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ON THE CONVERGENCE OF NEWTON'S METHOD UNDER $\omega^\star$-CONDITIONED SECOND DERIVATIVE

Abstract. We provide a new semilocal result for the quadratic convergence of Newton's method under $\omega^\star$-conditioned second Fréchet derivative on a Banach space. This way we can handle equations where the usual Lipschitz-type conditions are not verifiable. An application involving nonlinear integral equations and two boundary value problems is provided. It turns out that a similar result using ω-conditioned hypotheses can provide usable error estimates indicating only linear convergence for Newton's method.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet differentiable operator defined on an open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

The field of computational sciences has seen a considerable development in mathematics, engineering sciences, and economic equilibrium theory. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent states of the systems. For simplicity, assume that a time-invariant system is driven by an equation $\dot{x} = T(x)$, for a suitable operator T, where x is the state. Then the equilibrium states are determined by solving equation (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or

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real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative—when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We note that in computational sciences, the practice of numerical analysis for finding such solutions is essentially connected with variants of Newton's method.

Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0,\ x_0 \in \mathcal{D}) \tag{1.2}$$

is undoubtedly the most popular method for generating (under certain conditions) a sequence $\{x_n\}$ quadratically convergent to $x^\star$. Here, for $x \in \mathcal{D}$, $F'(x) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ (the space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$) denotes the Fréchet derivative of the operator F [4]. Kantorovich established the first semilocal convergence theorem for Newton's method (1.2) [20].

The main condition is given by

$$\|F''(x)\| \leq K, \qquad K \geq 0, \tag{1.3}$$

for x in some neighborhood of the initial guess x_0. Several authors have used Lipschitz conditions:

$$\|\mathcal{J}(F'(x) - F'(y))\| \leq \alpha(\|x - y\|) \quad \text{for } x, y \in \mathcal{D}, \tag{1.4}$$

where $\mathcal{J}$ is the identity operator or $F'(x_0)^{-1}$, and α a continuous nondecreasing function defined on the nonnegative axis. Some special cases are given by $\alpha(r) = \ell r^p$, $p \in [0, 1]$, and $\alpha(r) = \gamma(r)r$, where the function γ has the same properties as α. A survey of such results can be found in [4] and the references there (see also [1]–[3], [5]–[30]). As Ezquerro and Hernández have already noted in [16], [17], the verification of (1.4) at least for some type of problems is not possible. That is why motivated by (1.3), they introduced the condition

$$\|F''(x)\| \leq \omega(\|x\|) \quad \text{for } x \in \mathcal{D}, \tag{1.5}$$

where $\omega : \mathcal{I} = [0, +\infty) \to \mathcal{I}$ is a continuous function with $\omega(0) \geq 0$, which is either nondecreasing or nonincreasing. Ezquerro and Hernández [16], [17] call this type of estimate "ω-conditioned second derivative". Sufficient semilocal convergence conditions were given but the usable error bounds imply only linear convergence for Newton's method (1.2).

In order for us to rectify this drawback, instead of (1.5), we use the condition

$$\|F'(x_0)^{-1}F''(x)\| \le \omega^\star(\|x_0 - x\|) \quad \text{for all } x \in \mathcal{D}, \tag{1.6}$$

where $\omega^\star : \mathcal{I} = [0, +\infty) \to \mathcal{I}$ is a continuous function with $\omega^\star(0) \ge 0$, which is either nondecreasing or nonincreasing. We shall call this type of estimate "$\omega^\star$-conditioned second derivative". Our semilocal convergence analysis is based on an approach using more precise majorizing sequences.

The advantages are:

(1) The order of convergence of Newton's method is quadratic.
(2) The results are provided in affine invariant form (see [4] for an explanation of the advantages of affine versus nonaffine invariant results).

The paper is organized as follows. The semilocal convergence analysis of Newton's method under the $\omega^\star$-condition (1.6) is given in Section 2. In Section 3 we provide some applications involving nonlinear Hammerstein integral equations of second type, and two-point boundary value problems. In particular, we show the quadratic instead of linear convergence of Newton's method applied to an example already used in [17].

