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ON THE PRIMARY ORBITS OF STAR MAPS (SECOND PART: SPIRAL ORBITS)

Abstract. This paper is the second part of [2] and is devoted to the study of the spiral orbits of self maps of the 4-star with the branching point fixed, completing the characterization of the strongly directed primary orbits for such maps.

1. Introduction. In this paper we continue the work done in [2], in order to complete the characterization of the strongly directed primary orbits for self maps of the 4-star with the branching point fixed. Recall that *strongly directed* are those periodic orbits satisfying the following condition.

Generalized Directed Rule. *Let P be a periodic orbit of a map $f \in \mathcal{X}_n$. For each sequence $A_0, A_1, \ldots, A_{k-1}$ of overlapping arrows of P, we have $k \geq n$.*

To develop our study, in [2] we have classified the primary strongly directed orbits of the 4-star into several families, paying attention to several features of their shape. First, we looked at the number of coloured arrows the orbit has. For *twist* orbits, having no coloured arrows, and *single* orbits, with just one coloured arrow, a characterization independent of n is summarized in [2, Theorem A].

In the case $n = 4$, the Generalized Directed Rule imposes some crucial restrictions on the coloured arrows a primary strongly directed orbit can have. Namely, such an orbit cannot have more than three coloured arrows and it must be *colour compatible* (see [2, Theorem B]). For a directed orbit,

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to be colour compatible means that for each nonempty set of coloured arrows the sum of its colours is not a multiple of n. In particular, when $n = 4$, this means that the orbit cannot have two red arrows, nor can it have green and blue arrows simultaneously. Moreover, if it has three coloured arrows, then these arrows must all be of the same colour, green or blue.

The classification of the primary strongly directed orbits in the 4-star with more than one coloured arrow also depends on another feature of their shape: the existence of crossing arrows. We already know that such orbits with crossing arrows are *box* orbits (see [2, Theorem C]). Strongly directed orbits without crossing arrows will be called *spiral* in this paper.

The goal of this paper is to end the characterization of the primary strongly directed orbits by classifying the primary spiral orbits with more than one coloured arrow. They are characterized as *double* (Definition 4.1) and *triple* (Definition 5.3) orbits, in Theorems 4.9 and 5.14 respectively. This part of the characterization of the primary strongly directed orbits is far more technical than the previous work done in [2].

This paper is organized as follows. Through Section 2 we describe spiral orbits in general. In Section 3 we obtain a sufficient condition for spiral orbits to be primary. Sections 4 and 5 are devoted to the study of double and triple spiral orbits. Finally, in Section 6 we summarize the main results of this paper and [2].

2. Spiral orbits. In this section we start the study of the strongly directed periodic orbits having no crossing arrows, that is, *spiral* orbits. We recall that two arrows A and B such that $b(A) < b(B)$ are said to be *crossing* if $e(A) > e(B)$. Clearly, single orbits are spiral. On the contrary, directed orbits having only black arrows obviously have crossing arrows and, hence, are not spiral.

Since we want to characterize the spiral orbits which are primary, in view of the First Theorem (Theorem 2.3) of [1] and [2, Theorem B], in what follows we shall assume that P is a spiral orbit of an EP-adjusted map $f \in \mathcal{X}_4$ of period m with $\nu \leq 3$ coloured arrows. When necessary, we can also assume that P is colour compatible, again by [2, Theorem B].

We note that since P is directed and has some coloured arrow, $m > n = 4$. Since the arrows are not crossing and P is directed, we find that for each branch, br, either $\mathrm{sm}(br) = f(\mathrm{sm}(\sigma^{-1}(br)))$ or $\mathrm{sm}(br)$ is the end of a coloured arrow. Notice that the first condition is not satisfied for every branch because $m > n$. Hence there is one branch whose smallest point is the end of a coloured arrow. In particular, this implies that single orbits are the only spiral orbits with a unique coloured arrow. Thus, since single orbits have already been studied, in what follows we only need to study spiral orbits with $\nu \geq 2$ coloured arrows. So, from now on, $\nu \in \{2, 3\}$.

We start by fixing the notation. Let br_0 be a branch such that sm_0 is the end of a coloured arrow F_0. We label the points of P and the branches of $\mathbb{X}_n$ as follows:

$$x_i = f^i(sm_0) \quad \text{for each } i \in \mathbb{Z}_m, \qquad br_i = \sigma^i(br_0) \quad \text{for each } i \in \mathbb{Z}_n.$$

With this notation, we have $F_0 = (x_{m-1}, x_0)$ and if F_k with $k = 1, \ldots \nu-1$ denote the other coloured arrows, then there exist $p_k \in \{2, 3, \ldots, m-1\}$ such that $F_k = (x_{p_k-1}, x_{p_k})$ for each $k = 1, \ldots, \nu-1$. For convenience, when $\nu = 3$, we assume that $p_1 < p_2$.

If we denote by A_i the arrow beginning at x_i (that is, $b(A_i) = x_i$ and $e(A_i) = x_{i\oplus 1}$ for each $i \in \mathbb{Z}_m$) we see that $F_0 = A_{m-1}$ and $F_k = A_{p_k-1}$ for $k = 1, \ldots, \nu-1$. Let us set $\mathcal{C} = \{F_k : k \in \mathbb{Z}_\nu\}$ and, for $i, j \in \mathbb{Z}_m$, $i \neq j$, $\mathcal{C}_{ij} = \mathcal{C} \cap \{A_i, A_{i\oplus 1}, \ldots, A_{j\ominus 1}\}$. Obviously, $\mathcal{C}_{ji} = \mathcal{C} \setminus \mathcal{C}_{ij}$. The following lemma is a simple rewriting of [2, Lemma 2.4].

Lemma 2.1. *For $i, j \in \mathbb{Z}_m$, $i \neq j$, we have*

$$\operatorname{ind}(x_j) + \sum_{F \in \mathcal{C}_{ij}} c(F) \equiv \operatorname{ind}(x_i) + j \ominus_m i.$$

Set $p_0 = 0$ and $p_\nu = m$ to unify the notation. Then the strings of P are

$$S_k = \{x_i \in P : p_k \le i < p_{k+1}\}, \qquad k = 0, 1, \ldots, \nu-1,$$

of lengths $l(S_k) = p_{k+1} - p_k$.

We define $q : \mathbb{Z}_m \to \mathbb{Z}_\nu$ as follows:

$$\text{for each } i \in \mathbb{Z}_m, \qquad q(i) = k \qquad \text{if } x_i \in S_k.$$

We are going to see that each string of a spiral orbit spirals out, which justifies the name given to these orbits. Note that, by [2, Proposition 2.10], this property is satisfied by every directed primary orbit with some coloured arrow. Since we are only interested in the study of the strongly directed orbits which are primary, we could have imposed this property on our orbits. However, as we will see in Lemma 2.3, this would be redundant. We start by proving that the beginning of each string is not the largest point in its branch. This fact, which is evident for $x_0 = sm_0 < e(smA_{n-1})$, is stated for the other strings in the following lemma.

Lemma 2.2. *We have $x_{p_1} < x_{p_1-c(F_1)}$ and, if $\nu = 3$, there exists $d_2 < p_2$ such that $x_{p_2} < x_{p_2-d_2}$.*

Proof. Let $k \in \{1, \ldots, \nu-1\}$. If $c(F_k) \le l(S_{k-1})$, then the $c(F_k)$ arrows

$$A_{p_k-c(F_k)}, A_{p_k-c(F_k)+1}, \ldots, A_{p_k-1} = F_k$$

are all black but the last one and, by [2, Lemma 4.3], $x_{p_k} < x_{p_k-c(F_k)}$. In particular this is what happens when F_k is green and, by [2, Lemma 4.4], when $k = 1$. This proves the first statement of the lemma.

If $c(F_k) > l(S_{k-1})$ then, from the above, $k = 2$ and the arrows

$$A_0, \ldots, A_{p_1 - c(F_1) - 1}, A_{p_1}, \ldots, A_{p_2 - 1} = F_2, A_{p_2}, \ldots, A_{m-1}$$

are overlapping. Again by [2, Lemma 4.4], we have $p_2 - 1 - c(F_1) \geq c(F_2)$. That is, $p_2 > c(F_1) + c(F_2)$. Since $c(F_2) > p_2 - p_1$, we can consider the $c(F_2)$ consecutive arrows

$$A_{p_2 - c(F_1) - c(F_2)}, \ldots, A_{p_1 - c(F_1) - 1}, A_{p_1}, \ldots, A_{p_2 - 1} = F_2$$

in the above sequence of overlapping arrows. Then, again by [2, Lemma 4.3], we conclude that $x_{p_2} < x_{p_2 - c(F_1) - c(F_2)}$. ■

The next lemma already shows that each string spirals out.

