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M. A. W. Mahmoud, M. A. Atallah and M. Albassam (Jeddah)

ON CHARACTERISTIC FUNCTIONS OF kTH RECORD VALUES FROM THE GENERALIZED EXTREME VALUE DISTRIBUTION AND ITS CHARACTERIZATION

Abstract. Recurrence relations for the marginal, joint and conditional characteristic functions of kth record values from the generalized extreme value distribution are established. These relations are utilized to obtain recurrence relations for single, product and conditional moments of kth record values. Moreover, by making use of the recurrence relations the generalized extreme value distribution is characterized.

1. Introduction. Let $\{X_n, n \geq 1\}$ be an infinite sequence of independent and identically distributed random variables with common probability density function (pdf) $f(x)$ and cumulative distribution function (cdf) $F(x)$. Let $V_n = \min\{X_1, \ldots, X_n\}$, $n \geq 1$. We say X_i is a *lower record value* of this sequence if $V_i < V_{i-1}$, $i \geq 2$. Consider $L(n) = \min\{i : i > L(n-1)$, $X_i < X_{L(n-1)}\}$ for $n \geq 2$, and $L(1) = 1$. Then $\{X_{L(n)}, n \geq 1\}$ denotes the sequence of lower record values. Looking at the successive kth largest values in the sequence, Dziubdziela and Kopociński (1976) introduced the model of kth upper record values. Pawlas and Szynal (1998) discussed the concept of kth lower record values.

For a fixed $k \geq 1$, we define the sequence $\{L_k(n), n \geq 1\}$ of kth *lower record times* of $\{X_n,\, n \geq 1\}$ as follows:

$$L_k(1) = 1, \qquad L_k(n+1) = \min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}.$$

The sequence $\{X_{n;k},\, n \geq 1\}$, where $X_{n;k} = X_{k:L_k(n)+k-1}$, is called the sequence of kth *lower record values* of $\{X_n,\, n \geq 1\}$.

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It is known (see Pawlas and Szynal 1998) that the pdf of $X_{n;k}$ and the joint pdf of $(X_{m;k}, X_{n;k})$, $m < n$, are given respectively by

$$f_{X_{n;k}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1}f(x), \qquad n \geq 1,\ -\infty < x < \infty, \tag{1.1}$$

$$\begin{aligned} f_{X_{m;k},X_{n;k}}(x,y) = {} & \frac{k^n}{(m-1)!(n-m-1)!}[\ln F(x) - \ln F(y)]^{n-m-1} \\ & \times [-\ln F(x)]^{m-1}\,\frac{f(x)}{F(x)}[F(y)]^{k-1}f(y), \\ & x > y,\ 1 \leq m < n,\ n \geq 2. \end{aligned} \tag{1.2}$$

Record values appear in many statistical applications. They may be helpful as a model for successively largest values for highest water levels or highest temperatures. In the context of bioscience, they appear when we are interested in the behavior of human organs, like kidneys or lungs, or when studying the behavior of the ordered records of glocagine in the assessment of glucose level among diabetic patients.

The generalized extreme value distribution (GEV) has been discussed by Jenkinson (1955). Gumbel, Frechet and Weibull distributions are special cases of the GEV distribution. These distributions have been used in the analysis of data concerning floods, extreme sea levels and air pollution problems. For details, see Gumbel (1958), Jenkinson (1955) and Ahsanullah (1995).

Recurrence relations for single and product moments of record values from the GEV distribution are derived by Balakrishnan et al. (1993). Ahsanullah and Raqab (1999) established recurrence relations for the moment generating functions of record values from Pareto and Gumbel distributions. Pawlas and Szynal (2000) gave characterization conditions for the inverse Weibull distribution and generalized extreme value distributions based on moments of kth record values.

Recently Raqab (2003) obtained recurrence relations between the marginal and joint moment generating functions of lower record values from the generalized extreme value distribution.