2. Semilocal convergence analysis of Newton's method. We shall first introduce some preliminary conditions and results.

Let $x_0 \in \mathcal{D}$ be such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. We assume:

$$(\mathcal{C}_1) \qquad \|F'(x_0)^{-1}F(x_0)\| \le \eta,$$

$$(\mathcal{C}_2) \qquad \|F'(x_0)^{-1}F''(x)\| \le \omega^\star(\|x - x_0\|) \quad \text{for all } x \in \mathcal{D},$$

where $\omega^\star : \mathcal{I} \to \mathcal{I}$ is a continuous function with $\omega^\star(0) \ge 0$ such that

$$\omega^\star(ty) \le h(t)\omega^\star(y) \quad \text{for all } t \in [0,1],\ y \in \mathcal{D},$$

and $h : [0,1] \to \mathcal{I}$ a continuous nondecreasing function. Set

$$H = \int_0^1 h(t)\, dt.$$

$(\mathcal{C}_3)$ The function

$$g(r) = \Big(\overline{L}(r) + 4\overline{L}_0(r) + \sqrt{\overline{L}^2(r) + 8\overline{L}(r)\overline{L}_0(r)}\Big)\eta - 4$$

has a minimal positive zero, denoted by r_0.

Here,

$$(\mathcal{C}_4) \qquad \begin{gathered} \overline{L}(r) = \omega^\star(r + \|x_0\|), \qquad \overline{L}_0(r) = H\omega^\star(r). \\ \overline{L}_0(r) \le \overline{L}(r) \quad \text{for all } r \in \mathcal{I}_0 = [0, r_0]. \end{gathered}$$

$$(\mathcal{C}_5) \qquad \overline{U}(x_0, r_0) = \{x \in \mathcal{X} : \|x - x_0\| \le r_0\} \subseteq \mathcal{D}.$$

($\mathcal{C}_6$) The function

$$q(r)=\int_0^1\int_0^1 \omega^\star(s(r_0+t(r-r_0))\,ds(r_0+t(r-r_0))\,dt-1$$

has a minimal positive zero $r^\star \geq r_0$.

The importance of introducing the function h, so that we can have better error bounds, has been explained in [4], [7], [9].

We need two results on majorizing sequences before we present the main theorem.

LEMMA 2.1 ([5]). *Assume that there exist constants $L_0, L, \eta \geq 0$ with $L_0 \leq L$ such that*

$$q_0=\overline{L}\eta\begin{cases}\leq 1/2 & \text{if } L_0\neq 0,\\ <1/2 & \text{if } L_0=0,\end{cases} \tag{2.1}$$

where

$$\overline{L}=\frac{1}{8}\Big(L+4L_0+\sqrt{L^2+8L_0L}\Big). \tag{2.2}$$

Then the sequence $\{t_k\}$ $(k\geq 0)$ given by

$$t_0=0,\qquad t_1=\eta,\qquad t_{k+1}=t_k+\frac{L(t_k-t_{k-1})^2}{2(1-L_0t_k)}\qquad (k\geq 1), \tag{2.3}$$

is well defined, nondecreasing, bounded from above by $t^{\star\star}$, and converges to its unique least upper bound $t^\star\in[0,t^{\star\star}]$, where

$$t^{\star\star}=\frac{2\eta}{2-\delta}, \tag{2.4}$$

$$1\leq\delta=\frac{4L}{L+\sqrt{L^2+8L_0L}}<2\quad \text{for } L_0\neq 0. \tag{2.5}$$

Moreover, the following estimates hold:

$$L_0t^\star\leq 1, \tag{2.6}$$

$$0\leq t_{k+1}-t_k\leq\frac{\delta}{2}(t_k-t_{k-1})\leq\cdots\leq\left(\frac{\delta}{2}\right)^k\eta\qquad (k\geq 1), \tag{2.7}$$

$$t_{k+1}-t_k\leq\left(\frac{\delta}{2}\right)^k(2q_0)^{2^k-1}\eta\qquad (k\geq 0), \tag{2.8}$$

$$0\leq t^\star-t_k\leq\left(\frac{\delta}{2}\right)^k\frac{(2q_0)^{2^k-1}\eta}{1-(2q_0)^{2^k}}\qquad (2q_0<1,\ k\geq 0). \tag{2.9}$$

LEMMA 2.2. *Assume conditions ($\mathcal{C}_3$) and ($\mathcal{C}_4$) hold, and set*

$$L_0=\overline{L}_0(r_0)\quad \text{and}\quad L=\overline{L}(r_0).$$

Then the conclusions of Lemma 2.1 hold for the iteration $\{t_n\}$.

