulate initial boundary value problems for parabolic functional differential equations. Suppose that

$$\mathbf{F}:\varXi\to M_{n\times n},\qquad \mathbf{F}=[F_{ij}]_{i,j=1}^{n},$$
$$\mathbf{G}:\varXi\to\mathbb{R}^n,\quad \mathbf{G}=(G_1,\dots,G_n),\quad G:\varXi\to\mathbb{R},\quad \varphi:E_0\cup\partial_0E\to\mathbb{R}$$

are given functions. We consider the functional differential equation

$$\begin{aligned}(3)\qquad \partial_t z(t,x) = &\sum_{i,j=1}^{n} F_{ij}(t,x,z_{(t,x)})\partial_{x_i x_j} z(t,x)\\ &+\sum_{i=1}^{n} G_i(t,x,z_{(t,x)})\partial_{x_i} z(t,x) + G(t,x,z_{(t,x)})\end{aligned}$$

with the initial boundary condition (2). We assume that $\mathbf{F}$, $\mathbf{G}$, G satisfy the following condition (V): if $(t,x) \in E$, $w,\tilde{w} \in C(B,\mathbb{R})$ and $w(\tau,y) = \tilde{w}(\tau,y)$ for $(\tau,y) \in D[t,x]$ then $\mathbf{F}(t,x,w) = \mathbf{F}(t,x,\tilde{w})$, $\mathbf{G}(t,x,w) = \mathbf{G}(t,x,\tilde{w})$, $G(t,x,w) = G(t,x,\tilde{w})$. We consider classical solutions to (3), (2).

The paper is organized as follows. In Section 2 we propose a new method of investigating explicit or implicit difference schemes corresponding to initial boundary value problems for quasilinear functional differential equations. We formulate general difference functional problems with an unknown function of several variables. We give sufficient conditions for the existence and uniqueness of solutions of initial boundary value problems and we prove a theorem on error estimates of approximate solutions. The error is estimated by a solution of an initial problem for a nonlinear difference equation with an unknown function of one variable. In Section 3 we apply the above general results to quasilinear first order partial functional differential equations. Section 4 deals with explicit and implicit difference schemes for parabolic functional differential problems. We use general ideas for finite difference equations which were introduced in the monographs [7], [16], [17].

2. Functional difference equations. For $x \in \mathbb{R}^n$, $W \in M_{n\times n}$ where $x = (x_1,\ldots,x_n)$ and $W = [w_{ij}]_{i,j=1}^{n}$ we put

$$\|x\| = \sum_{i=1}^{n} |x_i|, \qquad \|W\| = \max\Big\{\sum_{j=1}^{n} |w_{ij}| : 1 \le i \le n\Big\}.$$

The norm in the space $C(B,\mathbb{R})$ is defined by $\|w\|_B = \max\{|w(\tau,y)| : (\tau,y) \in B\}$.

For any sets X and Y we denote by $F(X,Y)$ the set of all functions defined on X and taking values in Y. We define a mesh on $E_0 \cup E$ in the following way. Suppose that (h_0,h'), $h' = (h_1,\ldots,h_n)$, $h_i > 0$ for $0 \le i \le n$, are steps of the mesh. For $h = (h_0,h')$ and $(r,m) \in \mathbb{Z}^{1+n}$ where $m = (m_1,\ldots,m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \qquad x^{(m)} = (x_1^{(m_1)},\ldots,x_n^{(m_n)}) = (m_1h_1,\ldots,m_nh_n).$$

Let H be the set of all h such there are $K_0 \in \mathbb{Z}$ and $M = (M_1,\ldots,M_n) \in \mathbb{N}^n$ satisfying $K_0h_0 = b_0$ and $(M_1h_1,\ldots,M_nh_n) = b$. Let $K \in \mathbb{N}$ be defined by $Kh_0 \le a < (K+1)h_0$. Write

$$\mathbb{R}_h^{1+n} = \{(t^{(r)},x^{(m)}) : (r,m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_{0.h} = E_0 \cap \mathbb{R}^{1+n}_h, \quad E_h = E \cap \mathbb{R}^{1+n}_h, \quad \partial_0 E_h = \partial_0 E_h \cap \mathbb{R}^{1+n},$$

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h \setminus \partial_0 E_h : 0 \le r \le K-1\}, \quad I_h = \{t^{(r)} : 0 \le r \le K\}.$$

For $z \in F(E_{0.h} \cup E_h, \mathbb{R})$, $\omega \in F(I_h, \mathbb{R})$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and $\omega^{(r)} = \omega(t^{(r)})$. Set

$$\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_n) : \lambda_i \in \{-1, 0, 1\} \text{ for } 1 \le i \le n \text{ and } \|\lambda\| \le 2\},$$
$$\Lambda' = \Lambda \setminus \{\theta\}, \qquad \theta = (0, \ldots, 0) \in \mathbb{R}^n,$$

and $\chi = 1 + 2n^2$. Note that χ is the number of elements of Λ. Let $\psi : \Lambda \to \{1, \ldots, \chi\}$ be a function such that $\psi(\lambda) \ne \psi(\tilde\lambda)$ for $\lambda \ne \tilde\lambda$. We assume that $\prec$ is an order in Λ defined in the following way: if $\psi(\lambda) < \psi(\tilde\lambda)$ then $\lambda \prec \tilde\lambda$. Elements of the space $\mathbb{R}^\chi$ will be denoted by $\xi = \{\xi_\lambda\}_{\lambda \in \Lambda}$. Write

$$A_h = \{\, x^{(m)} : m = (m_1, \ldots, m_n) \in \Lambda \,\}.$$

For $\zeta : A_h \to \mathbb{R}$ we put $\zeta^{(m)} = \zeta(x^{(m)})$. If $z : E_{0.h} \cup E_h \to \mathbb{R}$ and $(t^{(r)} x^{(m)}) \in E_h \setminus \partial_0 E_h$ then the function $z_{\langle r,m \rangle} : A_h \to \mathbb{R}$ is defined by

$$z_{\langle r,m \rangle}(y) = z(t^{(r)}, x^{(m)} + y), \qquad y \in A_h.$$

Solutions of difference functional equations are elements of the space $F(E_{0.h} \cup E_h, \mathbb{R})$. Since equations (1) and (3) contain the functional variable $z_{(t,x)}$ which is an element of $C(D[t,x], \mathbb{R})$, we need an interpolating operator $T_h : F(E_{0.h} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$. Additional assumptions on T_h are formulated later in this section. For $z \in F(E_{0.h} \cup E_h, \mathbb{R})$ and $(t^{(r)}, x^{(m)}) \in E_h$ we write $(T_h z)_{[r,m]}$ instead of $(T_h z)_{(t^{(r)}, x^{(m)})}$. Set $\Omega_h = E'_h \times C(B, \mathbb{R})$ and suppose that

$$f_h : \Omega_h \to \mathbb{R}, \quad G_h : \Omega_h \to \mathbb{R}^\chi, \quad G_h = \{G_{h.\lambda}\}_{\lambda \in \Lambda}$$

are given functions. For $(t, x, w) \in \Omega_h$ and $\zeta \in F(A_h, \mathbb{R})$ we put

$$G_h(t, x, w) \circ \zeta = \sum_{\lambda \in \Lambda} G_{h.\lambda}(t, x, w) \zeta^{(\lambda)}.$$

Let $F_h : \Omega_h \times F(A_h, \mathbb{R}) \to \mathbb{R}$ be defined by

$$F_h(t, x, w, \zeta) = f_h(t, x, w) + G_h(t, x, w) \circ \zeta. \tag{4}$$

We will say that f_h and G_h satisfy the condition (V) if for each $(t, x) \in E'_h$ and $w, \tilde w \in C(B, \mathbb{R})$ such that $w(\tau, y) = \tilde w(\tau, y)$ for $(\tau, y) \in D[t, x]$ we have $f_h(t, x, w) = f(t, x, \tilde w)$ and $G_h(t, x, w) = G_h(t, x, \tilde w)$. For $z \in F(E_{0.h} \cup E_h, \mathbb{R})$ and $(t^{(r)} x^{(m)}) \in E'_h$ we put

$$\mathbf{F}_{\mathrm{ex}.h}[z]^{(r,m)} = F_h(t^{(r)}, x^{(m)}, (T_h)_{[r,m]}, z_{\langle r,m \rangle}). \tag{5}$$

Let δ_0 be the difference operator defined by

$$\delta_0 z^{(r,m)} = \frac{1}{h_0}[z^{(r+1,m)} - z^{(r,m)}].$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, we consider the functional difference equation

$$\delta_0 z^{(r,m)} = \mathbf{F}_{\mathrm{ex}.h}[z]^{(r,m)} \tag{6}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h. \tag{7}$$

If G_h, f_h satisfy the condition (V) then there exists exactly one solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (6), (7).