Let us consider the GEV distribution with pdf

$$f(x) = \begin{cases} (1-\alpha x)^{1/\alpha - 1} e^{-(1-\alpha x)^{1/\alpha}}, & x < 1/\alpha \text{ for } \alpha > 0 \\ & \text{and } x > 1/\alpha \text{ for } \alpha < 0, \\ e^{-x}e^{-e^{-x}}, & -\infty < x < \infty \text{ for } \alpha = 0. \end{cases} \tag{1.3}$$

One can observe that this distribution satisfies

$$(1-\alpha x)f(x) = F(x)(-\ln F(x)), \qquad -\infty < x < \infty. \tag{1.4}$$

The relation (1.4) is used to derive recurrence relations between the marginal and joint characteristic functions of kth record values from the GEV distribution. In this paper we derive recurrence relations for the marginal, joint and conditional characteristic functions of kth lower record value from the GEV distribution. These recurrence relations are used to obtain recurrence relations for single, product and conditional moments of kth lower record values as well as to characterize the GEV distribution.

2. Recurrence relation for the marginal characteristic function of kth record values. Let $\Phi_{n;k}(t)$ denote the characteristic function of kth lower record values $X_{n;k}$.

THEOREM 2.1. *For $n \geq 1$, $k \geq 1$ and $\alpha \neq 0$,*

$$n\Phi_{n+1;k}(t) = (n - it)\Phi_{n;k}(t) + i\alpha t\delta_{n;k}(t), \tag{2.1}$$

where

$$\delta_{n;k}(t) = \frac{1}{i}\frac{d}{dt}\Phi_{n;k}(t) \quad \text{and} \quad i = \sqrt{-1}.$$

Proof. We have

$$\Phi_{n;k}(t) = E(e^{itX_{n;k}}).$$

Thus

$$\Phi_{n;k}(t) = \int_{-\infty}^{\infty} e^{itx} f_{X_{n;k}}(x)\,dx.$$

By making use of (1.1) and (1.4), we get

$$\Phi_{n;k}(t) - \alpha\delta_{n;k}(t) = \frac{k^n}{(n-1)!}\int_{-\infty}^{\infty} e^{itx}(-\ln F(x))^n F^k(x)\,dx. \tag{2.2}$$

Integrating the right hand side of (2.2) yields

$$\begin{aligned}\Phi_{n;k}(t) - \alpha\delta_{n;k}(t) &= \frac{k^n e^{itx}}{it(n-1)!}(-\ln F(x))^n F^k(x)\Big|_{-\infty}^{\infty} \\ &\quad\times \frac{nk^n}{it(n-1)!}\int_{-\infty}^{\infty} e^{itx}(-\ln F(x))^{n-1}F^{k-1}(x)f(x)\,dx \\ &\quad- \frac{nk^{n+1}}{it(n)!}\int_{-\infty}^{\infty} e^{itx}(-\ln F(x))^{n}F^{k-1}(x)f(x)\,dx, \\ &= \frac{n}{it}\Phi_{n;k}(t) - \frac{n}{it}\Phi_{n+1;k}(t).\end{aligned} \tag{2.3}$$

After simplifying (2.3), we get (2.1).

Corollary 2.2. *For $n \geq 1$, and $r = 1, 2, \ldots$, the following equation is satisfied:*

$$\Phi^{(r)}_{n+1;k}(t) = \left[1 - \frac{it - r\alpha}{n}\right]\Phi^{(r)}_{n;k}(t) - \frac{ir}{n}\Phi^{(r-1)}_{n;k}(t) + \frac{i\alpha t}{n}\Phi^{(r+1)}_{n;k}(t), \tag{2.4}$$

where $\Phi^{(r)}_{n;k}(t)$ is the rth derivative of $\Phi_{n;k}(t)$.

Proof. This follows directly by differentiating (2.1) r times.

Remark 1.

1. By setting $t = 0$ and $k = 1$, equation (2.4) reduces to the recurrence relation for single moments of record values given in Balakrishnan et al. (1993).
2. The result of Pawlas and Szynal (1998) for single moments of kth record values is a special case of (2.4) obtained by setting $t = 0$.

3. Recurrence relations for the joint characteristic function of kth record values. Let $\Phi_{m,n;k}(t_1, t_2)$ denote the characteristic function of mth and nth lower kth record values $X_{m;k}$ and $X_{n;k}$.