Proof. Clearly, all hypotheses of Lemma 2.1 are satisfied for the above choices of L_0 and L. That completes the proof. ■

We can show the main semilocal convergence result for Newton's method under $\omega^\star$-conditions and $\omega^\star$ a nondecreasing function.

THEOREM 2.3. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a twice Fréchet differentiable operator. Let x_0 be such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Moreover, assume conditions $(\mathcal{C}_1)$–$(\mathcal{C}_6)$ hold, and $t^\star \leq r_0$. Then the sequence $\{x_n\}$ generated by Newton's method* (1.2) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \geq 0$, and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of the equation $F(x) = 0$. Moreover, the following estimates hold for all $n \geq 0$:*

$$\|x_n - x^\star\| \leq t^\star - t_n, \tag{2.10}$$

where L_0, L, and $t^\star$, t_n are given in Lemmas 2.1, 2.2, *respectively. Furthermore, the solution $x^\star$ is unique in $\mathcal{D}_0 = U(x_0, t^\star) \cap \mathcal{D}$.*

Proof. We shall show by induction that

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k \tag{2.11}$$

and

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k) \tag{2.12}$$

for all $k \geq 0$.

For every $z \in \overline{U}(x_1, t^\star - t_1)$,

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^\star - t_1 + t_1 - t_0 = t^\star - t_0,$$

so $z \in \overline{U}(x_0, t^\star - t_0)$. Since also

$$\|x_1 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \eta = t_1 - t_0,$$

estimates (2.11) and (2.12) hold for $k = 0$.

Assuming that (2.11) and (2.12) hold for all $i \leq k$, we have

$$\|x_{k+1} - x_0\| \leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} < t^\star,$$

and

$$\|x_k + \rho(x_{k+1} - x_k) - x_0\| \leq t_k + \rho(t_{k+1} - t_k) \leq t^\star, \quad \rho \in [0, 1].$$

We obtain $\|x_1 - x_0\| < t^\star \leq r_0$, that is, $x_1 \in \overline{U}(x_0, t^\star)$, and from the above $x_0 + \rho(x_1 - x_0) \in \overline{U}(x_0, t^\star)$ for $\rho \in [0, 1]$.

Let $x_k \in U(x_0, t^\star)$. We then get from $(\mathcal{C}_2)$ the estimate

$$\begin{aligned}
(2.13)\qquad & \|F'(x_0)^{-1}(F'(x_k) - F'(x_0))\| \\
&= \Big\| \int_0^1 F'(x_0)^{-1} F''(x_0 + t(x_k - x_0))(x_k - x_0)\, dt \Big\| \\
&\le \int_0^1 \omega^\star(t\|x_k - x_0\|)\|x_k - x_0\|\, dt \\
&\le \int_0^1 h(t)\, dt\, \omega^\star(\|x_k - x_0\|)\|x_k - x_0\| \le H\omega^\star(t_k)t_k \\
&\le H\omega^\star(t^\star)t_k \le H\omega^\star(r_0)t_k = L_0 t_k \le L_0 t^\star < 1.
\end{aligned}$$

It follows from (2.13) and the Banach lemma on invertible operators [4], [21] that $F'(x_k)^{-1}$ exists, and

$$(2.14)\qquad \|F'(x_k)^{-1}F'(x_0)\| \le (1 - L_0 t_k)^{-1}.$$

In view of (1.2) and Taylor's formula

$$F(x_k) = \int_0^1 F''(x_{k-1} + t(x_k - x_{k-1}))(1 - t)(x_k - x_{k-1})^2\, dt,$$

we get

$$\begin{aligned}
(2.15)\qquad & \|F'(x_0)^{-1}F(x_k)\| \\
&= \Big\| \int_0^1 F'(x_0)^{-1} F''(x_{k-1} + t(x_k - x_{k-1}))(1 - t)(x_k - x_{k-1})^2\, dt \Big\| \\
&\le \frac{1}{2} \int_0^1 \omega^\star(\|x_{k-1} - x_0 + t(x_k - x_{k-1}\|)\|x_k - x_{k-1}\|^2\, dt \\
&\le \frac{1}{2} \int_0^1 \omega^\star((1-t)\|x_{k-1} - x_0\| + t\|x_k - x_0\| + \|x_0\|), (t_k - t_{k-1})^2\, dt \\
&\le \frac{1}{2} \int_0^1 \omega^\star(t^\star + \|x_0\|)(t_k - t_{k-1})^2\, dt \\
&\le \frac{1}{2} \int_0^1 \omega^\star(r_0 + \|x_0\|)(t_k - t_{k-1})^2\, dt = \frac{1}{2} L(t_k - t_{k-1})^2.
\end{aligned}$$