The above problem (6), (7) is obtained in the following way. Explicit difference equations corresponding to (1) or (3) have the form

$$\delta_0 z^{(r,m)} = \boldsymbol{\Psi}_h(t^{(r)}, x^{(m)}, z(\cdot)) \tag{8}$$

where $\boldsymbol{\Psi}_h : E'_h \times F(E_{0.h} \cup E_h, \mathbb{R}) \to \mathbb{R}$ is an operator of Volterra type. Discretization of partial derivatives

$$\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z), \qquad \partial_{xx} z = [\partial_{x_i x_j} z]_{i,j=1}^n$$

leads to the following observation: the numbers $z^{(r,m+\lambda)}$ where $\lambda \in \Lambda$ appear in the definitions of the difference operators

$$\delta z = (\delta_1 z, \ldots, \delta_n z), \qquad \delta^{(2)} = [\delta_{ij} z]_{i,j=1}^n, \tag{9}$$

corresponding to these derivatives. It follows that the right hand side of (8) depends on the functional variable $z_{\langle r,m\rangle}$. Since (1) and (3) contain the functional variable $z_{(t,x)}$ which is an element of $C(D[t,x], \mathbb{R})$ we conclude that $\boldsymbol{\Psi}_h$ in (8) depends on $(T_h z)_{[r,m]}$. It is clear that assumptions on the functional variable $(T_h z)_{[r,m]}$ and on $z_{\langle r,m\rangle}$ are not the same in theorems on convergence of difference methods. So it is natural to consider the following explicit difference scheme for (1) or (3):

$$\delta_0 z^{(r,m)} = \boldsymbol{\Phi}_h(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, z_{\langle r,m\rangle}) \tag{10}$$

where $\boldsymbol{\Phi}_h : \Omega_h \times F(A_h, \mathbb{R}) \to \mathbb{R}$. We associate with (10) the initial boundary condition (7). Equations (1) and (3) are linear with respect to derivatives. It follows that explicit difference schemes for (1), (2) and (3), (2) are linear with respect to difference operators (9). Thus they have the form (6), (7) with $\mathbf{F}_{\mathrm{ex}.h}$ defined by (5). The functions $\varphi_h : E_{0.h} \cup E_h \to \mathbb{R}$, $h \in H$, approximate $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$.

Set

$$\mathbf{F}_{\mathrm{im}.h}[z]^{(r,m)} = F_h(t^{(r)}, x^{(m)}, (T_h)_{[r,m]}, z_{\langle r+1,m\rangle}). \tag{11}$$

The functional difference equation

$$\delta_0 z^{(r,m)} = \mathbf{F}_{\mathrm{im}.h}[z]^{(r,m)} \tag{12}$$

with the initial boundary condition (7) is considered to be an implicit difference method.

The above functional difference problems have the property: the numbers $z^{(r+1,m+\lambda)}$, $\lambda \in \Lambda$, appear in (12). So (12), (7) is an implicit functional difference equation. It is important that the functional variable $(T_h z)_{[r,m]}$ appears in (6) and in (12) in the classical sense.

For $z \in C(E_0 \cup E, \mathbb{R})$ and $z_h \in F(E_{0.h} \cup E_h, \mathbb{R})$ we define the seminorms

$$\|z\|_t = \max\{|z(\tau,y)| : (\tau,y) \in (E_0 \cup E) \cap ([-b_0,t] \times \mathbb{R}^n)\}, \qquad 0 \le t \le a,$$
$$\|z_h\|_{h.r} = \max\{|z_h(\tau,y)| : (\tau,y) \in (E_{0.h} \cup E_h) \cap ([-b_0,t^{(r)}] \times \mathbb{R}^n)\}, \; 0 \le r \le K.$$

Assumption $H[\varrho]$. The function $\varrho : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and it is nondecreasing with respect to both variables and for each $\eta \in \mathbb{R}_+$ there exists on $[0,a]$ the maximal solution of the Cauchy problem

$$\omega'(t) = \varrho(t,\omega(t)), \qquad \omega(0) = \eta. \tag{13}$$

Assumption $H[G_h, f_h]$. The functions $G_h : \Omega_h \to \mathbb{R}^\chi$, $f_h : \Omega_h \to \mathbb{R}$ satisfy the condition (V) and

1) for $(t,x,w) \in \Omega_h$ we have

$$G_{h.\lambda}(t,x,w) \ge 0 \quad \text{for } \lambda \in \Lambda' \quad \text{and} \quad \sum_{\lambda \in \Lambda} G_{h.\lambda}(t,x,w) \le 0, \tag{14}$$

2) there is $\varrho : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\varrho]$ is satisfied and

$$|f_h(t,x,w)| \le \varrho(t,\|w\|_B) \quad \text{for } (t,x,w) \in \Omega_h,\ h \in H,$$

3) there is $\tilde\eta \in \mathbb{R}_+$ such that

$$|\varphi_h^{(r,m)}| \le \tilde\eta \quad \text{on } E_{0.h} \quad \text{and} \quad |\varphi_h^{(r,m)}| \le \omega(t^{(r)},\tilde\eta) \quad \text{on } \partial_0 E_h$$

where $\omega(\cdot,\tilde\eta)$ is the maximal solution to (13) for $\eta = \tilde\eta$.

Assumption $H[T_h]$. The operator $T_h : F(E_{0.h} \cup E_h, \mathbb{R}) \to C(E_0 \cup E, \mathbb{R})$ satisfies the conditions:

1) for $z, \bar z \in F(E_{0.h} \cup E_h, \mathbb{R})$ we have

$$\|T_h[z] - T_h[\bar z]\|_{t^{(r)}} \le \|z - \bar z\|_{h.r}, \qquad 0 \le r \le K,$$

2) if $z : E_0 \cup E \to \mathbb{R}_+$ is of class C^1 then there is $\gamma_\star : H \to \mathbb{R}_+$ such that

$$\|T_h[z_h] - z\|_t \le \gamma_\star(h), \quad 0 \le t \le a, \qquad \lim_{h \to 0} \gamma_\star(h) = 0,$$

where z_h is the restriction of z to $E_{0.h} \cup E_h$,

3) if $\mathbf{0}_h \in F(E_{0.h} \cup E_h, \mathbb{R})$ is given by $\mathbf{0}_h(t,x) = 0$ for $(t,x) \in E_{0.h} \cup E_h$ then $T_h[\mathbf{0}_h](t,x) = 0$ for $(t,x) \in E_0 \cup E$.

An example of the operator T_h satisfying Assumption $H[T_h]$ can be found in [7, Chapter 5]. We begin with a theorem on the existence and estimates of solutions to (6), (7) and (12), (7).

THEOREM 2.1. *Suppose that Assumptions $H[G_h, f_h]$ and $H[T_h]$ are satisfied. Then:*

I. *There exists exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to* (12), (7) *and*

$$|v_h^{(r,m)}| \le \omega(t^{(r)}, \tilde{\eta}) \quad \text{on } E_h. \tag{15}$$

II. *If additionally the steps of the mesh satisfy the condition*

$$1 + h_0 G_{h.\theta}(t, x, w) \ge 0 \quad \text{on } \Omega_h, \tag{16}$$

then there is exactly one solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (6), (7) *and*

$$|u_h^{(r,m)}| \le \omega(t^{(r)}, \tilde{\eta}) \quad \text{on } E_h. \tag{17}$$

Proof. **I.** Suppose that $0 \le r < K$ is fixed and that the solution v_h to (12), (7) is given on $(E_{0.h} \cup E_h) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n)$. We prove that the values $v_h^{(r+1,m)}$, $-M \le m \le M$, exist and are unique. It is sufficient to show that there exists exactly one solution of the linear system

$$z^{(r+1,m)} = v_h^{(r,m)} + h_0\, F_h(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, z_{\langle r+1,m\rangle}) \quad \text{for } (t^{(r)}, x^{(m)}) \in E'_h, \tag{18}$$

$$z^{(r+1,m)} = \varphi_h^{(r+1,m)} \quad \text{for } (t^{(r+1)}, x^{(m)}) \in \partial_0 E_h, \tag{19}$$

where F_h is given by (4). It follows from (14) that the homogeneous system corresponding to (18), (19) has exactly one zero solution. Thus the solution v_h is given on $(E_{0.h} \cup E_h) \cap ([-b_0, t^{(r+1)}] \times \mathbb{R}^n)$ and it is unique. Since v_h is given on $E_{0.h}$, the proof of the existence and uniqueness of a solution to (12), (7) is completed by induction with respect to r, $0 \le r \le K$.

Now we prove (15). We conclude from (12) that

$$\begin{aligned} v_h^{(r+1,m)}&[1 + h_0 G_{h.\theta}(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]})] \\ &= v_h^{(r+1,m)} + h_0 \sum_{\lambda \in \Lambda'} G_{h.\lambda}(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}) v_h^{(r+1,m+\lambda)} \\ &\quad + h_0 f_h(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}) \end{aligned} \tag{20}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Let $\omega_h : I_h \to \mathbb{R}_+$ be defined by $\omega_h^{(r)} = \|v_h\|_{h.r}$, $0 \le r \le K$. It follows from condition 2) of Assumption $H[G_h, f_h]$ and from (14), (20) that

$$\omega_h^{(r+1)} \le \omega_h^{(r)} + h_0\, \varrho(t^{(r)}, \omega_h^{(r)}) \quad \text{for } 0 \le r \le K-1$$

and $\omega_h^{(r)} \le \tilde{\eta}$. The function $\omega(\cdot, \tilde{\eta})$ satisfies the recurrent inequality

$$\omega(t^{(r+1)}, \tilde{\eta}) \ge \omega(t^{(r)}, \tilde{\eta}) + h_0 \varrho(t^{(r)}, \omega(t^{(r)}, \tilde{\eta})), \quad 0 \le r \le K-1. \tag{21}$$

Since $\omega_h^{(0)} \le \omega(t^{(0)}, \tilde{\eta})$, we have $\omega_h^{(r)} \le \omega(t^{(r)}, \tilde{\eta})$ for $0 \le r \le K$ and (15) follows.