Lemma 2.3. *For each* $k = 0, 1, \ldots, \nu - 1$, *if* $l(S_k) > n$ *then* $x_{p_k + i} < x_{p_k + i + n}$ *for* $i = 0, 1, \ldots, l(S_k) - n - 1$.

Proof. Since the arrows are not crossing, it is enough to see that $x_{p_k} < x_{p_k + n}$. This is immediate for $k = 0$, because $x_n \sim x_0 = sm_0$. Let us see it for $k \in \{1, \ldots, \nu - 1\}$. On the contrary, assume that $x_{p_k} > x_{p_k + n}$. Then we claim that, for each $i = 0, 1, \ldots, p_k - 1$, there exists a $t \in \mathbb{Z}_n$ such that $x_i < x_{p_k + t}$ and we will get a contradiction.

Now we prove the claim. Clearly, for each branch br here is just one $t \in \mathbb{Z}_n$ such that $x_{p_k + t} \in br$. Hence, the statement is true for $i = 0$ because $x_0 = sm_0$. Now we will prove that if it is true for $i \in \{0, 1, \ldots, p_k - 2\}$, then it is also true for $i + 1$. Indeed, if x_i is the beginning of a black arrow, then since the arrows are not crossing, $x_i < x_{p_k + t}$ implies that $x_{i+1} < x_{p_k + t + 1}$ and we are done if $t \neq n - 1$. Otherwise, $x_{i+1} < x_{p_k + n} < x_{p_k}$. If x_i is the beginning of a coloured arrow, then $i = p_1 - 1$ (and $k = 2$). Thus, by Lemma 2.2, there exists $d_1 < p_1$ such that $x_{i+1} = x_{p_1} < x_{p_1 - d_1}$. Since the arrows beginning at x_j for $j = 0, 1, \ldots, p_1 - d_1 - 1$ are all black, from the above, there is a $t \in \mathbb{Z}_n$ such that $x_{p_1 - d_1} < x_{p_2 + t}$. This ends the proof of the claim.

Then we obtain a contradiction in the following way. By Lemma 2.2, $x_{p_k - d_k} > x_{p_k}$, and for all $t \in \mathbb{Z}_n \setminus \{0\}$, $x_{p_k - d_k} \sim x_{p_k} \not\sim x_{p_k + t}$. That is, the claim is false for $i = p_k - d_k$. ■

In what follows, for $a, b \in \mathbb{Z}$, when we write $a \lesseqqgtr b$ we mean that $a \leq b$ and $a \equiv b$. The symbol $\gtreqqless$ will also be used in the natural way. The next simple corollary follows from Lemmas 2.1 and 2.3 and summarizes the behaviour of each string of P.

Corollary 2.4. *If* $q(i) = q(j)$ *then* $x_i \leq x_j$ *if and only if* $i \lesseqqgtr j$.

To describe the spiral orbits, since we already know how the points of a string are situated with respect to the other points of the same string, it remains to study how each string is intertwined with the other ones. So, in the rest of the section we shall study the relative position of points of different strings. To do this we shall define certain numbers which will determine when and how points of different strings are in the same branch.

For $k \in \mathbb{Z}_\nu$, the *next* string to S_k will be $S_{k\oplus_\nu 1}$ and the *previous* one $S_{k\ominus_\nu 1}$. Observe that, since $\nu \in \{2,3\}$, given two different strings, they are always *consecutive*. The arrow F_k *separates* the string S_k from its previous one.

LEMMA 2.5. *Two consecutive strings have points in the same branch if and only if the sum of their lengths is greater than the colour of the arrow separating them.*

Proof. Assume that $x_i \sim x_j$ with $q(j) = q(i) \oplus_\nu 1$. Then observe that $x_j = f^r(x_i)$ with $r = j \ominus_m i$. Set $c = c(F_{q(j)})$. By Lemma 2.1 we have $r \equiv c$. So, since $c \in \mathbb{Z}_n$, we get $r \gneqq c$. On the other hand, by the definition of r, it is clear that $r < l(S_{q(i)}) + l(S_{q(j)})$.

Conversely, if $l(S_{k\ominus 1}) + l(S_k) > c = c(F_k)$ then we can find an $x \in S_{k\ominus 1}$ such that $f^c(x) \in S_k$. Indeed, if $c \geq l(S_{k\ominus 1})$ it is enough to take $x = x_{p_{k\ominus 1}}$ and if $c < l(S_{k\ominus 1})$ we take the $x \in S_{k\ominus 1}$ such that $f^c(x) = x_{p_k}$. Hence, $x \sim f^c(x)$ by [2, Lemma 2.4]. ∎

In order to analyze the relative positions of points of two consecutive strings, we fix $q \in \mathbb{Z}_\nu$ and look at the strings S_q and $S_{q\oplus 1}$. Let c be the colour of the arrow separating these strings, that is, $c = c(F_{q\oplus 1})$. In accordance with Lemma 2.5, to have points of both strings in the same branch it must happen that $c < l(S_q) + l(S_{q\oplus 1})$. Moreover, from the proof of the above lemma, we see that in such a case there is an $x \in S_q$ such that $f^c(x) \in S_{q\oplus 1}$ and $x \sim f^c(x)$. Hence $f^c(x) < x$ by the Generalized Directed Rule. With this in mind we can define the following numbers:

DEFINITION 2.6. Let $q \in \mathbb{Z}_\nu$.

(a) If $c(F_{q\oplus 1}) \geq l(S_q) + l(S_{q\oplus 1})$, then we set $r_q = c(F_{q\oplus 1})$.
(b) If $c(F_{q\oplus 1}) < l(S_q) + l(S_{q\oplus 1})$, then we set:
  (b.1) $i_q = \min\{i \in \mathbb{Z}_m : q(i) = q \text{ and } [0, x_i] \cap S_{q\oplus 1} \neq \emptyset\}$.
  (b.2) $j_q = \max\{j \in \mathbb{Z}_m : q(j) = q \oplus_\nu 1 \text{ and } x_j < x_{i_q}\}$.
  (b.3) $r_q = j_q \ominus_m i_q$.

The motivation for this definition will be found in what follows. In the next lemma we gather the basic properties of the numbers i_q, j_q and r_q when there are points of both strings in the same branch.

LEMMA 2.7. *Assume that* $q \in \mathbb{Z}_\nu$ *and* $c(F_{q\oplus 1}) < l(S_q) + l(S_{q\oplus 1})$. *Then*:

(a) $c(F_{q\oplus 1}) \lneqq r_q < l(S_q) + l(S_{q\oplus 1})$.
(b) $i_q = p_q$ *or* $j_q = p_{q\oplus 1}$.
(c) $i_q = p_q$ *is equivalent to* $r_q \geq l(S_q)$.
(d) $r_q + n < m$.

Proof. (a) This follows immediately from Definition 2.6 and Lemma 2.1 (see the first part of the proof of Lemma 2.5).

(b) If $i_q > p_q$ and $j_q > p_{q\oplus 1}$, then the two arrows (x_{i_q-1}, x_{i_q}) and (x_{j_q-1}, x_{j_q}) are black and, since they are not crossing, $x_{i_q-1} > x_{j_q-1}$, in contradiction with the definition of i_q.

(c) If $i_q = p_q$, then $r_q = j_q \ominus p_q$ and hence $r_q \geq l(S_q)$. Otherwise, by (b), $j_q = p_{q\oplus 1}$ and then $p_{q\oplus 1} \ominus r_q = i_q > p_q$. This implies that $r_q < l(S_q)$.

(d) Consider the $m - r_q$ overlapping arrows $A_{j_q}, A_{j_q\oplus 1}, \ldots, A_{i_q\ominus 1}$. Since P is strongly directed, we get $m - r_q \geq n$. The inequality is strict since at least one of the arrows is coloured. ∎

Note that, by Definition 2.6(a), the statement $r_q \geqslant c(F_{q\oplus 1})$ of Lemma 2.7(a) is always true. Thus, the next lemma follows straightforwardly from Lemma 2.1.