Using (1.2), Lemma 2.1, (2.14), and (2.15), we deduce

$$\begin{aligned}
\|x_{k+1} - x_k\| \le \|F'(x_k)^{-1}F'(x_0)\|\, \|F'(x_0)^{-1}F(x_k)\| &\le \frac{L(t_k - t_{k-1})^2}{2(1 - L_0 t_k)} \\
&= t_{k+1} - t_k,
\end{aligned}$$

which shows (2.11) for all k. Thus, for every $v \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$, we have

$$\|v - x_k\| \le \|v - x_{k+1}\| + \|x_{k+1} - x_k\| \le t^\star - t_{k+1} + t_{k+1} - t_k = t^\star - t_k,$$

which implies $v \in \overline{U}(x_k, t^\star - t_k)$. The induction for (2.11) and (2.12) is completed.

Lemma 2.2 implies that $\{t_n\}$ is a Cauchy sequence. By (2.11), and (2.12), the sequence $\{x_n\}$ is Cauchy too, in the Banach space $\mathcal{X}$, and so it converges to some $x^\star \in \overline{U}(x_0, t^\star)$ (since $\overline{U}(x_0, t^\star)$ is a closed set). In view of (2.15), by letting $k \to \infty$ we get $F(x^\star) = 0$.

Estimate (2.10) follows from (2.11) by using standard majorization techniques [4], [6], [7], [21].

Finally, to show uniqueness of $x^\star$ in $\mathcal{D}_0$, let us assume $y^\star$ is a solution of the equation $F(x) = 0$ in $\mathcal{D}_0$. We need the identity

$$\mathcal{M}(y^\star - x^\star) = F(y^\star) - F(x^\star), \tag{2.16}$$

where

$$\mathcal{M} = \int_0^1 F'(x_0)^{-1} F'(x^\star + \theta(y^\star - x^\star))\, d\theta.$$

Using $(\mathcal{C}_2)$, $(\mathcal{C}_6)$, and (2.15), we obtain

$$\begin{aligned}
&\|F'(x_0)^{-1}(F'(x_0) - \mathcal{M})\| \qquad (2.17)\\
&\le \Big\| \int_0^1 F'(x_0)^{-1}(F'(x^\star + t(y^\star - x^\star)) - F'(x_0))\, dt \Big\| \\
&\le \Big\| \int_0^1\!\!\int_0^1 F'(x_0)^{-1} F''(x_0 + s((1-t)(x^\star - x_0) \\
&\qquad + t(y^\star - x_0)))\, ds\, ((1-t)(x^\star - x_0) + t(y^\star - x_0))\, dt \Big\| \\
&< \int_0^1\!\!\int_0^1 \|F'(x_0)^{-1} F''(s((1-t)(x^\star - x_0) + t(y^\star - x_0)))\|\, ds\, (r_0 + t(r^\star - r_0))\, dt \\
&\le \int_0^1\!\!\int_0^1 \omega^\star(s(r_0 + t(r^\star - r_0)))\, ds\, (r_0 + t(r^\star - r_0))\, dt = 1.
\end{aligned}$$

In view of (2.17) and the Banach lemma on invertible operators, $\mathcal{M}^{-1}$ exists. It follows from (2.16) that $x^\star = y^\star$.

That completes the proof of Theorem 2.3. ∎

REMARK 2.4.

(a) The point $t^{\star\star}$ given in closed form by (2.4) can replace $t^\star$ in the hypotheses of Theorem 2.3.