II. It is clear that there exists exactly one solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (6), (7). We conclude from (6) that

$$(22)\quad u_h^{(r+1,m)} = u_h^{(r,m)}[1 + h_0 G_{h.\theta}(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]})] + h_0 \sum_{\lambda \in \Lambda'} G_{h.\lambda}(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]})\, u_h^{(r,m+\lambda)} + h_0 f_h(t^{(r)}, x^{(m)}, (T_h u_h)_{[r,m]})$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Let $\tilde{\omega}_h : I_h \to \mathbb{R}_+$ be defined by $\tilde{\omega}_h^{(r)} = \|u_h\|_{h.r}$, $0 \le r \le K$. It follows from condition 2) of Assumption $H[G_h, f_h]$ and from (14), (16), (22) that

$$\tilde{\omega}_h^{(r+1)} \le \tilde{\omega}_h^{(r)} + h_0 \varrho(t^{(r)}, \tilde{\omega}_h^{(r)}) \quad \text{for } 0 \le r \le K-1,$$

and $\tilde{\omega}_h^{(0)} \le \tilde{\eta}$. The above relations and (21) imply (17). This completes the proof of the theorem.

We will consider approximate solutions to (6), (7) and (12), (7). Suppose that the functions $z_h : E_{0.h} \cup E_h \to \mathbb{R}$, $h \in H$, and $\alpha_0, \gamma : H \to \mathbb{R}_+$ satisfy

$$(23)\qquad |\delta_0 z_h^{(r,m)} - \mathbf{F}_{\mathrm{ex}.h}[z_h]^{(r,m)}| \le \gamma(h) \quad \text{on } E'_h$$

and

$$(24)\qquad |z_h^{(r,m)} - \varphi_h^{(r,m)}| \le \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h,$$

$$(25)\qquad \lim_{h \to 0} \alpha_0(h) = 0, \qquad \lim_{h \to 0} \gamma(h) = 0.$$

The functions z_h, $h \in H$, satisfying the above relations are considered to be approximate solutions to (6), (7). If

$$(26)\qquad |\delta_0 z_h^{(r,m)} - \mathbf{F}_{\mathrm{im}.h}[z_h]^{(r,m)}| \le \gamma(h) \quad \text{on } E'_h$$

and conditions (24), (25) are satisfied then z_h, $h \in H$, are approximate solutions to (12), (7).

We give a theorem on estimates of differences between the exact and approximate solutions to (6), (7) and (12), (7). Suppose that Assumptions $H[G_h, f_h]$ and $H[T_h]$ are satisfied. Write $\tilde{C} = \omega(a, \tilde{\eta})$ and

$$K_{C(B,\mathbb{R})}[\tilde{C}] = \{w \in C(B, \mathbb{R}) : \|w\|_B \le \tilde{C}\}.$$

Assumption $H_0[\sigma]$. The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions:

1) σ is continuous and it is nondecreasing with respect to both variables,
2) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)), \qquad \omega(0) = 0,$$

is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$.

Theorem 2.2. *Suppose that Assumptions $H[G_h, f_h]$ and $H[T_h]$ are satisfied and there exist $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $Y_h \subset F(A_h, \mathbb{R})$ such that Assumption $H_0[\sigma]$ is satisfied and for $w, \tilde{w} \in K_{C(B,\mathbb{R})}[\tilde{C}]$ and $\zeta \in Y_h$ we have*

$$|F_h(t,x,w,\zeta) - F_h(t,x,\tilde{w},\zeta)| \le \sigma(t, \|w-\tilde{w}\|_B), \qquad (t,x) \in E'_h. \tag{27}$$

I. *Suppose that $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to* (12), (7) *and $\tilde{v}_h : E_{0.h} \cup E_h \to \mathbb{R}$ satisfies the conditions:*

(i) $\|\tilde{v}_h\|_{h.r} \le \tilde{C}$ *and* $(\tilde{v}_h)_{\langle r,m\rangle} \in Y_h$ *for* $0 \le r \le K$,

(ii) *conditions* (24)–(26) *are satisfied for* $z_h = \tilde{v}_h$.

Then there is $\alpha : H \to \mathbb{R}_+$ such that

$$|(v_h - \tilde{v}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{28}$$

II. *Suppose that the steps of the mesh satisfy* (16) *and $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to* (6), (7) *and $\tilde{u}_h : E_{0.h} \cup E_h \to \mathbb{R}$ satisfies the conditions:*

(i) $\|\tilde{u}_h\|_{h.r} \le \tilde{C}$ *and* $(\tilde{u}_h)_{\langle r,m\rangle} \in Y_h$ *for* $0 \le r \le K$,

(ii) *conditions* (23)–(25) *are satisfied with* $z_h = \tilde{u}_h$.

Then there is $\alpha : H \to \mathbb{R}_+$ such that

$$|(u_h - \tilde{u}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0. \tag{29}$$

Proof. **I.** The existence of the solution v_h to (12), (7) follows from Theorem 2.1. Let $\Gamma_{\mathrm{im}.h} : E'_h \to \mathbb{R}$ be defined by

$$\delta_0 \tilde{v}_h^{(r,m)} = \mathbf{F}_{\mathrm{im}.h}[\tilde{v}_h]^{(r,m)} + \Gamma_{\mathrm{im}.h}^{(r,m)}.$$

Then $|\Gamma_{\mathrm{im}.h}^{r,m)}| \le \gamma(h)$ on E'_h. It follows from (12) that

$$\begin{aligned}(\tilde{v}_h - v_h)^{(r+1,m)}&[1 - h_0\, G_{h.\theta}(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]})] \\ &= (\tilde{v}_h - v_h)^{(r,m)} \\ &\quad + h_0 \sum_{\lambda\in\Lambda'} G_{h.\lambda}(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]})(\tilde{v}_h - v_h)^{(r,m+\lambda)} + \Gamma_{\mathrm{im}.h}^{(r,m)} \\ &\quad + F_h(t^{(r)}, x^{(m)}, (T_h \tilde{v}_h)_{[r,m]}, (\tilde{v}_h)_{\langle r+1,m\rangle}) \\ &\quad - F_h(t^{(r)}, x^{(m)}, (T_h v_h)_{[r,m]}, (\tilde{v}_h)_{\langle r+1,m\rangle}).\end{aligned} \tag{30}$$

Let $\varepsilon_h : I_h \to \mathbb{R}_+$ be given by $\varepsilon_h^{(r)} = \|\tilde{v}_h - v_h\|_{h.r}$, $0 \le r \le K$. We conclude from Assumptions $H[G_h, f_h]$, $H[T_h]$ and $H[\sigma]$ and from (30) that

$$\varepsilon_h^{(r+1)} \le \varepsilon_h^{(r)} + h_0 \sigma(t^{(r)}, \varepsilon_h^{(r)}) + h_0 \gamma(h), \qquad 0 \le r \le K-1, \tag{31}$$

and $\varepsilon_h^{(0)} \le \alpha_0(h)$. Denote by $\omega(\cdot, h)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \sigma(t, \omega(t)) + \gamma(h), \qquad \omega(0) = \alpha_0(h). \tag{32}$$

It follows that $\omega(\cdot, h)$ is defined on $[0,a]$ and $\lim_{h\to 0}\omega(t,h)=0$ uniformly on $[0,a]$. Moreover we have

$$\omega(t^{(r+1)},h)\ge \omega(t^{(r)},h)+h_0\sigma(t^{(r)},\omega(t^{(r)},h))+h_0\gamma(h),\qquad 0\le r\le K-1.$$

This gives $\varepsilon_h^{(r)}\le \omega(t^{(r)},h)$ for $0\le r\le K$ and condition (28) is satisfied with $\alpha(h)=\omega(a,h)$.

II. It is clear that there exists a solution u_h to (6), (7). Let $\Gamma_{\mathrm{ex}.h}: E'_h\to\mathbb{R}$ be defined by

$$\delta_0\tilde{v}_h^{(r,m)}=\mathbf{F}_{\mathrm{ex}.h}[\tilde{v}_h]^{(r,m)}+\Gamma_{\mathrm{ex}.h}^{(r,m)}.$$

Then $|\Gamma_{\mathrm{ex}.h}^{r,m)}|\le\gamma(h)$ on E'_h. It follows from (6) that

$$\begin{aligned}(33)\qquad (\tilde{v}_h-v_h)^{(r+1,m)}&=(\tilde{v}_h-v_h)^{(r,m)}[1+h_0\,G_{h.\theta}(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]})]\\&\quad+h_0\sum_{\lambda\in\Lambda'}G_{h.\lambda}(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]})(\tilde{u}_h-u_h)^{(r,m+\lambda)}+\Gamma_{\mathrm{im}.h}^{(r,m)}\\&\quad+F_h(t^{(r)},x^{(m)},(T_h\tilde{u}_h)_{[r,m]},(\tilde{u}_h)_{\langle r,m\rangle})-F_h(t^{(r)},x^{(m)},(T_hu_h)_{[r,m]},(\tilde{u}_h)_{\langle r,m\rangle}).\end{aligned}$$

Let $\varepsilon_h: I_h\to\mathbb{R}_+$ be given by $\varepsilon_h^{(r)}=\|\tilde{u}_h-u_h\|_{h.r}$, $0\le r\le K$. We conclude from Assumptions $H[G_h,f_h]$, $H[T_h]$ and $H[\sigma]$ and from (16), (33) that the function ε_h satisfies (31) and $\varepsilon_h^{(0)}\le\alpha_0(h)$. Thus condition (29) is satisfied with $\alpha(h)=\omega(a,h)$ where $\omega(\cdot,h)$ is the maximal solution to (32).