LEMMA 2.8. *For each $q \in \mathbb{Z}_\nu$, if $q(i) = q$ and $q(j) = q \oplus_\nu 1$, then*

$$x_i \sim x_j \quad \text{if and only if} \quad j \ominus_m i \equiv r_q.$$

The following proposition already describes the relative positions of the points of the strings S_q and $S_{q\oplus 1}$.

PROPOSITION 2.9. *For each $q \in \mathbb{Z}_\nu$, if $q(i) = q$ and $q(j) = q \oplus_\nu 1$, then*

$$x_j < x_i \quad \text{if and only if} \quad j \ominus_m i \lesseqgtr r_q.$$

Proof. In the case $c(F_{q\oplus 1}) \geq l(S_q) + l(S_{q\oplus 1})$ there is nothing to prove because, by Lemma 2.5, for all $i, j \in \mathbb{Z}_m$ such that $q(i) = q$ and $q(j) = q \oplus 1$, we have $x_i \not\sim x_j$. Hence, by Lemma 2.8, $j \ominus i \not\equiv r_q$.

Assume that $c(F_{q\oplus 1}) < l(S_q) + l(S_{q\oplus 1})$ and set $w = \min(i - p_q,\, j - p_{q\oplus 1})$. Note that $q(i-w) = q$, $q(j-w) = q \oplus 1$ and either $i - w = p_q$ or $j - w = p_{q\oplus 1}$. Since the arrows are not crossing, $x_j < x_i$ if and only if $x_{j-w} < x_{i-w}$. On the other hand, $j \ominus i \lesseqgtr r_q$ is equivalent to $(j-w) \ominus (i-w) \lesseqgtr r_q$. So it is enough to prove the proposition when either $i = p_q$ or $j = p_{q\oplus 1}$.

Consider first the case $i = p_q$. If $x_j < x_{p_q}$ then clearly $i_q = p_q$. Thus, by Definition 2.6(b.2) and Corollary 2.4, $j \lesseqgtr j_q$. Since $j_q = i_q \oplus r_q$, we have $j \lesseqgtr p_q \oplus r_q$. Conversely, $j \lesseqgtr p_q \oplus r_q$ implies $l(S_q) \leq r_q$ because $p_{q\oplus 1} \leq j \leq p_q \oplus r_q$. Hence, $p_q = i_q$ by Lemma 2.7(c) and thus, $p_q \oplus r_q = j_q$. Therefore, $x_j \leq x_{p_q \oplus r_q} < x_{p_q}$ by Corollary 2.4 and Definition 2.6(b.2).

Consider now the case $j = p_{q\oplus 1}$. If $x_{p_{q\oplus 1}} < x_i$ then clearly $i \geq i_q$. Since also $j_q \geq p_{q\oplus 1}$, we have $\operatorname{ind}(x_i) \equiv \operatorname{ind}(x_{i_q}) + i - i_q$ and $\operatorname{ind}(x_{j_q}) \equiv \operatorname{ind}(x_{p_{q\oplus 1}}) + j_q - p_{q\oplus 1}$ by Lemma 2.1. Since $x_i \sim x_{p_{q\oplus 1}}$ and $x_{i_q} \sim x_{j_q}$, we have $0 \lesseqgtr j_q - p_{q\oplus 1} + i - i_q$. Hence, $p_{q\oplus 1} \ominus i \lesseqgtr j_q \ominus i_q = r_q$ as required.

Conversely, if $p_{q\oplus 1} \ominus i \lesseqgtr r_q$, then $x_{p_{q\oplus 1}} \sim x_i$ by Lemma 2.8. If we assume that $x_i < x_{p_{q\oplus 1}}$, then the $p_{q\oplus 1} \ominus i$ arrows

$$A_i, A_{i+1}, \ldots, A_{p_{q\oplus 1}\ominus 1}$$

are overlapping. Hence, $p_{q\oplus 1} \ominus i \geq n$ by the Generalized Directed Rule. Let $k \in \mathbb{Z}_n$ be such that $j_q - p_{q\oplus 1} \geqslant k$. Then $p_{q\oplus 1} \ominus i > k$ and, thus, $q(i+k) = q$. Since the arrows are not crossing and $p_{q\oplus 1} + k \lesseqgtr j_q$, we get $x_{i+k} < x_{p_{q\oplus 1}+k} \leq x_{j_q} < x_{i_q}$. Hence $i_q > i+k \geq i \geq p_q$ and, by Lemma 2.7(b), $j_q = p_{q\oplus 1}$. Therefore $p_{q\oplus 1} \ominus i > p_{q\oplus 1} \ominus i_q = r_q$; a contradiction. ∎

We already know that to have points of S_q above points of $S_{q\oplus 1}$, it must happen that $c(F_{q\oplus 1}) < l(S_q)+l(S_{q\oplus 1})$. Now we get another condition on the lengths of these strings in order to have points of S_q below points of $S_{q\oplus 1}$.

PROPOSITION 2.10. *Let $q \in \mathbb{Z}_\nu$. There exist $i, j \in \mathbb{Z}_m$ such that $q(i) = q$, $q(j) = q \oplus_\nu 1$ and $x_i < x_j$ if and only if $r_q + n < l(S_q) + l(S_{q\oplus 1})$. For each $i, j \in \mathbb{Z}_m$ such that $q(i) = q$ and $q(j) = q \oplus_\nu 1$ we have*

$$x_i < x_j \quad \textit{if and only if} \quad r_q + n \lesseqgtr j \ominus_m i.$$

Proof. Assume that there are $i, j \in \mathbb{Z}_m$ with $q(i) = q$, $q(j) = q \oplus 1$ and $x_i < x_j$. Then, by Lemma 2.8 and Proposition 2.9, $r_q + n \lesseqgtr j \ominus i < l(S_q)+l(S_{q\oplus 1})$. Conversely, assume that $r_q+n < l(S_q)+l(S_{q\oplus 1})$ (in particular this gives $c(F_{q\oplus 1}) < l(S_q) + l(S_{q\oplus 1})$). We consider two cases.

If $r_q + n \geq l(S_q)$, then $q(p_q \oplus (r_q + n)) = q \oplus 1$ and, by Lemma 2.8, $x_{p_q} \sim x_{p_q\oplus(r_q+n)}$. When $i_q > p_q$, we have $x_{p_q} < x_{p_q\oplus(r_q+n)}$ by Definition 2.6(b.1). If $i_q = p_q$, then $p_q \oplus r_q = j_q$ by Definition 2.6(b.3) and hence, $x_{p_q} < x_{(p_q\oplus r_q)+n}$ by Definition 2.6(b.2).

If $r_q + n < l(S_q)$, then $i_q > p_q$ by Lemma 2.7(c) and $j_q = p_{q\oplus 1}$ by Lemma 2.7(b). Furthermore, $q(p_{q\oplus 1} \ominus (r_q + n)) = q$ and, by Lemma 2.8, $x_{p_{q\oplus 1}\ominus(r_q+n)} \sim x_{p_{q\oplus 1}}$. By Definition 2.6(b.3) we have $p_{q\oplus 1} \ominus r_q = i_q$. Hence, $x_{p_{q\oplus 1}\ominus(r_q+n)} < x_{p_{q\oplus 1}}$ by Definition 2.6(b.1).

The last statement of the proposition follows trivially from Proposition 2.9 and Lemma 2.8. ∎

In the next lemma we state some special features of the numbers r_0 and $r_{\nu-1}$. In particular, statement (a) says that these numbers are (almost) always defined according to Definition 2.6(b).

LEMMA 2.11. (a) $c(F_1) < l(S_0) + l(S_1)$ *and, if $\nu = 2$ or P is colour compatible, then also $c(F_0) < l(S_{\nu-1}) + l(S_0)$.*

(b) $j_0 = p_1$, $r_0 < l(S_0)$, $i_0 = p_1 - r_0$ *and* $x_{p_1} < x_{p_1 - r_0}$.

(c) $l(S_{\nu-1}) < r_{\nu-1} + n$.

Proof. (a) [2, Lemma 4.4] with $j = p_1 - 1$ shows that $l(S_0) > c(F_1)$. Therefore, $c(F_1) < l(S_0)+l(S_1)$. If $\nu = 2$ then $c(F_0) < n < m = l(S_1)+l(S_0)$. When P is colour compatible and $\nu = 3$, since all the coloured arrows are of the same colour (see [2, Definition 4.11]), from $l(S_0) > c(F_1)$ we get $c(F_0) = c(F_1) < l(S_{\nu-1}) + l(S_0)$.