Theorem 3.1. *For $1 \leq m < n - 1$ and $\alpha \neq 0$, the joint characteristic functions of kth lower record values from* GEV(α) *defined in* (1.3) *satisfy the recurrence relation*

$$m\Phi_{m+1,n;k}(t_1, t_2) = (m - it_1)\Phi_{m,n;k}(t_1, t_2) + i\alpha t_1\delta_{m,n;k}(t_1, t_2), \tag{3.1}$$

where

$$\delta_{m,n;k}(t_1, t_2) = \frac{1}{i}\frac{\partial}{\partial t_1}\Phi_{m,n;k}(t_1, t_2).$$

Proof. On making use of (1.2) and (1.4), we get

$$\Phi_{m,n;k}(t_1, t_2) - \alpha\delta_{m,n;k}(t_1, t_2) = c\int_{-\infty}^{\infty} e^{it_2 y} I(y) F^{k-1}(y) f(y)\, dy, \tag{3.2}$$

where

$$c = \frac{k^n}{(m-1)!(n-m-1)!}$$

and

$$I(y) = \int_{y}^{\infty} e^{it_1 x}(-\ln F(x))^m[\ln F(x) - \ln F(y)]^{n-m-1}\, dx. \tag{3.3}$$

Integrating (3.3) by parts (with e^{it_1x} differentiated) gives

$$(3.4)\qquad I(y)=\frac{m}{it_1}\int_y^\infty e^{it_1x}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}\frac{f(x)}{F(x)}\,dx$$
$$-\frac{n-m-1}{it_1}\int_y^\infty e^{it_1x}(-\ln F(x))^{m}[\ln F(x)-\ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}\,dx.$$

Making use of (3.4) in (3.2), after some simplification we get (3.1).

Corollary 3.2. *For $1\le m<n-1$, $r,s=0,1,2,\ldots$ and $\alpha\neq 0$, the following is satisfied:*

$$(3.5)\qquad \Phi^{(r+1,s)}_{m+1,n;k}(t_1,t_2)=\left[1-\frac{it_1-r\alpha}{m}\right]\Phi^{(r+1,s)}_{m,n;k}(t_1,t_2)$$
$$-\frac{ir}{m}\Phi^{(r,s)}_{m,n;k}(t_1,t_2)+\frac{\alpha t_1}{m}\Phi^{(r+2,s)}_{m,n;k}(t_1,t_2),$$

where

$$\Phi^{(r,s)}_{m,n;k}(t_1,t_2)=\frac{\partial^{r+s}}{\partial^r t_1\partial^s t_2}\Phi_{m,n;k}(t_1,t_2).$$

Proof. This follows directly by differentiating (3.1) $r+1$ times with respect to t_1 and s times with respect to t_2.

Remark 2.

1. By setting $t_1=t_2=0$ and $k=1$, the result of Balakrishnan et al. (1993) is a special case of (3.5).
2. Setting $t_2=0$ in Theorem 3.1 and Corollary 3.2, the results of Theorem 2.1 and Corollary 2.2 are obtained on replacing r by $r+1$.

4. Recurrence relations for the conditional characteristic functions. Let $f_{n|m;k}(y|x)$ and $\phi_{n|m;k}(t)$ be the conditional pdf and the conditional characteristic function of $X_{n;k}$ given $X_{m;k}=x$, respectively. Then we can prove the following theorem.

Theorem 4.1. *For $1\le m\le n-1$ and $\alpha\neq 0$, the conditional characteristic functions of kth lower record values from* GEV(α) *defined in* (3.1) *satisfy the following recurrence relation:*

$$(4.1)\qquad (n-m)\phi_{n+1|m;k}(t)+k[\ln F(x)]\phi_{n|m;k}(t)$$
$$=[n-m+k\ln F(x)-it]\phi_{n|m;k}(t)+i\alpha t\delta_{n|m;k},$$

where

$$\delta_{n|m;k}(t)=\frac{1}{i}\frac{d}{dt}\Phi_{n|m;k}(t).$$

Proof. Using (1.1) and (1.2), one can show that

$$f_{n|m;k}(y|x) = c_1[\ln F(x) - \ln F(y)]^{n-m-1}F^{k-1}(y)f(y), \quad -\infty < y < x < \infty, \tag{4.2}$$

where

$$c_1 = \frac{k^{n-m}}{(n-m-1)!F^k(x)}.$$

Making use of (1.4) and (4.2), we get

$$\begin{aligned}\phi_{n|m;k}(t) &- \alpha\delta_{n|m;k}(t)\\ &= c_1 \int_{-\infty}^{x} e^{ity}(1-\alpha y)[\ln F(x) - \ln F(y)]^{n-m-1}F^{k-1}(y)f(y)\,dy\\ &= c_1 \int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m-1}F^{k}(y)(-\ln F(y))\,dy,\end{aligned}$$

which can be written as

$$\phi_{n|m;k}(t) - \alpha\delta_{n|m;k}(t) = I_1 - I_2, \tag{4.3}$$

where

$$I_1 = c_1 \int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m}F^{k}(y)\,dy, \tag{4.4}$$

$$I_2 = c_1[\ln F(x)] \int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m-1}F^{k}(y)\,dy. \tag{4.5}$$