This completes the proof of the theorem.

Remark 2.3. Suppose that all the assumptions of Theorem 2.2 hold with $\sigma(t,p)=Lp$ on $[0,a]\times\mathbb{R}_+$ where $L\in\mathbb{R}_+$, so the operator F_h satisfies the Lipschitz condition with respect to the functional variable on $E'_h\times K_{C(B,\mathbb{R})}[\tilde{C}]\times Y_h$. Then $|(v_h-\tilde{v}_h)^{(r,m)}|\le\tilde{\alpha}(h)$ on E_h and $|(u_h-\tilde{v}_h)^{(r,m)}|\le\tilde{\alpha}(h)$ on E_h where

$$(34)\qquad \tilde{\alpha}(h)=\alpha_0(h)e^{La}+\gamma(h)\frac{e^{La}-1}{L}\quad\text{if } L>0,$$

$$(35)\qquad \tilde{\alpha}(h)=\alpha_0(h)+a\gamma(h)\quad\text{if } L=0.$$

The above estimates are obtained by solving problem (32) with $\sigma(t,p)=Lp$.

It follows that we have obtained the same estimates of errors for the implicit and for the explicit difference equations.

3. First order partial functional differential equations. We formulate difference methods for (1), (2). Consider the operator $U_h:\Omega_h\times F(A_h,\mathbb{R})\to\mathbb{R}$ defined in the following way. Let $(t^{(r)},x^{(m)},w,\zeta)\in\Omega_h\times F(A_h,\mathbb{R})$. Write

$$J_+^{(r,m)}[w]=\{i\in\{1,\dots,n\}: f_i(t^{(r)},x^{(m)},w)\ge 0\},$$

$$J_-^{(r,m)}[w]=\{1,\dots,n\}\setminus J_+^{(r,m)}[w]$$

and

$$U_h(t^{(r)},x^{(m)},w,\zeta)=\sum_{i=1}^{n}f_i(t^{(r)},x^{(m)},w)\delta_j\zeta^{(\theta)}+g(t^{(r)},x^{(m)},w).$$

The expressions $(\delta_1\zeta^{(\theta)},\dots,\delta_n\zeta^{(\theta)})$ are given in the following way. Set $e_i=(0,\dots,0,1,0,\dots,0)\in\mathbb{R}^n$ with 1 in the ith place and

$$\delta_j\zeta^{(\theta)}=\frac{1}{h_j}[\zeta^{(e_j)}-\zeta^{(\theta)}]\qquad\text{for } j\in J_+^{(r,m)}[w],$$
$$\delta_j\zeta^{(\theta)}=\frac{1}{h_j}[\zeta^{(\theta)}-\zeta^{(-e_j)}]\qquad\text{for } j\in J_-^{(r,m)}[w].$$

For $z\in F(E_{0.h}\cup E_h,\mathbb{R})$ and $(t^{(r)},x^{(m)})\in E'_h$ we write

$$\mathbb{U}_{\mathrm{ex}.h}[z]^{(r,m)}=U_h(t^{(r)},x^{(m)},(T_hz)_{[r,m]},z_{\langle r,m\rangle}),$$
$$\mathbb{U}_{\mathrm{im}.h}[z]^{(r,m)}=U_h(t^{(r)},x^{(m)},(T_hz)_{[r,m]},z_{\langle r+1,m\rangle}).$$

Given $\varphi_h:E_{0.h}\cup E_h\to\mathbb{R}$, we approximate the classical solution to (1), (2) with the solution of the functional difference equation

$$\delta_0z^{(r,m)}=\mathbb{U}_{\mathrm{ex}.h}[z]^{(r,m)}\tag{36}$$

with the initial boundary condition

$$z^{(r,m)}=\varphi_h^{(r,m)}\qquad\text{on } E_{0.h}\cup\partial_0E_h.\tag{37}$$

The difference equation

$$\delta_0z^{(r,m)}=\mathbb{U}_{\mathrm{im}.h}[z]^{(r,m)}\tag{38}$$

with initial boundary condition (37) is considered to be an implicit difference scheme for (1), (2).

Assumption $H_0[\mathbf{f},g,\varphi]$. The functions $\mathbf{f}:\Xi\to\mathbb{R}^n$, $g:\Xi\to\mathbb{R}$ and $\varphi:E_0\cup\partial_0E\to\mathbb{R}$ are such that

1) $\mathbf{f}$ and g are continuous and satisfy the condition (V),
2) there is $\varrho:[0,a]\times\mathbb{R}_+\to\mathbb{R}_+$ such that of Assumption $H[\varrho]$ holds and
$$|g(t,x,w)|\le\varrho(t,\|w\|_B)\qquad\text{on } \Xi,$$
3) $\varphi\in C(E_0\cup\partial_0E,\mathbb{R})$, $\varphi_h\in F(E_{0.h}\cup\partial_0E_h,\mathbb{R})$ and there is $\alpha_0:H\to\mathbb{R}_+$ such that
$$|\varphi(t,x)-\varphi_h(t,x)|\le\alpha_0(h)\quad\text{on } E_{0.h}\cup\partial_0E_h\quad\text{and}\quad\lim_{h\to0}\alpha_0(h)=0,$$
4) the constant $\tilde\eta\in\mathbb{R}_+$ is defined by the relations

$$|\varphi(t,x)|\le\tilde\eta\quad\text{on } E_0,\qquad|\varphi_h(t,x)|\le\tilde\eta\quad\text{on } E_{0.h},\tag{39}$$
$$|\varphi(t,x)|\le\omega(t,\tilde\eta)\quad\text{on } \partial_0E,\qquad|\varphi_h(t,x)|\le\omega(t,\tilde\eta)\quad\text{on } \partial_0E_h,\tag{40}$$

where $\omega(\cdot,\tilde\eta)$ is the maximal solution to (13) for $\eta=\tilde\eta$.

We give estimates of solutions to (1), (2).

LEMMA 3.1. *If Assumption $H_0[\mathbf{f}, g, \varphi]$ is satisfied and $\tilde{z} : E_0 \cup E \to \mathbb{R}$ is a solution to* (1), (2) *and $\tilde{z}$ is of class C^1 then*

$$|\tilde{z}(t, x)| \le \omega(t, \tilde{\eta}) \quad \text{on } E. \tag{41}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\cdot, \tilde{\eta}, \varepsilon)$ the maximal solution of the Cauchy problem

$$\omega'(t) = \varrho(t, \omega(t)) + \varepsilon, \qquad \omega(0) = \tilde{\eta} + \varepsilon. \tag{42}$$

Then $\omega(\,\cdot\,, \tilde{\eta}, \varepsilon)$ is defined on $[0, a]$ and

$$\lim_{\varepsilon \to 0} \omega(t, \tilde{\eta}, \varepsilon) = \omega(t, \tilde{\eta}) \quad \text{uniformly on } [0, a]. \tag{43}$$

Write $\tilde{\omega}(t) = \|\tilde{z}\|_t$, $t \in [0, a]$. We prove that

$$\tilde{\omega}(t) < \omega(t, \tilde{\eta}, \varepsilon) \quad \text{for } t \in [0, a]. \tag{44}$$

Suppose for contradiction that assertion (44) fails to be true. Then there is $(\tilde{t}, \tilde{x}) \in (0, a] \times (-b, b)$ such that

$$\tilde{\omega}(t) < \omega(t, \tilde{\eta}, \varepsilon) \quad \text{for } t \in [0, \tilde{t}) \quad \text{and} \quad \tilde{\omega}(\tilde{t}) = \omega(\tilde{t}, \tilde{\eta}, \varepsilon) = |\tilde{z}(\tilde{t}, \tilde{x})|.$$

Two cases are possible: either (i) $\tilde{z}(\tilde{t}, \tilde{x}) = \omega(\tilde{t}, \tilde{\eta}, \varepsilon)$ or (ii) $\tilde{z}(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon)$. In the first case,

$$D_-\tilde{\omega}(\tilde{t}) \ge \omega'(\tilde{t}, \tilde{\eta}, \varepsilon) \tag{45}$$

where D_- is the left hand lower Dini derivative. Since $\partial_x \tilde{z}(\tilde{t}, \tilde{x}) = \theta$, we have

$$D_-\tilde{\omega}(\tilde{t}) \le \partial_t \tilde{z}(\tilde{t}, \tilde{x}) = g(\tilde{t}, \tilde{x}, \tilde{z}_{(\tilde{t}, \tilde{x})}) \le \varrho(\tilde{t}, \omega(\tilde{t}, \tilde{\eta}, \varepsilon)) < \omega'(\tilde{t}, \tilde{\eta}, \varepsilon),$$

which contradicts (45). The case $\tilde{z}(\tilde{t}, \tilde{x}) = -\omega(\tilde{t}, \tilde{\eta}, \varepsilon)$ can be treated in a similar way. Thus (44) is proved. Letting ε tend to zero in (44) we obtain (41). This completes the proof.