(b) Since $x_0 = sm_0$, we have $i_0 > p_0 = 0$ by Definition 2.6(b.1). Therefore, by Lemma 2.7(b,c), $j_0 = p_1$ and $r_0 < l(S_0)$. Hence, $i_0 = p_1 - r_0$ and $x_{p_1} < x_{p_1 - r_0}$ by Definition 2.6(b).

(c) If $r_{\nu-1} + n \leq l(S_{\nu-1})$, then $q(m - r_{\nu-1} - n) = \nu - 1$ and, by Proposition 2.10, $x_{m-r_{\nu-1}-n} < x_0$, contrary to $x_0 = sm_0$. ∎

The relationships between the different numbers r_q are given in the next three propositions.

Proposition 2.12. *If $\nu = 2$, then $r_0 + r_1 = m - n$.*

Proof. By Lemma 2.11(a), Definition 2.6(b.1) and Proposition 2.10, it is clear that $p_1 = i_1$ if and only if $r_0 + n \leq l(S_0) = p_1$, and in that case, $x_{p_1 - r_0 - n} < x_{p_1}$. Therefore, if $r_0 + n \leq l(S_0)$, then also by Proposition 2.10 and Definition 2.6(b), we have $p_1 - r_0 - n = j_1 = p_1 \oplus r_1 = p_1 + r_1 - m$. That is, $m - n = r_0 + r_1$.

Assume now that $r_0 + n > l(S_0)$. Then $i_1 > p_1$ and, by Lemma 2.7(b) and Definition 2.6(b.3), we have $j_1 = p_0 = 0$ and $i_1 = j_1 \ominus r_1 = m - r_1$. On the other hand, since $q(r_0 + n) = 1$ by Lemma 2.7(d), we also have $x_0 < x_{r_0+n}$ by Proposition 2.10. Hence, $x_0 < x_{m-r_1} \leq x_{r_0+n}$ by Definition 2.6(b.1). Then, from $x_0 < x_{m-r_1}$, we get $r_0 + n \lesseqgtr m - r_1$ by Proposition 2.10. From $x_{m-r_1} \leq x_{r_0+n}$, we get $m - r_1 \lesseqgtr r_0 + n$ by Corollary 2.4. This ends the proof of the proposition. ∎

Proposition 2.13. *If $\nu = 3$ then $r_0 + r_1 + r_2 \lesseqgtr m - n$.*

Proof. Assume first that $c(F_{q\oplus1}) \geq l(S_q) + l(S_{q\oplus1})$ for some $q \in \mathbb{Z}_\nu$. Note that, by Lemma 2.11(a), $q \geq 1$. Then, by Definition 2.6, $r_q = c(F_{q\oplus1})$. Since $c(F_{q\oplus1}) > l(S_q)$, there is an $x \in P$ such that $x < x_{p_q}$. Indeed, if such an x does not exist, then by [2, Lemma 4.4] applied to the sequence

$$A_{p_q}, A_{p_q+1}, \ldots, A_{m-1}, A_0, \ldots, A_{p_q-1}$$

with $j = l(S_q) - 1$ we obtain a contradiction. Moreover, $x \notin S_q \cup S_{q\oplus1}$ by Corollary 2.4 and Lemma 2.5. Therefore, there is a point from S_{q-1} which is smaller than x_{p_q}. In particular, by Lemma 2.5, $c(F_q) < l(S_{q-1}) + l(S_q)$. Moreover, by Proposition 2.10, $l(S_{q-1}) + l(S_q) > r_{q-1} + n > r_{q-1} + c(F_{q\oplus1})$. Hence $p_{q\oplus1} \ominus (r_{q-1} + r_q) > p_{q-1}$. On the other hand, from $c(F_{q\oplus1}) > l(S_q)$ we obtain $p_q > p_{q\oplus1} \ominus c(F_{q\oplus1})$. So we have $p_{q-1} < p_{q\oplus1} \ominus (r_{q-1} + r_q) < p_q$, that is, $q(p_{q\oplus1} \ominus (r_{q-1} + r_q)) = q - 1$. Thus, by Lemmas 2.7(a) and 2.1, $x_{p_{q\oplus1}} \sim x_{p_{q\oplus1} \ominus (r_{q-1}+r_q)}$.

Let us show that $x_{p_{q\oplus1}} < x_{p_{q\oplus1} \ominus (r_{q-1}+r_q)}$. Otherwise, the $r_{q-1} + r_q$ arrows

$$A_{p_{q\oplus1} \ominus (r_{q-1}+r_q)}, \ldots, A_{p_{q\oplus1} \ominus 1}$$

would be overlapping. By Definition 2.6(b) we have $p_{q\oplus1} \ominus 1 \geq j_{q-1} \geq p_q > p_{q\oplus1} \ominus c(F_{q\oplus1})$ and $p_q > i_{q-1} = j_{q-1} - r_{q-1} > p_{q\oplus1} \ominus (c(F_{q\oplus1}) + r_{q-1})$. Hence

$A_{i_{q-1}}, \dots, A_{j_{q-1}-1}$ are r_{q-1} consecutive arrows in the above sequence. Since $x_{i_{q-1}} > x_{j_{q-1}}$, the $r_q = c(F_{q\oplus 1})$ arrows

$$A_{p_{q\oplus 1}\ominus(r_{q-1}+r_q)}, \dots, A_{i_{q-1}-1}, A_{j_{q-1}}, \dots, A_{p_{q\oplus 1}\ominus 1}$$

would be overlapping, in contradiction with the Generalized Directed Rule. Hence $x_{p_{q\oplus 1}} < x_{p_{q\oplus 1}\ominus(r_{q-1}+r_q)}$ and, by Proposition 2.10, $r_{q\oplus 1} + n \lesseqgtr m - (r_{q-1} + r_q)$. That is, $r_{q-1} + r_q + r_{q\oplus 1} \lesseqgtr m - n$.

Assume now that $c(F_{q\oplus 1}) < l(S_q) + l(S_{q\oplus 1})$ for each $q \in \mathbb{Z}_\nu$. If $i_1 = p_1$ then, by Lemma 2.7, Definition 2.6(b.3) and Lemma 2.11(b), we have $x_{p_1+r_1} < x_{p_1} < x_{p_1-r_0}$ with $q(p_1 + r_1) = 2$ and $q(p_1 - r_0) = 0$. Hence, by Proposition 2.10, $r_2 + n \lesseqgtr m - r_0 - r_1$. That is, $r_0 + r_1 + r_2 \lesseqgtr m - n$.

If $i_1 > p_1$ then $j_1 = p_2$, and we have to consider two more cases. If $i_2 = p_2$ then $x_{p_2+r_2-m} < x_{p_2} < x_{p_2-r_1}$, with $q(p_2 + r_2 - m) = 0$ and $q(p_2 - r_1) = 1$, by Lemma 2.7 and Definition 2.6(b). Hence, by Proposition 2.10, $r_0 + n \lesseqgtr m - r_1 - r_2$. That is, $r_0 + r_1 + r_2 \lesseqgtr m - n$. If $i_2 > p_2$, we have $r_q < l(S_q)$ for all $q \in \mathbb{Z}_\nu$, by Lemmas 2.7(c) and 2.11(b). Adding up these three inequalities we get $r_0 + r_1 + r_2 < m$. On the other hand, by [2, Lemma 2.6] and Lemma 2.7(a) we have $r_0 + r_1 + r_2 \equiv m$. Thus, $r_0 + r_1 + r_2 \lesseqgtr m - n$. ∎

Whereas the above proposition gives an upper bound on the sum of all the r_q, the following one gives a lower bound for the same sum, valid only when there are points of each string below points of the next string.