(b) It follows from $(\mathcal{C}_2)$ and (2.17) that the uniqueness of the solution $x^\star$ is guaranteed in $U(x_0, r^\star)$ if

$$\int_0^1 \omega^\star(sr^\star)\, ds\, r^\star \leq 1,$$

or

$$H\omega^\star(r^\star)r^\star \leq 1.$$

In $(\mathcal{C}_2)$, we assumed that $\omega^\star$ is a nondecreasing function. However, a result can be given under the condition that $\omega^\star$ is a nonincreasing function. Indeed, let us assume:

$(\mathcal{B}_1) \equiv (\mathcal{C}_1)$.

$(\mathcal{B}_2)$ $$\|F'(x_0)^{-1}F''(x)\| \leq \omega^\star(\|x - x_0\|) \quad \text{for all } x \in \mathcal{D},$$

where $\omega^\star : \mathcal{I} \to \mathcal{I}$ is a continuous nonincreasing function with $\omega^\star(0) \geq 0$.

$(\mathcal{B}_3)$ The function g defined in $(\mathcal{C}_3)$ has a minimal positive zero, denoted by r_0 with $r_0 \leq \|x_0\|$, where

$$\overline{L}(r) = \omega^\star(\|x_0\| - r), \qquad \overline{L}_0 = \overline{L}_0(r) = \omega^\star(0), \quad r \in [0, \|x_0\|].$$

$(\mathcal{B}_4) \equiv (\mathcal{C}_4)$.

$(\mathcal{B}_5)$ $$\omega^\star(0)r_0 \leq 1.$$

Set now

$$r^\star = r_0 + 2(1 - \omega^\star(0)r_0).$$

Then we can show the following semilocal result for Newton's method under $\omega^\star$-conditions and for $\omega^\star$ a nonincreasing function.

THEOREM 2.5. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be a twice Fréchet differentiable operator. Let x_0 be such that $F'(x_0)^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Moreover, assume conditions $(\mathcal{B}_1)$–$(\mathcal{B}_5)$ hold, and $t^\star \leq r_0$. Then the conclusions of Theorem* 2.3 *hold.*

Proof. We follow the proof of Theorem 2.3 until

$$\begin{aligned}\|F'(x_0)^{-1}(F'(x_k) - F'(x_0))\| &\leq \int_0^1 \omega^\star(t\|x_k - x_0\|)\|x_k - x_0\|\, dt \\ &\leq \int_0^1 \omega^\star(0)t_k\, dt = L_0 t_k \leq L_0 t^\star < 1.\end{aligned}$$

Then we have, instead of (2.15),

$$(2.18)\quad \begin{aligned}&\|F'(x_0)^{-1}F(x_k)\|\\ &\le \frac{1}{2}\int_0^1 \omega^\star(\|x_{k-1}-x_0+t(x_k-x_{k-1}\|))\|x_k-x_{k-1}\|^2\,dt\\ &\le \frac{1}{2}\int_0^1 \omega^\star(\|x_0\|-(1-t)\|x_{k-1}-x_0\|-t\|x_k-x_0\|)\|x_k-x_{k-1}\|^2\,dt\\ &\le \frac{1}{2}\int_0^1 \omega^\star(\|x_0\|-r_0)\|x_k-x_{k-1}\|^2\,dt=\frac{1}{2}L(t_k-t_{k-1})^2.\end{aligned}$$

Finally, for the uniqueness part, we have

$$(2.19)\quad \begin{aligned}&\|F'(x_0)^{-1}(F'(x_0)-\mathcal{M})\|\\ &\le \int_0^1\int_0^1 \omega^\star(s\|(1-t)(x^\star-x_0)\|+t\|y^\star-x_0\|)\,ds\,(r_0+t(r^\star-r_0))\,dt\\ &\le \int_0^1 \omega^\star(0)(r_0+t(r^\star-r_0))\,dt=\omega^\star(0)r_0+\frac{1}{2}(r^\star-r_0)=1.\end{aligned}$$

That completes the proof of Theorem 2.5. ■

Remark 2.6. We shall compare our results with earlier ones. Let us introduce the set of conditions used in [16], [17] but in affine invariant form.

$(\mathcal{A}_1)\equiv(\mathcal{C}_1)$.

$(\mathcal{A}_2)$ $$\|F'(x_0)^{-1}F''(x)\|\le\omega(\|x\|)\quad\text{for all } x\in\mathcal{D},$$

where $\omega:\mathcal{I}\to\mathcal{I}$ is a continuous nondecreasing or nonincreasing function with $\omega(0)\ge 0$.