LEMMA 3.2. *Suppose that Assumption $H[T_h]$ and $H_0[\mathbf{f}, g, \varphi]$ are satisfied. Then:*

I. *There exists exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to* (38), (37), *and*

$$|v_h^{(r,m)}| \le \omega(t^{(r)}, \tilde{\eta}) \quad \text{on } E_h.$$

II. *Moreover, if*

$$1 - h_0 \sum_{i=1}^{n} \frac{1}{h_i} |f_i(t, x, w)| \ge 0 \quad \text{on } \Xi \tag{46}$$

then the solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (36), (37) *satisfies the condition*

$$|u_h^{(r,m)}| \le \omega(t^{(r)}, \tilde{\eta}) \quad \text{on } E_h.$$

Proof. We apply Theorem 2.1. Define $G_h : \Omega_h \to \mathbb{R}^{\chi}$, $G_h = \{G_{h.\lambda}\}_{\lambda\in\Lambda}$, and $f_h : \Omega_h \to \mathbb{R}$ in the following way. Let $(t^{(r)}, x^{(m)}, w) \in \Omega_h$. Write

$$\Lambda_+^{(r,m)}[w] = \{\lambda \in \Lambda : \text{there is } j \in J_+^{(r,m)}[w] \text{ such that } \lambda = e_j\},$$
$$\Lambda_-^{(r,m)}[w] = \{\lambda \in \Lambda : \text{there is } j \in J_-^{(r,m)}[w] \text{ such that } \lambda = -e_j\}.$$

Set

$$f_h(t^{(r)}, x^{(m)}, w) = g(t^{(r)}, x^{(m)}, w),$$
$$G_{h.\theta}(t^{(r)}, x^{(m)}, w) = -\sum_{j=1}^{n} \frac{1}{h_j} |f_j(t^{(r)}, x^{(m)}, w)|,$$
$$G_{h.e_j}(t^{(r)}, x^{(m)}, w) = \frac{1}{h_j} f_j(t^{(r)}, x^{(m)}, w) \qquad \text{for } j \in J_+^{(r,m)}[w],$$
$$G_{h.-e_j}(t^{(r)}, x^{(m)}, w) = -\frac{1}{h_j} f_j(t^{(r)}, x^{(m)}, w) \quad \text{for } j \in J_-^{(r,m)}[w],$$
$$G_{h.\lambda}(t^{(r)}, x^{(m)}, w) = 0 \quad \text{for } \lambda \in \Lambda \setminus [\Lambda_{i.+}^{(r,m)}[w] \cup \Lambda_{i.-}^{(r,m)}[w] \cup \{\theta\}].$$

Then Assumption $H[G_h]$ is satisfied and problems (36), (37) and (38), (37) are equivalent to (6), (7) and (12), (7) respectively. Our lemma follows from Theorem 2.1.

Remark 3.3. The stability of difference equations generated by hyperbolic systems of conservation laws is strictly connected with the so-called Courant–Friedrichs–Levy (CFL) conditions (see [6, Chapter III]). The (CFL) conditions for the quasilinear equation (1) have the form (46).

Suppose that Assumptions $H[T_h]$ and $H_0[\mathbf{f}, g, \varphi]$ are satisfied. Write $\tilde{C} = \omega(a, \tilde{\eta})$ and

$$X[\tilde{C}] = \{w \in C(B, \mathbb{R}) : \|w\|_B \le \tilde{C}\}. \tag{47}$$

Assumption $H[\sigma]$. The function $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:

1) σ is continuous and it is nondecreasing with respect to both variables,
2) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $c \ge 1$ the maximal solution of the Cauchy problem
$$\omega'(t) = c\sigma(t, \omega(t)), \qquad \omega(0) = 0,$$
is $\tilde{\omega}(t) = 0$ for $t \in [0, a]$.

Assumption $H[\mathbf{f}, g, \varphi]$. The functions $\mathbf{f} : \Xi \to \mathbb{R}^n$ and $g : \Xi \to \mathbb{R}$, $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ satisfy Assumption $H_0[\mathbf{f}, g, \varphi]$ and there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\sigma]$ is satisfied and

$$\|\mathbf{f}(t, x, w) - \mathbf{f}(t, x, \tilde{w}\| \le \sigma(t, \|w - \tilde{w}\|_B), \tag{48}$$
$$|g(t, x, w) - g(t, x, \tilde{w}| \le \sigma(t, \|w - \tilde{w}\|_B), \tag{49}$$

where $(t, x) \in E$, $w, \tilde{w} \in X[\tilde{C}]$.

REMARK 3.4. It is important that we have assumed inequalities (48), (49) for $w, \tilde{w} \in X[\tilde{C}]$. It is clear that there are differential integral equations and differential equations with deviated variables such that Assumption $H[\mathbf{f}, g, \sigma]$ holds and estimates (48), (49) are not satisfied on Ξ.

THEOREM 3.5. *Suppose that Assumptions $H[T_h]$ and $H[\mathbf{f}, g, \varphi]$ are satisfied and $\tilde{z} : E_0 \cup E \to \mathbb{R}$ is a solution to* (1), (2) *and $\tilde{z}$ is of class C^1 and $\tilde{z}_h$ is the restriction of $\tilde{z}$ to the set $E_{0.h} \cup E_h$. Then:*

I. *There exists exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to* (38), (37) *and there is $\alpha : H \to \mathbb{R}_+$ such that*

$$|(v_h - \tilde{z}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0. \tag{50}$$

II. *If condition* (46) *is satisfied then there is $\alpha : H \to \mathbb{R}_+$ such that*

$$|(u_h - \tilde{z}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \alpha(h) = 0, \tag{51}$$

where $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ is a solution to (36), (37).

Proof. We apply Theorem 2.2 to prove (50) and (51). Let $\tilde{c} \in \mathbb{R}_+$ be such that

$$\|\partial_x \tilde{z}(t, x)\| \le \tilde{c} \quad \text{on } E.$$

Denote by Y_h the class of all $\zeta \in F(A_h, \mathbb{R})$ such that

$$\left| \frac{1}{h_j} [\zeta^{(e_j)} - \zeta^{(\theta)}] \right| \le \tilde{c}, \qquad \left| \frac{1}{h_j} [\zeta^{(\theta)} - \zeta^{(-e_j)}] \right| \le \tilde{c}, \qquad j = 1, \ldots, n.$$

Then we have

$$\|\tilde{z}_h\|_{h.r} \le \tilde{C} \quad \text{and} \quad (\tilde{z}_h)_{\langle r,m \rangle} \in Y_h \quad \text{for } 0 \le r \le K.$$

It follows from Assumption $H[\mathbf{f}, g, \varphi]$ that for $w, \tilde{w} \in X_h[\bar{C}]$ and $\zeta \in Y_h$ we have

$$|U_h(t, x, w, \zeta) - U_h(t, x, \tilde{w}, \zeta)| \le (1 + \tilde{c})\sigma(t, \|w - \tilde{w}\|_B), \qquad (t, x) \in E'_h.$$

It is clear that condition (46) for equation (36) is equivalent to (15) for (6). Thus all the assumptions of Theorem 2.2 are satisfied and assertions (50), (51) follow.

REMARK 3.6. Suppose that all the assumptions of Theorem 3.5 are satisfied with $\sigma(t, p) = \tilde{L}p$ on $[0, a] \times \mathbb{R}_+$ where $\tilde{L} \in \mathbb{R}_+$. Then there is $L \in \mathbb{R}_+$ such that $|(\tilde{z}_h - v_h)^{(r,m)}| \le \tilde{\alpha}(h)$ on E_h and $|(\tilde{z}_h - u_h)^{(r,m)}| \le \tilde{\alpha}(h)$ on E_h where $\tilde{\alpha} : H \to \mathbb{R}_+$ is given by (34), (35).

Now we present numerical examples. Put $n = 2$. Solutions of the initial boundary value problems below are defined on $E = [0, 0.5] \times [-1, 1] \times [-1, 1]$.

Consider the differential equation with deviated variables

$$\begin{aligned}&\partial_t z(t,x,y)\\&=4x\{1+\cos[z(t,0.25(x+\sqrt{3}y),0.25(\sqrt{3}x-y))-z(t,0.5x,0.5y)]\}\partial_x z(t,x,y)\\&\quad+4y\{1-\cos[z(t,0.25\sqrt{2}(x-y),0.25\sqrt{2}(x+y))-z(t,0.5x,0.5y)]\}\partial_y z(t,x,y)\\&\quad+z(t,x,y)(x^2+y^2-1-16x^2t)\end{aligned}$$

with the initial boundary conditions

$$z(0,x,y)=1,\qquad (x,y)\in[-1,1]\times[-1,1], \tag{52}$$

$$z(t,-1,y)=z(t,1,y)=e^{ty^2},\qquad t\in[0,0.5],\ y\in[-1,1], \tag{53}$$

$$z(t,x,-1)=z(t,x,1)=e^{tx^2},\qquad t\in[0,0.5],\ x\in[-1,1]. \tag{54}$$

The solution of the above problem is $v(t,x,y)=\exp[t(x^2+y^2-1)]$. The following tables show maximal values of errors for several step sizes.

Table 1. Errors for explicit Euler method

h_0	$h_1=h_2$	Maximal error	Time [s]
2^{-8}	2^{-4}	$6.49082661\cdot 10^{-3}$	0.063
2^{-9}	2^{-5}	$3.35139036\cdot 10^{-3}$	0.502
2^{-10}	2^{-6}	$1.70254707\cdot 10^{-3}$	4.039
2^{-11}	2^{-7}	$8.58257280\cdot 10^{-4}$	32.574

Table 2. Explicit Euler method, violated CFL condition

h_0	$h_1=h_2$	Maximal error
2^{-6}	2^{-4}	$2.31260490\cdot 10^{0}$
2^{-7}	2^{-5}	$7.77142720\cdot 10^{7}$
2^{-8}	2^{-6}	$5.43515795\cdot 10^{21}$
2^{-9}	2^{-7}	$+\infty$

Now we consider the implicit Euler method and the (CFL) condition is not satisfied.