Integrating (4.3) and (4.4) by parts (with e^{ity} integrated and the rest of the integrand differentiated), we get

$$I_1 = \frac{n-m}{it}\phi_{n|m;k}(t) - \frac{n-m}{it}\phi_{n+1|m;k}(t), \tag{4.6}$$

$$I_2 = \frac{k\ln F(x)}{it}\phi_{n|m+1;k}(t) - \frac{k}{it}[\ln F(x)]\phi_{n|m;k}(t). \tag{4.7}$$

Upon using (4.3), (4.6) and (4.7), we obtain (4.1). This completes the proof.

Now, let $f_{m|n;k}(x|y)$ and $\phi_{m|n;k}(t)$ be the conditional pdf and the conditional characteristic function of $X_{m;k}$ given $X_{n;k} = y$.

THEOREM 4.2. *For $1 \le m < n-1$ and $\alpha \neq 0$,*

$$m\Phi_{m+1|n;k}(t) = (m - it_1)\Phi_{m|n;k}(t) + i\alpha t\delta_{m|n;k}(t), \tag{4.8}$$

where

$$\delta_{m|n;k}(t) = \frac{1}{i}\frac{d}{dt}\Phi_{m|n;k}(t).$$

Proof. On utilization (1.1) and (1.2), we get

$$f_{m|n}(x|y) = c_2(-\ln F(x))^{m-1}[\ln F(x) - \ln F(y)]^{n-m-1}\frac{f(x)}{F(x)}, \quad -\infty < y < x < \infty. \tag{4.9}$$

where

$$c_2 = \frac{(n-1)!}{(m-1)!(n-m-1)!(-\ln F(y))^{n-1}}.$$

Upon using (1.4) and (4.9), one can see that

$$\Phi_{m|n;k}(t) - \alpha\delta_{m|n;k}(t) = c_2 \int_y^\infty e^{itx}(-\ln F(x))^m[\ln F(x) - \ln F(y)]^{n-m-1}\,dx.$$

Integrating by parts on the right hand side (with e^{ity} integrated and the rest of the integrand differentiated), after some simplifications we obtain the required result.

The following can be derived from Theorem 4.2:

COROLLARY 4.3. *For $1 \le m < n-1$, $r, s = 0, 1, 2, \ldots$ and $\alpha \ne 0$,*

$$\Phi^{(r)}_{m+1|n;k}(t) = \left[1 - \frac{it - r\alpha}{m}\right]\Phi^{(r)}_{m|n;k}(t) - \frac{ir}{m}\Phi^{(r-1)}_{m|n;k}(t) + \alpha t\Phi^{(r+1)}_{m|n;k}(t), \tag{4.10}$$

$$\begin{aligned} a\Phi^{(r)}_{n+1|m;k}(t) + b\Phi^{(r)}_{n|m+1;k}(t) &= [a + b - it + r\alpha]\Phi^{(r)}_{n|m;k}(t) \\ &= [a + b - it + ir] - ir\Phi^{(r-1)}_{n|m;k}(t) + \alpha t\Phi^{(r+1)}_{n|m;k}(t), \end{aligned} \tag{4.11}$$

where $a = n - m$ and $b = k\ln F(x)$.

REMARK 3.

1. By letting $\alpha \to 0$ in (4.1), (4.8), (4.10) and (4.11), results for the Gumbel distribution are obtained.
2. Setting $t = 0$ in (4.10) and (4.11), one gets recurrence relations between the conditional moments of lower kth record values.

5. Characterization. In this section, we characterize the generalized extreme value distribution based on the recurrence relations for marginal, joint and conditional characteristic functions.