$(\mathcal{A}_3)$ The equation
$$3\eta\varphi(t)t-2\eta^2\varphi(t)-2t+2\eta=0$$
has a minimal positive solution R, where
$$\varphi(t)=\begin{cases}\omega(\|x_0\|+t) & \text{if } \omega \text{ is nondecreasing},\\ \omega(\|x_0\|-t) & \text{if } \omega \text{ is nonincreasing}.\end{cases}$$

Note that R must be smaller than $\|x_0\|$ if ω is nonincreasing.

$(\mathcal{A}_4)$ $$\overline{U}(x_0,R)\subseteq\mathcal{D}.$$

$(\mathcal{A}_5)$ $$\alpha_0=\eta\varphi(R)\in(0,1/2).$$

$(\mathcal{A}_6)\equiv(\mathcal{C}_6)$.

Under $(\mathcal{A}_1)$–$(\mathcal{A}_6)$, the linear convergence of the iteration $\{x_n\}$ to a unique solution of the equation $F(x)=0$ in $U(x_0,R)$ was established.

The advantages of our approach over the corresponding ones in [16], [17] are already stated in the introduction of this paper.

Finally, note that verifying the set of conditions $(\mathcal{C}_1)$–$(\mathcal{C}_6)$ (or $(\mathcal{B}_1)$–$(\mathcal{B}_5)$) presents the same type of difficulties as $(\mathcal{A}_1)$–$(\mathcal{A}_6)$.

REMARK 2.7. The estimate $L_0 \le L$ in Lemma 2.1 is only needed to show the quadratic convergence of the iteration $\{t_n\}$. However, in Theorem 2.5, we have $L_0 \ge L$, which guarantees only linear convergence. But, in view of estimate (2.18), the parameters L_0 and L in Theorem 2.5 can be defined by $L = L_0 = \omega^\star(0)$. This way the quadratic convergence of the iteration $\{t_n\}$ is recovered according to Theorem 2.5. Note that in this case, we replace r_0 by $\|x_0\|$ in the hypothesis $t^\star \le r_0$, whereas $(\mathcal{B}_3)$ is replaced by

$$(\mathcal{B}_3') \qquad 2\omega^\star(0)\eta \le 1$$

in Theorem 2.5.

3. Applications. We shall apply our results to integral equations of Hammerstein type. Such equations have already been studied in [4], [16], [17]. The error bounds in [16], [17] indicate only linear convergence for Newton's method (1.2). Here, we shall show that under our sufficient convergence conditions, Newton's method converges quadratically.

EXAMPLE 3.1. To make our study as self-contained as possible, we will repeat some terminology and results from [16], [17]. We consider the nonlinear Hammerstein equation of the second kind

$$x(s) = y(s) + \int_a^b \mathcal{G}(s,t)\mathcal{K}(t,x(t))\,dt, \qquad s \in [a,b], \tag{3.1}$$

where $\mathcal{G}(s,t)$ is the Green's kernel given by

$$\mathcal{G}(s,t) = \begin{cases} \dfrac{(b-s)(t-a)}{b-a} & \text{if } t \le s, \\ \dfrac{(s-a)(b-t)}{b-a} & \text{if } s \le t, \end{cases} \tag{3.2}$$

$\mathcal{K}(t,u)$ is a continuous function for $t \in [a,b]$, $-\infty < u < +\infty$, $y \in \mathcal{C}[a,b]$, and x is the unknown function sought in $\mathcal{C}[a,b]$.

In particular, we shall consider a special case of (3.1) given by

$$x(s) = y(s) + \int_a^b \mathcal{G}(s,t)(x(t)^{1+p} + \lambda x(t)^2)\,dt, \qquad p \in [0,1],\ \lambda \in \mathbb{R}. \tag{3.3}$$

where y is a continuous function such that $y(s) > 0$ for all $s \in [a,b]$.