Table 3. Errors for implicit Euler method

h_0	$h_1=h_2$	Maximal error	Time [s]
2^{-8}	2^{-6}	$6.11531734\cdot 10^{-3}$	0.982
2^{-9}	2^{-7}	$3.10254097\cdot 10^{-3}$	8.194
2^{-10}	2^{-8}	$1.56092644\cdot 10^{-3}$	63.151
2^{-11}	2^{-9}	$8.04424286\cdot 10^{-4}$	510.758

Consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x,y) &= 4x\Big\{1+\sin\Big[2t\int_{-1}^{x} sz(t,s,y)\,ds - z(t,x,y)\Big]\Big\}\partial_x z(t,x,y)\\ &\quad + 4y\Big\{1-\sin\Big[2t\int_{-1}^{y} sz(t,x,s)\,ds - z(t,x,y)\Big]\Big\}\partial_y z(t,x,y)\\ &= f(t,x,y)z(t,x,y)\end{aligned}$$

with the initial boundary conditions (52)–(54) where $f(t,x,y) = x^2+y^2-1-8t(x^2+y^2)+8tx^2\sin[\exp(ty^2)]-8ty^2\sin[\exp(tx^2)]$. The solution of the above problem is $v(t,x,y) = \exp[t(x^2+y^2-1)]$. The following tables show maximal error values for several step sizes.

Table 4. Errors for explicit Euler method

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-8}	2^{-4}	$6.63816929 \cdot 10^{-3}$	0.063
2^{-9}	2^{-5}	$3.43430042 \cdot 10^{-3}$	0.502
2^{-10}	2^{-6}	$1.74641609 \cdot 10^{-3}$	4.039
2^{-11}	2^{-7}	$8.80718231 \cdot 10^{-4}$	32.574

Table 5. Explicit Euler method, violated CFL condition

h_0	$h_1 = h_2$	Maximal error
2^{-6}	2^{-4}	$4.39485836 \cdot 10^{0}$
2^{-7}	2^{-5}	$1.76221740 \cdot 10^{7}$
2^{-8}	2^{-6}	$1.99352761 \cdot 10^{19}$
2^{-9}	2^{-7}	$+\infty$

Now we consider the implicit Euler method and the (CFL) condition is not satisfied.

Table 6. Errors for implicit Euler method

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-8}	2^{-6}	$6.07287884 \cdot 10^{-3}$	1.353
2^{-9}	2^{-7}	$3.11011076 \cdot 10^{-3}$	11.014
2^{-10}	2^{-8}	$1.57845020 \cdot 10^{-3}$	88.759
2^{-11}	2^{-9}	$7.96794891 \cdot 10^{-4}$	715.743

Our considerations reveal the following relations between explicit and implicit difference methods for (1), (2). Assumptions on the regularity of given functions are the same in the theorems on convergence of explicit and implicit difference schemes. We need condition (46) on the mesh for explicit difference methods, but not for implicit methods. Error estimates are the same for both methods. Tables 2, 3 and 5, 6 show that there are implicit difference methods which are convergent, while the corresponding explicit schemes are not.

4. Parabolic functional differential equations. We formulate difference methods for (3), (2). Consider the operator $W_h : \Omega_h \times F(A_h, \mathbb{R}) \to \mathbb{R}$ defined in the following way. Let $(t^{(r)}, x^{(m)}, w, \zeta) \in \Omega_h \times F(A_h, \mathbb{R})$. Write

$$S_{+}^{(r,m)}[w] = \{(i,j) : 1 \le i, j \le n,\ i \ne j,\ F_{ij}(t^{(r)}, x^{(m)}, w) \ge 0\},$$
$$S_{-}^{(r,m)}[w] = \{(i,j) : 1 \le i, j \le n,\ i \ne j,\ F_{ij}(t^{(r)}, x^{(m)}, w) < 0\},$$

and

$$\delta_i^{+}\zeta^{(\theta)} = \frac{1}{h_i}[\zeta^{(e_i)} - \zeta^{(\theta)}], \qquad \delta_i^{-}\zeta^{(\theta)} = \frac{1}{h_i}[\zeta^{(\theta)} - \zeta^{(-e_i)}], \qquad 1 \le i \le n.$$

Set

$$(55)\quad W_h(t^{(r)}, x^{(m)}, w, \zeta) = \sum_{i,j=1}^{n} F_{ij}(t^{(r)}, x^{(m)}, w)\delta_{ij}\zeta^{(\theta)} + \sum_{i=1}^{n} G_i(t^{(r)}, x^{(m)}, w)\delta_i\zeta^{(\theta)} + G(t^{(r)}, x^{(m)}, w).$$

The expressions $\delta\zeta^{(\theta)} = (\delta_1\zeta^{(\theta)}, \ldots, \delta_n\zeta^{(\theta)})$ and $\delta^{(2)}\zeta^{(\theta)} = [\delta_{ij}\zeta^{(\theta)}]_{i,j=1}^{n}$ are defined in the following way:

$$\delta_i\zeta^{(\theta)} = \frac{1}{2h_i}[\zeta^{(e_i)} - \zeta^{(-e_i)}], \qquad \delta_{ii}\zeta^{(\theta)} = \delta_i^{+}\delta_i^{-}\zeta^{(\theta)} \qquad \text{for } i = 1, \ldots, n.$$

and

$$\delta_{ij}\zeta^{(\theta)} = \begin{cases} \frac{1}{2}[\delta_i^{+}\delta_j^{+}\zeta^{(\theta)} + \delta_i^{-}\delta_j^{-}\zeta^{(\theta)}] & \text{for } (i,j) \in S_{+}^{(r,m)}[w], \\ \frac{1}{2}[\delta_i^{+}\delta_j^{-}\zeta^{(\theta)} + \delta_i^{-}\delta_j^{+}\zeta^{(\theta)}] & \text{for } (i,j) \in S_{-}^{(r,m)}[w]. \end{cases}$$

For $z \in F(E_{0.h} \cup E_h)$ and $(t^{(r)}, x^{(m)}) \in E'_h$ we put

$$\mathbb{W}_{\mathrm{ex}.h}[z]^{(r,m)} = W_h(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, z_{\langle r,m\rangle}),$$
$$\mathbb{W}_{\mathrm{im}.h}[z]^{(r,m)} = W_h(t^{(r)}, x^{(m)}, (T_h z)_{[r,m]}, z_{\langle r+1,m\rangle}).$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$, we approximate classical solutions to (3), (2) with solutions of the functional difference equation

$$(56)\qquad \delta_0 z^{(r,m)} = \mathbb{W}_{\mathrm{ex}.h}[z]^{(r,m)}$$

with the initial boundary condition

$$(57)\qquad z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h} \cup \partial_0 E_h.$$

The functional difference equation

$$\delta_0 z^{(r,m)} = \mathbb{W}_{\mathrm{im}.h}[z]^{(r,m)} \tag{58}$$

with the initial boundary condition (57) is considered to be an implicit difference scheme for (3), (2).

We first construct estimates of solutions to (3), (2). A function $z \in C(E_0 \cup E, \mathbb{R})$ will be called *of class* $C^{1.2}$ if $z(\cdot, x) : [-b_0, a] \to \mathbb{R}$ is of class C^1 for $x \in [-b, b]$ and $z(t, \cdot) : [-b, b] \to \mathbb{R}$ is of class C^2 for $t \in [-b_0, a]$.

ASSUMPTION $H_0[\mathbf{F}, \varphi]$. The functions $\mathbf{F} : \Xi \to M_{n\times n}$ and $\mathbf{G} : \Xi \to \mathbb{R}^n$, $G : \Xi \to \mathbb{R}$ are continuous and they satisfy the condition (V) and

1) the matrix $\mathbf{F}$ is symmetric and for $(t, x, w) \in \Xi$ we have

$$\sum_{i,j=1}^{n} F_{ij}(t, x, w) y_i y_j \ge 0, \qquad y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \tag{59}$$

2) there is $\varrho : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that Assumption $H[\varrho]$ is satisfied and

$$\big|G(t, x, w)\big| \le \varrho(t, \|w\|_B) \quad \text{on } \Xi, \tag{60}$$

3) $\varphi \in C(E_0 \cup \partial_0 E, \mathbb{R})$, $\varphi_h \in \mathbf{F}(E_{0.h} \cup \partial_0 E_h, \mathbb{R})$ and there is $\alpha_0 : H \to \mathbb{R}_+$ such that

$$|\varphi(t, x) - \varphi_h(t, x)| \le \alpha_0(h) \quad \text{on } E_{0.h} \cup \partial_0 E_h \quad \text{and} \quad \lim_{h\to 0} \alpha_0(h) = 0,$$

4) $\tilde{\eta} \in \mathbb{R}_+$ is such that

$$|\varphi(t, x)| \le \tilde{\eta} \quad \text{on } E_0 \quad \text{and} \quad |\varphi_h(t, x)| \le \tilde{\eta} \quad \text{on } E_{0.h},$$
$$|\varphi(t, x)| \le \omega(t, \tilde{\eta}) \quad \text{on } \partial_0 E \quad \text{and} \quad |\varphi_h(t, x)| \le \omega(t, \tilde{\eta}) \quad \text{on } \partial_0 E_h,$$

where $\omega(\,\cdot\,\tilde{\eta})$ is the maximal solution to (13) with $\eta = \tilde{\eta}$.