PROPOSITION 2.14. *If $\nu = 3$ and $r_q + n < l(S_q) + l(S_{q\oplus 1})$ for all $q \in \mathbb{Z}_\nu$, then $r_0 + r_1 + r_2 \gtreqless m - 2n$.*

Proof. Assume first that $r_1 + n \le l(S_1)$. Then $q(p_2 - r_1 - n) = 1$. By Lemma 2.11(c), $m - p_2 = l(S_2) < r_2 + n < l(S_2) + l(S_0) = m - p_2 + p_1$ and, hence, $q(p_2 - m + r_2 + n) = 0$. Thus, $x_{p_2-r_1-n} < x_{p_2} < x_{p_2-m+r_2+n}$ by Proposition 2.10. Therefore, $m - r_1 - r_2 - 2n \lesseqgtr r_0$ by Proposition 2.9. That is, $r_0 + r_1 + r_2 \gtreqless m - 2n$.

When $r_1 + n > l(S_1)$, we consider two cases. If $r_0 + n \le l(S_0)$, then $x_{p_1-r_0-n} < x_{p_1} < x_{p_1+r_1+n}$ by Proposition 2.10, with $q(p_1 - r_0 - n) = 0$ and $q(p_1 + r_1 + n) = 2$. Hence, $m - r_0 - r_1 - 2n \lesseqgtr r_2$ by Proposition 2.9. That is, $r_0 + r_1 + r_2 \gtreqless m - 2n$. If $r_0 + n > l(S_0)$, since by Lemma 2.11(c) we have $r_2 + n > l(S_2)$, by adding up we get $(r_0 + n) + (r_1 + n) + (r_2 + n) > m$. That is, $r_0 + r_1 + r_2 > m - 3n$. Since $r_0 + r_1 + r_2 \equiv m$ by [2, Lemma 2.6] and Lemma 2.7(a), we have $r_0 + r_1 + r_2 \gtreqless m - 2n$. ∎

In view of Proposition 2.9 we can say that, for every $x_i \in S_q$, r_q is the maximum number of steps needed to go from x_i to any point $x_j \in S_{q\oplus 1}$ which is below x_i (if there is such a point). Actually, Definition 2.6(b) guarantees that if there are points of both strings in the same branch, we can find $x \in S_q$ and $y \in S_{q\oplus 1}$ such that $y < x$ and $f^{r_q}(x) = y$. Analogously, Proposition 2.10 allows us to give a similar meaning to the number $m - (r_q + n)$

when we interchange the roles of these strings. Moreover, in the proof of that proposition we see that if $r_q + n < l(S_q) + l(S_{q\oplus 1})$, then there are points $x \in S_{q\oplus 1}$ and $y \in S_q$ such that $y < x$ and $f^{m-(r_q+n)}(x) = y$. To simplify the use of the numbers r_q and $m - (r_q + n)$ we define the functions χ and d as follows.

First consider the $\nu \times \nu$ lower triangular matrix $\chi = (\chi_{qk})$ with entries

$$\chi_{qk} = \chi(q,k) = \begin{cases} 0 & \text{if } q \le k, \\ 1 & \text{if } q > k. \end{cases}$$

Let $i, j \in \mathbb{Z}_m$ have $q(j) = q(i) \oplus_\nu 1$. Then, by Proposition 2.9,

$$x_j < x_i \quad \text{if and only if} \quad j - i + \chi(q(i), q(j)) \cdot m \lesseqgtr r_{q(i)},$$

and, by Proposition 2.10,

$$x_i < x_j \quad \text{if and only if} \quad r_{q(i)} + n \lesseqgtr j - i + \chi(q(i), q(j)) \cdot m.$$

The following definition will help us in dealing with the above two conditions.

Definition 2.15. For $q, k \in \mathbb{Z}_\nu$, we set

$$d(q,k) = \begin{cases} 0 & \text{if } k = q, \\ \chi(q,k) \cdot m - r_q & \text{if } k = q \oplus 1 \text{ and } c(F_k) < l(S_q) + l(S_k), \\ r_k + n - \chi(k,q) \cdot m & \text{if } k = q \ominus 1 \text{ and } r_k + n < l(S_k) + l(S_q). \end{cases}$$

Remark 2.16. Observe that, for $k \neq q$, we do not define $d(q,k)$ unless there are points of S_k below points of S_q. Hence, from now on, when we write $d(q,k)$ we assume that it is defined. From Lemma 2.11(a) we find that $d(0,1) = -r_0$ is always defined.

When $\nu = 2$, again by Lemma 2.11(a) we see that $d(1,0) = m - r_1$ is also defined. Moreover, since $l(S_0) + l(S_1) = m$, by Lemma 2.7(d) both $d(0,1)$ and $d(1,0)$ are defined in another way: $d(0,1) = r_1 + n - m$ and $d(1,0) = r_0 + n$. Clearly, by Proposition 2.12, both definitions coincide. ■

The following result summarizes all previous ones about the relative positions of the points of P.

Proposition 2.17. *For each $i, j \in \mathbb{Z}_m$, $x_i \ge x_j$ if and only if*

$$i \gtreqless j + d(q(i), q(j)).$$

Proof. The statement follows directly from Corollary 2.4, Definition 2.15 and the comments preceding it; we should have in mind that, since $\nu \le 3$, any two different strings are always consecutive. ■

3. A sufficient condition for primarity. We continue studying spiral orbits P of EP-adjusted maps $f \in \mathcal{X}_4$ with $\nu \in \{2, 3\}$ coloured arrows. At the end of this section we shall get a sufficient condition for such orbits to be primary. To do this we must obtain some properties of the length of loops in the EP-graph of the map.

We will use the notation from the previous sections. We take the EP-basic intervals labelled by their largest endpoint. That is, for each $i \in \mathbb{Z}_m$, we set $I_i = [a, x_i]$ with $a \in EP$, $x_i \in P$, $a < x_i$ and $(a, x_i) \cap P = \emptyset$.

Since the arrows are not crossing, $I_i \to I_{i\oplus 1}$ for each $i \in \mathbb{Z}_m$. That is, in the EP-graph of f we can always find the *fundamental* loop $I_0 \to I_1 \to \ldots \to I_{m-1} \to I_0$, which has length m and is associated to P.

Since f is EP-adjusted, if $I_i = [a, x_i] \to I_j = [b, x_j]$ there is a unique $x_s \in \{a, x_i\} \cap P$ such that $x_j \le x_{s\oplus 1}$. In particular, if $s = m-1$ then $j = 0$. We now get our first inequality for a single step in the EP-graph of f.

LEMMA 3.1. *If* $I_i \to I_j$ *then*

$$i + 1 \geqq j + d(q(i), q(s)) + \chi(q(s), q(s \oplus_m 1)) \cdot m + d(q(s \oplus_m 1), q(j)).$$

Proof. Since $x_s \le x_i$ and $x_j \le x_{s\oplus 1}$, by Proposition 2.17,

$$i \geqq s + d(q(i), q(s)) \quad \text{and} \quad s \oplus 1 \geqq j + d(q(s \oplus 1), q(j)).$$

Since $s \oplus 1 = s + 1 - \chi(q(s), q(s \oplus 1)) \cdot m$ the lemma is proved. ∎

Now, for any path $I_{i_0} \to I_{i_1} \to \ldots \to I_{i_{l-1}} \to I_{i_l}$ in the EP-graph of f, we denote by x_{s_k} the endpoint of I_{i_k} such that $x_{i_{k+1}} \le x_{s_k \oplus 1}$ $(k = 0, 1, \ldots, l-1)$. From the above lemma (applied to each step of the path) we obtain the fundamental inequality for loops. We restrict ourselves to loops of length m because these are the only loops relevant to the primarity of P. Certainly we could easily do a more general work (including any path), but here it seems unnecessary.

LEMMA 3.2. *If* $I_{i_0} \to I_{i_1} \to \ldots \to I_{i_{m-1}} \to I_{i_0}$ *is a loop of length* m *in the* EP*-graph of* f, *then*

$$m \geqq \sum_{k=0}^{m-1} (d(q(i_k), q(s_k)) + \chi(q(s_k), q(s_k \oplus 1)) \cdot m + d(q(s_k \oplus 1), q(i_{k\oplus 1}))). \tag{1}$$

Proof. By Lemma 3.1, we have

$$i_k + 1 \geqq i_{k\oplus 1} + d(q(i_k), q(s_k)) + \chi(q(s_k), q(s_k \oplus 1)) \cdot m + d(q(s_k \oplus 1), q(i_{k\oplus 1}))$$

for each $k \in \mathbb{Z}_m$. Adding up these inequalities proves the lemma. ∎

Formula (1) from the above lemma is basic in our study. However, it is cumbersome to work with because of its complicated right hand side. To conveniently handle this expression, we have chosen to make use of a special combinatorial graph with the strings of P as vertices. To do this we extend the notion of graph we have been using until now.