PROPOSITION (Lin 1986). *Let n_0 be any fixed non-negative integer, $-\infty < a < b < \infty$, and $g(x) > 0$ be an absolutely continuous function with $g'(x) \ne 0$ on (a, b). Then the sequence of functions $\{(g(x))^n e^{-g(x)},\ n \ge n_0\}$ is complete in $L(a, b)$ iff $g(x)$ is strictly monotone on (a, b).*

THEOREM 5.1. *A necessary and sufficient condition for a random variable X to be distributed according to* (1.4) *is that* (2.1) *holds.*

Proof. The necessity is proved in Theorem 2.1.

For the sufficiency, assume that (2.1) holds. Then

$$\frac{nk^{n+1}}{n!}\int_{-\infty}^{\infty} e^{ixt}(-\ln F(x))^n F^{k-1}(x)f(x)\,dx = (n-it)\Phi_{n;k}(t) + i\alpha t\delta_{n;k}(t).$$

Integrating the left hand side by parts (with $F^{k-1}(x)f(x)$ integrated and the rest of the integrand differentiated), we get

$$n\Phi_{n;k}(t) - it\int_{-\infty}^{\infty} e^{ixt}(-\ln F(x))^n F^{k}(x)\,dx = (n-it)\Phi_{n;k}(t) + i\alpha t\delta_{n;k}(t).$$

This leads to

$$\int_{-\infty}^{\infty} e^{ixt}(-\ln F(x))^{n-1} F^{k-1}(x)[(1-\alpha x)f(x) + F(x)\ln F(x)]\,dx = 0.$$

Upon using the Proposition, we get $(1-\alpha x)f(x) = -F(x)\ln F(x)$, and this completes the proof.

THEOREM 5.2. *For $1 \le m < n-1$, the following statements are equivalent:*

(i) $X \sim \mathrm{GEV}(\alpha), \alpha \neq 0$.
(ii) $m\Phi_{m+1,n;k}(t_1,t_2) = (m-it_1)\Phi_{m,n;k}(t_1,t_2) + i\alpha t_1\delta_{m,n;k}(t_1,t_2)$.

Proof. (i)⇒(ii) is shown in Theorem 3.1. Conversely, assume that (ii) holds. This leads to

$$\frac{mk^n}{m!(n-m-2)!}\int_{-\infty}^{\infty} e^{it_2 y} I(y)F^{k-1}(y)f(y)\,dy = (m-it_1)\Phi_{m,n;k}(t_1,t_2) + i\alpha t_1\delta_{m,n;k}(t_1,t_2), \tag{5.1}$$

where

$$I(y) = \int_{y}^{\infty} e^{it_1 x}(-\ln F(x))^m[\ln F(x) - \ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}\,dx.$$

Using integration by parts (with $[\ln F(x) - \ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}$ integrated and the rest of the integrand differentiated), $I(y)$ can be written as follows:

$$\begin{aligned} I(y) = {}& \frac{m}{(n-m-1)} \\ &\times \int_{y}^{\infty} e^{it_1 x}(-\ln F(x))^{m-1}[\ln F(x) - \ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}\,dx \\ &- \frac{it_1}{(n-m-1)}\int_{y}^{\infty} e^{it_1 x}(-\ln F(x))^{m}[\ln F(x) - \ln F(y)]^{n-m-1}\,dx. \end{aligned} \tag{5.2}$$

Upon using (5.1) and (5.2), we get

$$\int_{-\infty}^{\infty}\int_{y}^{\infty} e^{it_1x}e^{it_2y}(-\ln F(x))^m[\ln F(x)-\ln F(y)]^{n-m-1}F^{k-1}(y)f(y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{y}^{\infty} e^{it_1x}e^{it_2y}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}\frac{f(x)}{F(x)} \times F^{k-1}(y)f(y)\,dx\,dy$$

$$-\alpha\int_{-\infty}^{\infty}\int_{y}^{\infty} xe^{it_1x}e^{it_2y}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}\frac{f(x)}{F(x)} \times F^{k-1}(y)f(y)\,dx\,dy,$$

which leads to

$$\int_{-\infty}^{\infty}\int_{y}^{\infty} e^{it_1x}e^{it_2y}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}F^{k-1}(y)f(y) \times\left[(1-\alpha x)\frac{f(x)}{F(x)}+\ln F(x)\right]dx\,dy=0.$$

Upon using the Proposition, we get

$$(1-\alpha x)\frac{f(x)}{F(x)}=-\ln F(x),$$

so $(1-\alpha x)f(x)=-F(x)\ln F(x)$, which proves by (1.4) that $f(x)$ has the form (1.3).