LEMMA 4.1. *If Assumption* $H_0[\mathbf{F}, \varphi]$ *is satisfied and* $\tilde{z} : E_0 \cup E \to \mathbb{R}$ *is a solution to* (3), (2) *and* $\tilde{z}$ *is of class* $C^{1.2}$ *then*

$$|\tilde{z}(t, x)| \le \omega(t, \tilde{\eta}) \quad \textit{on } E. \tag{61}$$

Proof. For $\varepsilon > 0$ we denote by $\omega(\cdot, \tilde{\eta}, \varepsilon)$ the maximal solution of the Cauchy problem (13). The function $\omega(\cdot, \tilde{\eta}, \varepsilon)$ is defined on $[0, a]$ and it satisfies condition (43). Write $\tilde{\omega}(t) = \|\tilde{z}\|_t$, $t \in [0, a]$. We prove that

$$\tilde{\omega}(t) < \omega(t, \tilde{\eta}, \varepsilon) \quad \text{for } t \in [0, a]. \tag{62}$$

Suppose for contradiction that assertion (62) fails to be true. Then there exists $(\tilde{t}, \tilde{x}) \in (0, a] \times (-b,\, b)$ such that

$$\tilde{\omega}(t) < \omega(t, \tilde{\eta}, \varepsilon) \quad \text{for } t \in [0, \tilde{t}), \qquad \tilde{\omega}(\tilde{t}) = \omega(\tilde{t}, \tilde{\eta}, \varepsilon) = |\tilde{z}(\tilde{t}, \tilde{x})|.$$

Suppose that $\tilde{\omega}(\tilde{t}) = \tilde{z}(\tilde{t}, \tilde{x})$. Then

$$D_-\tilde{\omega}(\tilde{t}) \ge \omega'(\tilde{t}, \tilde{\eta}, \varepsilon). \tag{63}$$

Moreover $\partial_x \tilde{z}(\tilde{t},\tilde{x}) = \theta$ and

$$(64)\qquad \sum_{i,j=1}^{n} \partial_{x_i x_j} \tilde{z}(\tilde{t},\tilde{x}) y_i y_j \le 0 \quad \text{for } y = (y_1,\dots,y_n) \in \mathbb{R}^n.$$

We conclude from (59), (60), (64) that

$$D_-\tilde{\omega}(\tilde{t}) \le \partial_t \tilde{z}(\tilde{t},\tilde{x}) \le G(\tilde{t},\tilde{x},\tilde{z}_{(\tilde{t},\tilde{x})}) \le \varrho(\tilde{t},\omega(\tilde{t},\tilde{\eta},\varepsilon)) < \omega'(\tilde{t},\tilde{\eta},\varepsilon),$$

which contradicts (62). The case $\tilde{\omega}(\tilde{t}) = -\tilde{z}(\tilde{t},\tilde{x})$ can be treated in a similar way. Thus inequality (62) is proved. Letting ε tend to zero in (62) we obtain (61).

Assumption $H[\mathbf{F},\varphi]$. Assumption $H_0[\mathbf{F},\varphi]$ is satisfied and the steps of the mesh satisfy

$$(65)\qquad \frac{1}{h_i} F_{ii}(P) - \sum_{\substack{j=1\\ j\ne i}}^{n} \frac{1}{h_j}|F_{ij}(P)| - \frac{1}{2}|G_i(P)| \ge 0,$$
$$P = (t,x,w) \in \Xi,\ i = 1,\dots,n.$$

Remark 4.2. Suppose that there is $\tilde{a} > 0$ such that

$$F_{ii}(P) - \sum_{\substack{j=1\\ j\ne i}}^{n} |F_{ij}(P)| \ge \tilde{a}, \qquad P \in \Xi,\ i = 1,\dots,n.$$

Then condition (59) is satisfied (see [19]) and there is $\varepsilon_0 > 0$ such that for $\|h'\| < \varepsilon_0$ and for $h_1 = h_2,\dots,h_n$ inequalities (65) hold.

Lemma 4.3. *Suppose that Assumptions $H[T_h]$ and $H[\mathbf{F},\varphi]$ are satisfied. Then:*

I. *There exists exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to* (58), (57) *and*

$$|v_h^{(r,m)}| \le \omega(t^{(r)},\bar{\eta}) \quad \text{on } E_h.$$

II. *If additionally*

$$(66)\qquad 1 - 2h_0 \sum_{i=1}^{n} \frac{1}{h_i^2} F_{ii}(P) + \sum_{\substack{i,j=1\\ j\ne i}}^{n} \frac{1}{h_j}|F_{ij}(P)| \ge 0,$$

where $P = (t,x,w) \in \Xi$, then the solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ to (56), (57) *satisfies the condition*

$$|u_h^{(r,m)}| \le \omega(t^{(r)},\tilde{\eta}) \quad \text{on } E_h.$$

Proof. We apply Theorem 2.1. Consider the functions $f_h : \Omega_h \to \mathbb{R}$ and $G_h : \Omega_h \to \mathbb{R}^\chi$, $G_h = \{G_{h.\lambda}\}_{\lambda\in\Lambda}$, defined in the following way. Let $(t^{(r)}, x^{(m)}, w) \in \Omega_h$. Write

$$\Lambda_0^{(r,m)}[w] = \{\lambda \in \Lambda : \text{there is } i,\ 1 \le i \le n,\ \text{such that } \lambda = e_i \text{ or } \lambda = -e_i\},$$

$$\varLambda_I^{(r,m)}[w] = \{\lambda \in \varLambda : \text{there is } (i,j) \in S_+^{(r,m)}[w] \text{ such that } \lambda = e_i + e_j \text{ or } \lambda = -e_i - e_j\},$$

$$\varLambda_{II}^{(r,m)}[w] = \{\lambda \in \varLambda : \text{there is } (i,j) \in S_-^{(r,m)}[w] \text{ such that } \lambda = e_i - e_j \text{ or } \lambda = -e_i + e_j\},$$

$$\tilde{\varLambda}^{(r,m)}[w] = \varLambda \setminus \{\varLambda_0^{(r,m)}[w] \cup \varLambda_I^{(r,m)}[w] \cup \varLambda_{II}^{(r,m)}[w] \cup \{\theta\}\}.$$

Set $P = (t,x,w) \in \varOmega$ and

$$f_h(P) = G(P), \qquad G_{h.\theta}(P) = -2\sum_{i=1}^{n} \frac{1}{h_i^2} F_{ii}(P) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{h_i h_j} |F_{ij}(P)|,$$

$$G_{h.e_i}(P) = \frac{1}{h_i^2} F_{ii}(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} |F_{ij}(P)| + \frac{1}{2h_i} G_i(P),$$

$$G_{h.-e_i}(P) = \frac{1}{h_i^2} F_{ii}(P) - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{h_i h_j} |F_{ij}(P)| - \frac{1}{2h_i} G_i(P),$$

$$G_{h.e_i+e_j}(P) = G_{h.-e_i-e_j}(P) = -\frac{1}{2h_i h_j} F_{ij}(P), \qquad (i,j) \in S_+^{(r,m)}[w],$$

$$G_{h.e_i-e_j}(P) = G_{h.-e_i+e_j}(P) = \frac{1}{2h_i h_j} F_{ij}(P), \qquad (i,j) \in S_-^{(r,m)}[w],$$

$$G_{h.\lambda}(P) = 0 \qquad \text{for } \lambda \in \tilde{\varLambda}[w]^{(r,m)}.$$

Then Assumption $H[G_h, f_h]$ is satisfied and problems (56), (57) and (58), (57) are equivalent to (6), (7) and (12), (7) respectively. Our lemma follows from Theorem 2.1.

Suppose that Assumptions $H[T_h]$ and $H[\mathbf{F}, \varphi]$ are satisfied. Write $\tilde{C} = \omega(a, \tilde{\eta})$. Let $X[\tilde{C}]$ be defined by (47).

Assumption $H[\mathbf{F}, \sigma]$. The functions $\mathbf{F} : \varXi \to M_{n\times n}$, $\mathbf{G} : \varXi \to \mathbb{R}^n$, $G : \varXi \to \mathbb{R}$ and $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ satisfy Assumption $H_0[\mathbf{F}, \varphi]$ and there is $\sigma : [0,a] \times \mathbb{R}_+$ such that Assumption $H[\sigma]$ holds and the expressions

$$\|\mathbf{F}(t,x,w) - \mathbf{F}(t,x,\tilde{w})\|, \ \|\mathbf{G}(t,x,w) - \mathbf{G}(t,x,\tilde{w})\|, \ |G(t,x,w) - G(t,x,\tilde{w})|$$

are bounded from above by $\sigma(t, \|w - \tilde{w}\|_B)$ for all $(t,x) \in E$ and $w, \tilde{w} \in X[\tilde{C}]$.

Remark 4.4. It is important that we have assumed nonlinear estimates for $w, \tilde{w} \in X[\tilde{C}]$. It is clear that there are differential integral equations and differential equations with deviated variables such that Assumption $H[\mathbf{F}, \sigma]$ holds and the nonlinear estimates are not satisfied on $\varXi$.