Let V be a finite set whose elements will be called *vertices* and let S be a finite set whose elements will be called *labels*. Then $\Gamma = (V, S, U)$ is a *generalized oriented labelled graph* (*gol* graph, for short) if $U \subset V \times V \times S$. The elements of U are the (*labelled*) *arrows* of the graph. Usually, an element $(u, v, e) \in U$ will be graphically represented by $u \xrightarrow{e} v$.

Now we define the graph which will help us to handle the expression on the right hand side of (1).

Definition 3.3. The *qP-graph* of f is the gol graph $(\mathbb{Z}_\nu, \{d, \chi\}, U)$ such that

(i) $(i, j, d) \in U$ if and only if $d(i, j)$ is defined.
(ii) $(i, j, \chi) \in U$ if and only if $j = i$ or $j = i \oplus_\nu 1$.

The loop $0 \xrightarrow{\chi} 1 \xrightarrow{\chi} \dots \xrightarrow{\chi} \nu - 1 \xrightarrow{\chi} 0$ in the qP-graph of f will be called *fundamental*. Any loop of a gol graph of the form $i \xrightarrow{e} i$ with $i \in V$ and $e \in S$ will be called *trivial*.

Definition 3.4. To each arrow of the qP-graph of f there is assigned a *weight* in the following way: $\psi(i, j, d) = d(i, j)$ and $\psi(i, j, \chi) = \chi_{ij} \cdot m$.

For each path $\lambda = i_0 \xrightarrow{e_0} i_1 \xrightarrow{e_1} \dots \xrightarrow{e_{l-2}} i_{l-1} \xrightarrow{e_{l-1}} i_l$ of length l in the qP-graph of f we define its *weight* $\Psi(\lambda)$ as the sum of the weights of all steps. That is,

$$\Psi(\lambda) = \sum_{k=0}^{l-1} \psi(i_k, i_{k+1}, e_k).$$

Remark 3.5. By the definitions we have $\psi(i, j, \chi) = 0$ except if $i = \nu - 1$ and $j = 0$. In this case $\psi(\nu - 1, 0, \chi) = m$. Therefore, the fundamental loop in the qP-graph of f has weight m. Moreover, by straightforward computations we can obtain Table I, which gives the weights of loops in some qP-graphs of f.

Now, we are going to study the relation between the EP-graph and the qP-graph of f.

Definition 3.6. Each arrow $I_i \to I_j$ in the EP-graph of f *generates* the following path $\varphi(i, j)$ of three arrows in the qP-graph of f:

$$q(i) \xrightarrow{d} q(s) \xrightarrow{\chi} q(s \oplus_m 1) \xrightarrow{d} q(j).$$

Then each path $\omega = I_{i_0} \to I_{i_1} \to \dots \to I_{i_{l-1}} \to I_{i_l}$ of length l in the EP-graph of f *generates* a path

$$\Phi(\omega) = \varphi(i_0, i_1)\varphi(i_1, i_2) \dots \varphi(i_{l-1}, i_l)$$

of length $3l$ in the qP-graph of f, which is the concatenation of the l paths of three arrows generated by the arrows of ω.

Table I. The weights of the elementary loops of the qP-graphs of f

Case $\nu = 2$*:*		
	$\lambda_0 = 0 \xrightarrow{\chi} 1 \xrightarrow{\chi} 0$	$\Psi(\lambda_0) = m$ (fundamental loop)
	$\lambda_1 = 0 \xrightarrow{\chi} 1 \xrightarrow{d} 0$	$\Psi(\lambda_1) = r_0 + n$
	$\lambda_2 = 0 \xrightarrow{d} 1 \xrightarrow{\chi} 0$	$\Psi(\lambda_2) = m - r_0$
	$\lambda_3 = 0 \xrightarrow{d} 1 \xrightarrow{d} 0$	$\Psi(\lambda_3) = n$
Case $\nu = 3$*:*		
	$\lambda_0 = 0 \xrightarrow{\chi} 1 \xrightarrow{\chi} 2 \xrightarrow{\chi} 0$	$\Psi(\lambda_0) = m$ (fundamental loop)
	$\lambda_1 = 0 \xrightarrow{\chi} 1 \xrightarrow{\chi} 2 \xrightarrow{d} 0$	$\Psi(\lambda_1) = m - r_2$
	$\lambda_2 = 0 \xrightarrow{\chi} 1 \xrightarrow{d} 2 \xrightarrow{\chi} 0$	$\Psi(\lambda_2) = m - r_1$
	$\lambda_3 = 0 \xrightarrow{\chi} 1 \xrightarrow{d} 2 \xrightarrow{d} 0$	$\Psi(\lambda_3) = m - r_1 - r_2$
	$\lambda_4 = 0 \xrightarrow{d} 1 \xrightarrow{\chi} 2 \xrightarrow{\chi} 0$	$\Psi(\lambda_4) = m - r_0$
	$\lambda_5 = 0 \xrightarrow{d} 1 \xrightarrow{\chi} 2 \xrightarrow{d} 0$	$\Psi(\lambda_5) = m - r_0 - r_2$
	$\lambda_6 = 0 \xrightarrow{d} 1 \xrightarrow{d} 2 \xrightarrow{\chi} 0$	$\Psi(\lambda_6) = m - r_0 - r_1$
	$\lambda_7 = 0 \xrightarrow{d} 1 \xrightarrow{d} 2 \xrightarrow{d} 0$	$\Psi(\lambda_7) = m - r_0 - r_1 - r_2$
	$\lambda_8 = 0 \xrightarrow{d} 2 \xrightarrow{d} 1 \xrightarrow{d} 0$	$\Psi(\lambda_8) = r_0 + r_1 + r_2 + 3n - m$
	$\lambda_9 = 0 \xrightarrow{\chi} 1 \xrightarrow{d} 0$	$\Psi(\lambda_9) = r_0 + n$
	$\lambda_{10} = 0 \xrightarrow{d} 1 \xrightarrow{d} 0$	$\Psi(\lambda_{10}) = n$
	$\lambda_{11} = 1 \xrightarrow{\chi} 2 \xrightarrow{d} 1$	$\Psi(\lambda_{11}) = r_1 + n$
	$\lambda_{12} = 1 \xrightarrow{d} 2 \xrightarrow{d} 1$	$\Psi(\lambda_{12}) = n$
	$\lambda_{13} = 0 \xrightarrow{d} 2 \xrightarrow{\chi} 0$	$\Psi(\lambda_{13}) = r_2 + n$
	$\lambda_{14} = 0 \xrightarrow{d} 2 \xrightarrow{d} 0$	$\Psi(\lambda_{14}) = n$

With these definitions we can restate Lemmas 3.1 and 3.2 as follows.

LEMMA 3.7. *If* $I_i \to I_j$*, then* $i + 1 \geqslant j + \Psi(\varphi(i,j))$*.*

LEMMA 3.8. *If* ω *is a loop of length* m *of the* EP*-graph of* f*, then* $m \geqslant \Psi(\Phi(\omega))$*.*

Recall that in any graph (in gol graphs too) an *elementary* loop is a loop that has no repeated vertices, and that each loop can be obtained as a concatenation of elementary loops. So we will focus our attention on the elementary loops of the qP-graph of f.

REMARK 3.9. The loops shown in Table I are all the nontrivial elementary loops in all possible qP-graphs of f.

LEMMA 3.10. *If* λ *is an elementary loop of the* qP*-graph of* f*, then* $\Psi(\lambda) \geq 0$*. Moreover* $\Psi(\lambda) = 0$ *if and only if* λ *is trivial.*

Proof. If λ is trivial it is clear that $\Psi(\lambda) = 0$ because for each $i \in \mathbb{Z}_\nu$ we have $\chi(i,i) = 0$ and $d(i,i) = 0$. Now we show that if λ is not trivial then $\Psi(\lambda) > 0$.

In the case $\nu = 2$, by Remark 3.9 it is enough to look at the list of loops and weights given in Table I, taking into account Proposition 2.12.