Equation (3.3) is equivalent to the two-point boundary value problem

$$\begin{aligned} x'' &= -x^{1+p} - \lambda x^2, \\ x(a) &= \nu(a), \quad x(b) = \nu(b), \end{aligned} \tag{3.4}$$

where ν is a continuous function. Note that (3.3) is equivalent to solving the equation $F(x) = 0$ on

$$\mathcal{D} = \{x \in \mathcal{C}[a,b] : x(s) > 0,\ s \in [a,b]\}, \tag{3.5}$$

where $F : \mathcal{C}[a,b] \to \mathcal{C}[a,b]$ with

$$F(x)(s) = x(s) - y(s) - \int_a^b \mathcal{G}(s,t)(x(t)^{1+p} + \lambda x(t)^2)\,dt. \tag{3.6}$$

Functions of type (3.6) are of interest, since the usual studies do not apply, due to the fact that F is neither Lipschitz continuous nor p-Hölder continuous.

We shall compute the quantities and functions needed in Lemma 2.1 and Remark 2.7, so we can verify the hypotheses of Theorem 2.5. Let us consider the max norm, and set

$$N = \max_{s\in[a,b]} \int_a^b |\mathcal{G}(s,t)|\,dt. \tag{3.7}$$

We have

$$[F'(x)\phi](s) = \phi(s) - \int_a^b \mathcal{G}(s,t)((1+p)x(t)^p + 2\lambda x(t))\phi(t)\,dt. \tag{3.8}$$

Let $x_0(\cdot)$ be fixed. Then we have

$$\|I - F'(x_0)\| \le N((1+p)\|x_0\|^p + 2|\lambda|\,\|x_0\|), \tag{3.9}$$

and if

$$N((1+p)\|x_0\|^p + 2|\lambda|\,\|x_0\|) < 1, \tag{3.10}$$

the existence of $F'(x_0)^{-1}$ is guaranteed by the Banach lemma on invertible operators, and

$$\|F'(x_0)^{-1}\| \le \beta = (1 - N((1+p)\|x_0\|^p + 2|\lambda|\,\|x_0\|))^{-1}. \tag{3.11}$$

Moreover, in view of (3.6), we get

$$\|F(x_0)\| \le \|x_0 - y\| + N(\|x_0\|^{1+p} + |\lambda|\,\|x_0\|^p).$$

Consequently, we obtain

$$\|F'(x_0)^{-1}F(x_0)\| \le \eta = \frac{\|x_0 - y\| + N(\|x_0\|^{1+p} + |\lambda|\,\|x_0\|^p)}{1 - N((1+p)\|x_0\|^p + 2|\lambda|\,\|x_0\|)}. \tag{3.12}$$

We also have

$$[F''(x)\phi z](s) = -\int_a^b \mathcal{G}(s,t)((1+p)px(t)^{p-1} + 2\lambda)z(t)\phi(t)\,dt. \tag{3.13}$$

So, for $(\mathcal{B}_2)$ to be satisfied, we define

$$\omega^\star(r) = \beta N((1+p)p(r + \|x_0\|)^{p-1} + 2|\lambda|). \tag{3.14}$$

We consider the special case:

$$x_0(s) = y(s) = 1, \quad [a,b] = [0,1], \quad \lambda = p = 1/2.$$

Then, using (3.12), (3.14), and Remark 2.7, we get

$$\eta = \frac{3}{11}, \quad L_0 = L = \omega^\star(0) = \frac{7}{22}, \quad r_0 = 1 \quad \text{and} \quad r^\star = \frac{30}{22}.$$

According to Lemma 2.1,

$$\delta = 1, \quad t^{\star\star} = 2\eta = \frac{6}{11} < 1 = r_0,$$

and $(\mathcal{B}_3')$ is satisfied, since

$$2\omega^\star(0)\eta = 2\,\frac{7}{22}\,\frac{3}{11} = .173553719 < .5.$$

Hence, Theorem 2.5 guarantees the existence and uniqueness of a solution $x^\star$ for equation (3.3). Moreover, according to Lemma 2.1, Theorem 2.5, and Remark 2.7 the convergence of the iteration $\{x_n\}$ to $x^\star$ is quadratic. Note that for the same equation in [16], [17], the convergence was shown to only be linear.

Finally, we provide an example with a nondecreasing function $\omega^\star$.