THEOREM 5.3. *For $1\le m<n-1$, $k=1,2,\dots$ and $\alpha\ne 0$, the following statement are equivalent:*

(i) $X\sim \mathrm{GEV}(\alpha)$.
(ii) *Equation* (4.8) *holds.*

Proof. (i)⇒(ii) is proved in Theorem 4.2.

To prove (ii)⇒(i), we have

$$\frac{(n-1)!}{(m-1)!(n-m-2)!(-\ln F(y))^{n-1}}\int_{y}^{\infty} e^{itx}(-\ln F(x))^m \times[\ln F(x)-\ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}\,dx$$
$$=(m-it)\Phi_{m|n;k}(t)+i\alpha t\delta_{m|n;k}(t).$$

Integrating the left hand side by parts (with $[\ln F(x)-\ln F(y)]^{n-m-2}\frac{f(x)}{F(x)}$

integrated and the rest of the integrand differentiated) gives

$$\int_y^\infty e^{itx}(-\ln F(x))^m[\ln F(x)-\ln F(y)]^{n-m-1}\,dx$$
$$-\int_y^\infty e^{itx}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}\frac{f(x)}{F(x)}\,dx$$
$$+\alpha\int_y^\infty xe^{itx}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}\frac{f(x)}{F(x)}\,dx=0,$$

which leads to

$$\int_y^\infty e^{itx}(-\ln F(x))^{m-1}[\ln F(x)-\ln F(y)]^{n-m-1}$$
$$\times[(1-\alpha x)f(x)+F(x)\ln F(x)]\,dx=0.$$

Upon using the Proposition, we get

$$(1-\alpha x)f(x)=-F(x)\ln F(x)$$

This completes the proof.

THEOREM 5.4. *For $1\le m<n-1$, $k=1,2,\dots$ and $\alpha\neq 0$, the following statement are equivalent:*

(i) $Y\sim \mathrm{GEV}(\alpha)$ *and* $\alpha\neq 0$.
(ii) *Equation* (4.1) *holds.*

Proof. (i)⇒(ii) is proved in Theorem 4.1.

To prove the converse, assume that (4.1) is satisfied. Then

$$\frac{k^{n-m+1}}{(n-m-1)!F^k(x)}\int_{-\infty}^x e^{ity}[\ln F(x)-\ln F(y)]^{n-m}F^{k-1}(y)f(y)\,dy$$
$$+k\ln F(x)\frac{k^{n-m-1}}{(n-m-2)!F^k(x)}\int_{-\infty}^x e^{ity}[\ln F(x)-\ln F(y)]^{n-m-2}$$
$$\times F^{k-1}(y)f(y)\,dy=[n-m+k\ln F(x)-it]\varPhi_{n|m;k}(t)+i\alpha t\delta_{n|m;k}(t).$$

After integrating the two integrals on the left hand side by parts, it can be shown that

$$(n-m)\varPhi_{n|m;k}(t)-itJ_1+k\ln F(x)\varPhi_{n|m;k}(t)+itJ_2$$
$$=[n-m+k\ln F(x)-it]\varPhi_{n|m;k}(t)+i\alpha t\delta_{n|m;k}(t),$$

where

$$J_1 = \frac{k^{n-m}}{(n-m-1)!F^k(x)} \int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m} F^k(y)\,dy,$$

$$J_2 = \frac{k^{n-m}}{(n-m-1)!F^k(x)}[\ln F(x)] \int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m-1} F^k(y)\,dy.$$

This leads to

$$\int_{-\infty}^{x} e^{ity}[\ln F(x) - \ln F(y)]^{n-m-1} F^{k-1}(y) \times [F(y)\ln F(y) + (1-\alpha y)f(y)]\,dy = 0.$$

Using the Proposition, it can be shown that

$$(1-\alpha y)f(y) = F(y)\ln F(y),$$

which implies that (i) is true and the proof is complete.

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M. A. W. Mahmoud, M. A. Atallah, M. Albassam
Department of Statistics
King Abdulaziz University
Jeddah, Saudi Arabia
E-mail: mawmahmoud11@hotmail.com
asu_mahmoud@yahoo.com
malbassam@kau.edu.sa

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