Also, in the case $\nu = 3$ the loops λ_i ($i = 0, 1, \ldots, 14$) listed in Table I are all the elementary nontrivial loops that a qP-graph of f can have. In view of their weights it is clear that $\Psi(\lambda_i) > 0$ for $i = 9, 10, \ldots, 14$ and $i = 0$. Proposition 2.13 assures the same for $i = 1, 2, \ldots, 7$ and Proposition 2.14 shows that $\Psi(\lambda_8) > 0$. ∎

We are, at last, ready to give the promised sufficient condition for the primarity of P. Let Λ be the set of all elementary nontrivial loops different from the fundamental one of the qP-graph of f. For a spiral orbit P of period m we consider the following condition:

Primarity Condition.

$$m \not\geqslant \sum_{\lambda \in \Lambda} \beta_\lambda \Psi(\lambda) \qquad \textit{with } \beta_\lambda \in \mathbb{Z}_n.$$

The name given to the above condition is justified by the following theorem.

Theorem 3.11. *If P is a spiral orbit of an EP-adjusted map $f \in \mathcal{X}_4$ of period m with $\nu \in \{2,3\}$ coloured arrows and satisfies the Primarity Condition, then P is primary.*

Proof. Let $\omega = I_{i_0} \to I_{i_1} \to \ldots \to I_{i_{m-1}} \to I_{i_0}$ be a loop of length m in the EP-graph of f. We are going to show that ω must be the fundamental loop. Since f is EP-adjusted and P is associated to the fundamental loop, by [1, Proposition 1.10] and the First Theorem (Theorem 2.3 of [1]), this is enough to prove that P is primary.

We write $\Phi(\omega)$ as a concatenation of elementary loops. Let n_0 be the number of times that the fundamental loop of the qP-graph of f is repeated in this expression. For each $\lambda \in \Lambda$, let n_λ be the number of times that the loop λ is repeated in this expression. Then, by Lemmas 3.8 and 3.10, we have

$$m \geqslant \Psi(\Phi(\omega)) = n_0 m + \sum_{\lambda \in \Lambda} n_\lambda \Psi(\lambda).$$

Since, for each $\lambda \in \Lambda$ we can set $n_\lambda = n'_\lambda n + b_\lambda$ with $n'_\lambda \geq 0$ and $b_\lambda \in \mathbb{Z}_n$, we get $n_\lambda \Psi(\lambda) = n'_\lambda \Psi(\lambda) \cdot n + b_\lambda \Psi(\lambda) \geqslant b_\lambda \Psi(\lambda)$ by Lemma 3.10. Therefore,

$$m \geqslant n_0 m + \sum_{\lambda \in \Lambda} b_\lambda \Psi(\lambda)$$

with $b_\lambda \in \mathbb{Z}_n$. Since $\Psi(\lambda) > 0$, we have $n_0 \le 1$. On the other hand, the hypothesis implies $n_0 > 0$. Hence $n_0 = 1$ and, therefore, $n_\lambda = 0$ for all $\lambda \in \Lambda$.

This means that in $\Phi(\omega)$ there are no arrows of the form (i, j, d) with $i \neq j$, and that there appears one and only one of the form $(\nu - 1, 0, \chi)$ (of course, there will also be many trivial loops). That is, from Definition 3.6 it follows that for each arrow $I_{i_k} \to I_{i_{k\oplus 1}}$ of the loop ω we have $q(i_k) = q(s_k)$ and $q(s_k \oplus 1) = q(i_{k\oplus 1})$. Also, there is one and only one $k' \in \mathbb{Z}_m$ for which $s_{k'} = m - 1$. Hence, by Lemma 3.1 (or 3.7), for each $k \in \mathbb{Z}_m$,

$$i_{k\oplus 1} \le i_k + 1 \quad \text{if} \quad k \neq k' \text{ and } i_{k'\oplus 1} \le i_{k'} + 1 - m.$$

Neither of these inequalities can be strict, for otherwise, adding them up we would get $0 < 0$. Hence, $i_{k\oplus 1} = i_k + 1$ if $k \neq k'$ and $0 \le i_{k'\oplus 1} = i_{k'} + 1 - m$. This last relationship implies $i_{k'} = m - 1$ and $i_{k'\oplus 1} = 0$. From this, by using the other equalities in an ordered way, we deduce that $\{i_0, i_1, \ldots, i_{m-1}\}$ is a cyclic permutation of $\{0, 1, \ldots, m-1\}$ (namely, $i_k = k \oplus (m - 1 - k')$ for each $k \in \mathbb{Z}_m$) and, hence, ω is the fundamental loop. ■

In the next sections we shall see that the Primarity Condition is also necessary for the primarity of P. In fact, in each case ($\nu = 2$ or 3) we obtain necessary conditions for primarity which are equivalent to (but easier to handle than) the Primarity Condition.

4. Double orbits. In this section we characterize the primary spiral orbits with exactly two coloured arrows. To this end we keep the notation of the previous sections. In particular we assume that P is a spiral orbit of period m of an EP-adjusted map f with $\nu = 2$ coloured arrows. Remember that, in this case, both $d(0, 1)$ and $d(1, 0)$ are defined (see Remark 2.16).

We shall show that primary spiral orbits with two coloured arrows are precisely those defined as follows.

DEFINITION 4.1. A spiral orbit with two coloured arrows is called *double* if

$$(2) \qquad m \not\geqq \beta_q(m - r_q) \quad \text{for any } q \in \mathbb{Z}_2 \text{ and } \beta_q \in \mathbb{Z}_n.$$

The fact that double orbits are primary follows from the following result and Theorem 3.11.

LEMMA 4.2. *Let P be a spiral orbit with $\nu = 2$. Then the Primarity Condition is equivalent to* (2).

Proof. For $\nu = 2$, by Remark 3.9, the qP-graph of f has three elementary nontrivial loops different from the fundamental one, whose weights are n, $m - r_0$ and, by Proposition 2.12, $r_0 + n = m - r_1$. Then the Primarity

Condition is written as follows:

$$\text{for any } \beta, \beta_0, \beta_1 \in \mathbb{Z}_n, \qquad m \ngeqq \beta n + \beta_0(m - r_0) + \beta_1(m - r_1).$$

Setting $\beta = \beta_{q\oplus 1} = 0$, we get (2).

Conversely, if (2) holds, we have

$$m \ngeqq \beta n + \beta_0(m - r_0) + \beta_1(m - r_1)$$

when $\beta_0 = 0$ or $\beta_1 = 0$. If $\beta_0\beta_1 > 0$, by Proposition 2.12, we have

$$\beta_0(m - r_0) + \beta_1(m - r_1) \geq m - r_0 + m - r_1 = m + n > m.$$

Hence the Primarity Condition is also satisfied. ∎

To complete the characterization of the primary spiral orbits with two coloured arrows, we still need some more technical results, which will be obtained in the following lemmas.

Since, by Proposition 2.12, r_0 and r_1 determine each other, from now on we simplify the notation by setting $r = r_0$. Then $r_1 = m - (n + r)$. Remember also that $j_0 = p_1 = l(S_0) > r$, $i_0 = p_1 - r$ and $x_{p_1} < x_{p_1 - r}$ by Lemma 2.11(b).

In what follows we will use the labelling of EP-basic intervals introduced in the previous section.

Lemma 4.3. *$I_{p_1-r-1} \to I_{p_1}$.*

Proof. Since $i_0 = p_1 - r$, we have $I_{p_1-r-1} = [a, x_{p_1-r-1}]$ with $a = 0$ or $a \in S_0$ by Definition 2.6(b.1). Hence, $x_{p_1-r} > x_{p_1} > f(a)$. ∎

Lemma 4.4. *If $m - r < p_1$, then $I_{m-r-1} \to I_0$.*

Proof. Since $m - r < p_1$, we have $q(m - 1 - r) = 0$. By Corollary 2.4, if $x_{m-1} < x_i$ then $q(i) = 0$ and, by Proposition 2.9, $i \geqq m - r - 1$. Hence, $I_{m-r-1} = [x_{m-1}, x_{m-1-r}]$. Since (x_{m-r-1}, x_{m-r}) is black and (x_{m-1}, x_0) is coloured, the lemma is proved. ∎

Lemma 4.5. *Assume that $n + r \leq p_1$. If $I_{p_1-1} = [a, x_{p_1-1}]$, then $I_{p_1-1} \to I_{p_1-n-r}$, except when simultaneously $a = x_{m-1}$ and $p_1 = n + r$.*