Example 3.2. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$. Set $x_0 = .99$, $\gamma \in (0,1]$, and $\mathcal{D} = U(.99, 1-\gamma)$. Define a function F on $\mathcal{D}$ by

$$F(x) = x^3 - \gamma. \tag{3.15}$$

Using (3.15) and $(\mathcal{C}_1)$–$(\mathcal{C}_6)$, we get

$$\omega^\star(r) = \frac{2}{.99^2} r = 2.040608101r, \quad h(t) = t, \quad H = \frac{1}{2},$$

$$\overline{L}_0(r) = \frac{1}{.99^2} r = 1.020304051r < \overline{L}(r) = 2.040608101(r + .99),$$

$$\eta = \frac{1}{3 \times .99^2}(.99^3 - \gamma),$$

$$g(r) = 6.121824303\eta r + 2.020202020\eta + .00001\eta\sqrt{208204071100r^2 + 247346436500r + 40812162030} - 4,$$

and

$$q(r) = .34010135(r - r_0)^2 + 1.02030405r_0(r - r_0) + 1.02030405r_0^2,$$

Note that $\omega^\star$ is a nondecreasing function.

For $\gamma = .01$, and using Maple 13, we obtain

$$\eta = .3265989865, \quad r_0 = .7294057602 \quad \text{and} \quad r^\star = 1.229434953.$$

Hence, the conclusions of our Theorem 2.3 apply to the function (3.15). Moreover, the sequence defined by (1.2) starting at x_0 converges to $x^\star = .215443469$.

Remark 3.3. In Example 3.2, if we set $x_0 = 1$, we show that this choice is not possible for $\eta = \frac{1}{3}(1-\gamma)$. Indeed, by (3.15) and $(\mathcal{C}_1)$–$(\mathcal{C}_6)$, we get

$$\omega^\star(r) = 2r, \quad h(t) = t, \quad H = 1/2,$$
$$\overline{L}_0(r) = r < \overline{L}(r) = 2(r+1), \quad \eta = \tfrac{1}{3}(1-\gamma),$$
$$g(r) = 2(3r + 1 + \sqrt{5r^2 + 6r + 1})\eta - 4,$$
$$q(r) = \frac{r^3 - r_0^3}{3(r - r_0)} - 1 = r^2 + rr_0 + r_0^2 - 1.$$

Using Maple 13, we obtain

$$r_0(\eta) = \frac{3 - \sqrt{5 + 4\eta}}{2\eta} \quad \text{and} \quad r^\star(r_0) = \frac{\sqrt{4 - 3r_0^2} - r_0}{2}.$$

The solution set of the inequality $r^\star(\eta) \geq r_0$ in $\mathbb{R}^+$ is

$$I_{r_0} = [0, \sqrt{3}/3] \simeq [0, .5773502692]$$

(see also Fig. 1). Consequently, we can find all $\eta > 0$ such that $r_0(\eta) \in I_{r_0}$,

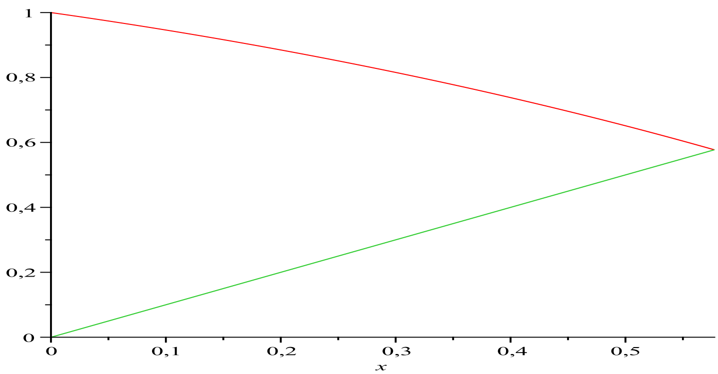

Fig. 1. The functions $r^\star(x)$ and x on $[0, .5773502692]$

and get $\eta \in (.3840179681, \infty)$. This contradicts the hypothesis $\eta = \frac{1}{3}(1-\gamma) \in [0, 0.333333)$. The choice of starting point $x_0 = 1$ is possible if we replace η by $\eta^\star > .3840179681$.

Conclusion. Under $\omega^\star$-conditioned second Fréchet derivative, we provided a new semilocal convergence analysis for Newton's method to approximate nonlinear equations in Banach spaces. Our analysis uses more precise majorizing sequences than [16], [17] and provides quadratic instead of linear convergence. An example using a Hammerstein integral equation is also provided to validate the theoretical results.

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