THEOREM 4.5. *Suppose that Assumptions $H[T_h]$ and $H[\mathbf{F},\sigma]$ are satisfied and $\tilde{z} : E_0 \cup E \to \mathbb{R}$ is a solution to* (3), (2) *of class $C^{1.2}$, and $\tilde{z}_h$ is the restriction of $\tilde{z}$ to the set $E_{0.h} \cup E_h$. Then:*

I. *There exists exactly one solution $v_h : E_{0.h} \cup E_h \to \mathbb{R}$ to* (58), (57) *and there is $\alpha : H \to \mathbb{R}_+$ such that*

$$(67)\qquad |(v_h - \tilde{z}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0.$$

II. *If condition* (66) *is satisfied then there is $\alpha : H \to \mathbb{R}_+$ such that*

$$(68)\qquad |(u_h - \tilde{z}_h)^{(r,m)}| \le \alpha(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h\to 0} \alpha(h) = 0,$$

where $u_h : E_{0.h} \cup E_h \to \mathbb{R}_+$ is a solution to (56), (57).

Proof. We apply Theorem 2.2 to prove (67), (68). Let $\tilde{c} \in \mathbb{R}_+$ be such that

$$\|\partial_x \tilde{z}(t,x)\| \le \tilde{c}, \quad \|\partial_{xx} \tilde{z}(t,x)\| \le \tilde{c} \qquad \text{for } (t,x) \in E.$$

Denote by Y_h the class of all $\zeta \in \mathbf{F}(A_h, \mathbb{R})$ satisfying

$$\frac{1}{2}|\delta_i^+ \zeta^{(\theta)} + \delta_i^{(\theta)}| \le \tilde{c}, \qquad i = 1, \dots, n,$$

$$\frac{1}{2}|\delta_i^+ \delta_j^+ \zeta^{(\theta)} + \delta_i^- \delta_j^- \zeta^{(\theta)}| \le \tilde{c}, \ \ \frac{1}{2}|\delta_i^+ \delta_j^- \zeta^{(\theta)} + \delta_i^- \delta_j^+ \zeta^{(\theta)}| \le \tilde{c}, \ \ i,j = 1, \dots, n.$$

Then

$$\|\tilde{z}_h\|_{h.r} \le \tilde{C}, \qquad (\tilde{z}_h)_{\langle r,m\rangle} \in Y_h \qquad \text{for } r = 0, 1, \dots, K.$$

It follows from Assumptions $H[T_h]$ and $H[\mathbf{F},\sigma]$ that there is $\gamma : H \to \mathbb{R}_+$ such that

$$|\delta_0 \tilde{z}_h^{(r,m)} - \mathbb{W}_{\mathrm{im}.h}[\tilde{z}_h]^{(r,m)}| \le \gamma(h) \qquad \text{on } E'_h,$$
$$|\delta_0 \tilde{z}_h^{(r,m)} - \mathbb{W}_{\mathrm{ex}.h}[\tilde{z}_h]^{(r,m)}| \le \gamma(h) \qquad \text{on } E'_h.$$

We conclude from Assumption $H[\mathbf{F},\sigma]$ that there is $\bar{c} > 0$ such that the operator W_h given by (55) satisfies

$$|W_h(t,x,w,\zeta) - W_h(t,x,\tilde{w},\zeta)| \le (1+\bar{c})\sigma(t, \|w - \tilde{w}\|_B)$$

for all $(t,x) \in E'_h$, $w, \tilde{w} \in X[\tilde{C}]$ and $\zeta \in Y_h$. It is clear that condition (66) for equation (56) is equivalent to (15) for equation (6). Thus all the assumptions of Theorem 2.2 are satisfied and assertions (67), (68) follow.

REMARK 4.6. Suppose that all the assumptions of Theorem 4.5 are satisfied with $\sigma(t,p) = \tilde{L}p$ on $[0,a] \times \mathbb{R}_+$ where $\tilde{L} \in \mathbb{R}_+$. Then there is $L \in \mathbb{R}_+$ such that $|(\tilde{z}_h - v_h)^{(r,m)}| \le \tilde{\alpha}(h)$ on E_h and $|(\tilde{z}_h - u_h)^{(r,m)}| \le \tilde{\alpha}(h)$ on E_h where $\tilde{\alpha} : H \to \mathbb{R}_+$ is given by (34), (35).

Now we give numerical examples. Put $n = 2$. Solutions of the initial boundary value problems are defined on $E = [0, 0.5] \times [-1,1] \times [-1,1]$.

Consider the differential equation with deviated variables

$$\begin{aligned}\partial_t z(t,x,y) = &\{2+\cos[z(t,0.5(x+y),0.5(x-y)) - e^{txy}]\}\partial_{xx} z(t,x,y)\\ &+ \{2+\cos[z(t,0.5(x-y),0.5(x+y)) - e^{-txy}]\}\partial_{yy} z(t,x,y)\\ &+ \partial_{xy} z(t,x,y)\sin[z(0.25t,x,y) - z(t,0.5x,0.5y)]\\ &+ f(t,x,y)z(t,x,y),\\ f(t,x,y) = &\, x^2 - y^2 - 12t^2(x^2+y^2),\end{aligned}$$

with the initial boundary conditions

$$z(0,x,y) = 1, \quad (x,y) \in [-1,1]\times[-1,1], \tag{69}$$

$$z(t,-1,y) = z(t,1,y) = e^{t(1-y^2)}, \quad t \in [0,0.5],\ y \in [-1,1], \tag{70}$$

$$z(t,x,-1) = z(t,x,1) = e^{t(x^2-1)}, \quad t \in [0,0.5],\ x \in [-1,1]. \tag{71}$$

The solution of the above problem is $v(t,x,y) = \exp[t(x^2-y^2)]$. The following tables show maximal values of errors for several step sizes.

Table 7. Errors for explicit Euler method

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-8}	2^{-1}	$6.09265984 \cdot 10^{-3}$	0.115
2^{-10}	2^{-3}	$3.10700168 \cdot 10^{-3}$	1.142
2^{-12}	2^{-4}	$1.56094035 \cdot 10^{-3}$	19.312
2^{-14}	2^{-5}	$7.81394600 \cdot 10^{-4}$	318.221

Table 8. Errors for explicit Euler scheme, violated CFL condition

h_0	$h_1 = h_2$	Maximal error
2^{-5}	2^{-2}	$3.80077434 \cdot 10^{1}$
2^{-7}	2^{-3}	$3.37466385 \cdot 10^{17}$
2^{-9}	2^{-4}	$7.36222251 \cdot 10^{88}$
2^{-11}	2^{-5}	$+\infty$

Now we consider the implicit Euler method with steps of the mesh given in Table 8.

Table 9. Errors for implicit Euler scheme

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-5}	2^{-2}	$2.70180118 \cdot 10^{-2}$	0.112
2^{-7}	2^{-3}	$1.46550838 \cdot 10^{-2}$	1.251
2^{-9}	2^{-4}	$7.47607123 \cdot 10^{-3}$	24.920
2^{-11}	2^{-5}	$3.75674966 \cdot 10^{-3}$	941.878

Consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x,y) = &\Big\{2+\cos\Big[2t\int_0^x sz(t,s,y)\,ds - z(t,x,y)\Big]\Big\}\partial_{xx}z(t,x,y)\\ &+\Big\{2+\cos\Big[2t\int_0^y sz(t,x,s)\,ds + z(t,x,y)\Big]\Big\}\partial_{yy}z(t,x,y)\\ &+\partial_{xy}z(t,x,y)\sin\Big[1+(x^2-y^2)\int_0^t z(\tau,x,y)\,d\tau - z(t,x,y)\Big]\\ &+f(t,x,y)z(t,x,y)\end{aligned}$$

with initial boundary conditions (69)–(71) where

$$\begin{aligned}f(t,x,y)) = x^2-y^2-8t^2(x^2+y^2)-2t(1+2x^2t)\cos e^{-ty^2}\\ -2t(-1+2y^2t)\cos e^{tx^2}.\end{aligned}$$

The function $v(t,x,y)=e^{t(x^2-y^2)}$ is a solution of the above problem. The following tables show maximal values of errors for several step sizes.

Table 10. Errors for explicit Euler method

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-8}	2^{-1}	$5.63773239 \cdot 10^{-3}$	0.006
2^{-10}	2^{-3}	$2.87933980 \cdot 10^{-3}$	0.052
2^{-12}	2^{-4}	$1.44708568 \cdot 10^{-3}$	0.766
2^{-14}	2^{-5}	$7.24464180 \cdot 10^{-4}$	12.727

Table 11. Errors for explicit Euler scheme, violated CFL condition

h_0	$h_1 = h_2$	Maximal error
2^{-5}	2^{-2}	$9.57986240 \cdot 10^{0}$
2^{-7}	2^{-3}	$6.01045114 \cdot 10^{16}$
2^{-9}	2^{-4}	$2.40913798 \cdot 10^{87}$
2^{-11}	2^{-5}	$+\infty$

Now we consider the implicit Euler method with steps of the mesh given in Table 11.

Table 12. Errors for implicit Euler scheme

h_0	$h_1 = h_2$	Maximal error	Time [s]
2^{-5}	2^{-2}	$2.69819681 \cdot 10^{-2}$	0. 938
2^{-7}	2^{-3}	$1.47067642 \cdot 10^{-2}$	0.827
2^{-9}	2^{-4}	$7.51138801 \cdot 10^{-3}$	22.291
2^{-11}	2^{-5}	$3.77561536 \cdot 10^{-3}$	899.428

Our considerations show the following relations between explicit and implicit difference methods for (3), (2). Assumptions on the regularity of given functions are the same in the theorems on convergence of explicit and implicit difference schemes. We need condition (46) on the mesh for explicit difference methods, but not for implicit ones. Error estimates are the same for both methods. Tables 8, 9 and 11, 12 show that there are implicit difference methods which are convergent, while the corresponding explicit schemes are not.

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