Proof. Since $n + r \leq p_1$, we have $q(p_1 - n - r) = 0$ and, by Proposition 2.10, $x_{p_1} > x_{p_1-n-r}$. Furthermore, the arrow A_{p_1-1} is coloured. Therefore, $I_{p_1-1} \to I_{p_1-n-r}$ except if the arrow $(a, f(a))$ has the same colour as A_{p_1-1} and $f(a) \geq x_{p_1-n-r}$. In this case, $(a, f(a)) = F_0$; that is, $a = x_{m-1}$ and $f(a) = x_0 = x_{p_1-n-r}$. Hence, $p_1 = n + r$. ∎

Lemma 4.6. *Assume that $n \leq p_1$. If $I_{p_1-1} = [a, x_{p_1-1}]$ and $(a, f(a))$ is black, then $I_{p_1-1} \to I_{p_1-n}$.*

Proof. If $a \in S_0$, then $a = x_{p_1-1-n}$ by Corollary 2.4 and so $f(a) = x_{p_1-n}$. If $a \in S_1$, then $a = x_{p_1-1+r}$ by Proposition 2.9. Since $(a, f(a))$ is black,

$f(a) = x_{p_1+r} \in S_1$. Therefore, $f(a) > x_{p_1-n}$ by Proposition 2.10. Since the arrow A_{p_1-1} is coloured, the lemma is proved in both cases. ∎

LEMMA 4.7. *If $r_1 < p_1$, then $I_{m-1} = [x_{r_1-1}, x_{m-1}] \to I_{r_1}$. If, furthermore, $m - n \geq p_1$, then $I_{m-1} \to I_{m-n}$.*

Proof. By Lemma 2.7(a) we have $r_1 > 0$. Hence $q(r_1 - 1) = 0$. Therefore, by Proposition 2.10, $x_{r_1-1} < x_{m-1}$. Then, again by Proposition 2.10 and Corollary 2.4, we infer that $(x_{r_1-1}, x_{m-1}) \cap P = \emptyset$. Hence, $I_{m-1} = [x_{r_1-1}, x_{m-1}]$. Since (x_{r_1-1}, x_{r_1}) is black and (x_{m-1}, x_0) is coloured, $I_{m-1} \to I_{r_1}$.

If $m - n \geq p_1$ then, by Proposition 2.9, we have $x_{m-n} < x_{r_1}$. Hence, $I_{m-1} \to I_{m-n}$. ∎

LEMMA 4.8. *Assume that $n + r \leq p_1$. If $r_1 \geq p_1$, then $I_{p_1-1+n+r} = [x_{p_1-1}, x_{p_1-1+n+r}]$ and $I_{p_1-1+n+r} \to I_{p_1-(n+r)}$.*

Proof. Since $m - n - r = r_1 \geq p_1$ we get $q(p_1 + n + r - 1) = 1$. Then, by Proposition 2.10, $x_{p_1-1} < x_{p_1-1+n+r}$ and, as above, we obtain $(x_{p_1-1}, x_{p_1-1+n+r}) \cap P = \emptyset$. Hence, $I_{p_1-1+n+r} = [x_{p_1-1}, x_{p_1-1+n+r}]$. Moreover, $A_{p_1-1+n+r}$ must have a different colour than A_{p_1-1} (which is coloured) because, otherwise, $e(A_{p_1-1+n+r}) = x_0$ and then these two arrows would be crossing. By Proposition 2.10 we have $x_{p_1} > x_{p_1-(n+r)}$. Hence, $I_{p_1-1+n+r} \to I_{p_1-(n+r)}$. ∎

Now we can state the main result of this section.

THEOREM 4.9. *A spiral orbit with two coloured arrows is primary if and only if it is double.*

Proof. Let P be a spiral orbit of a map $f \in \mathcal{X}_4$ with two coloured arrows. By the First Theorem (Theorem 2.3 of [1]) we may assume that f is EP-adjusted. As mentioned before, the fact that (2) is sufficient for P to be primary follows immediately from Lemma 4.2 and Theorem 3.11. Thus, double orbits are primary.

Now we will show that if P is primary then it satisfies (2) and so it is double. Assume then that P is primary.

By Lemma 2.7(a) and Proposition 2.12 we have $c(F_{q\oplus 1}) \equiv r_q \equiv m - r_{q\oplus 1}$ for each $q \in \mathbb{Z}_2$. Then, by [2, Proposition 4.7], we already know that $m \not\equiv \beta_q(m - r_q)$ for each $q \in \mathbb{Z}_2$ and $\beta_q \in \{0, 1\}$.

Now we shall prove that if for some $q \in \mathbb{Z}_2$ there is some $\beta_q \in \mathbb{Z}_n \setminus \{0, 1\}$ such that $m \gtreqqless \beta_q(m - r_q)$, then we can find a nonrepetitive loop of length m in the EP-graph of f, different from the fundamental one and going through some nonbranching interval. This will end the proof of the theorem because, by [3, Lemma 2.2] and [1, Proposition 1.11], there exists a periodic orbit of period m different from P. Hence, by the First Theorem (Theorem 2.3

of [1]), P is not primary. We shall build the desired loop by concatenation of suitable elementary loops. We consider two cases:

CASE (a): *$m \gtreqqless \beta_0(m - r_0)$ for some $\beta_0 \in \mathbb{Z}_n \setminus \{0, 1\}$.* By Lemma 4.3, we may consider the elementary loop

$$\lambda = I_0 \to \ldots \to I_{p_1-r-1} \to I_{p_1} \to \ldots \to I_{m-1} \to I_0$$

of length $m-r$. This loop can be concatenated with the branching loop, α. If $m > \beta_0(m-r)$, since $m \equiv \beta_0(m-r)$, we have $m = \beta_0(m-r) + ln$ with $l \geq 1$. Hence, the loop $\lambda^{\beta_0}\alpha^l$ has length m and is nonrepetitive. Moreover, by Lemma 2.7(d), $m - r > n$, hence λ goes through some basic interval not containing 0. Thus we are done. Assume now that $m = \beta_0(m-r)$. Since $\beta_0 > 1$, it follows that $r = (\beta_0 - 1)(m-r)$. Hence, since $p_1 > r$, we have $p_1 > m - r$. Therefore, by Lemma 4.4, we also have the elementary loop

$$\gamma = I_0 \to I_1 \to \ldots \to I_{m-r-1} \to I_0$$

of length $m-r$. Thus, the loop $\lambda^{(\beta_0-1)}\gamma$ is the one we are looking for.

CASE (b): *$m \gtreqqless \beta_1(m - r_1)$ for some $\beta_1 \in \mathbb{Z}_n \setminus \{0, 1\}$.* If x_{p_1} is the smallest point in its branch we can change the labelling of the points of the orbit in such a way that x_{p_1} is relabelled as x_0 and we are done by Case (a). Therefore, we may assume that $(0, x_{p_1}) \cap P \neq \emptyset$. By Corollary 2.4, $(0, x_{p_1}) \cap P \subset S_0$. Hence, $i_1 = p_1$ and, by Lemma 2.7(c), we see that $m - n - r = r_1 \geq l(S_1) = m - p_1$. Therefore, $p_1 \geq n + r$. Now we consider two subcases.

SUBCASE (b.i): *$I_{p_1-1} = [a, x_{p_1-1}]$ with $a \neq x_{m-1}$.* The interval I_{p_1-1} is not branching, because x_{p_1-1} is the beginning of a coloured arrow. Hence, a is the beginning of a black arrow. By Lemmas 4.5 and 4.6 we have the elementary loops

$$\lambda_p = I_{p_1-1} \to I_{p_1-(n+r)} \to \ldots \to I_{p_1-1}$$

and

$$\mu_p = I_{p_1-1} \to I_{p_1-n} \to \ldots \to I_{p_1-1}$$

of lengths $n + r = m - r_1$ and n, respectively. If $m > \beta_1(m - r_1)$, since $m \equiv \beta_1(m - r_1)$, there exists $l > 0$ such that the loop $\lambda_p^{\beta_1}\mu_p^l$ has length m and, of course, is nonrepetitive. Since this loop obviously goes through I_{p_1-1} which is not branching, we are done.

Assume now that $m = \beta_1(m - r_1)$. We still have two possibilities. If $r_1 < p_1$, then Lemma 4.7 gives the elementary loop

$$\lambda_m = I_{m-1} \to I_{r_1} \to \ldots \to I_{m-1}$$

of length $m - r_1 = n + r$. This loop is different from λ_p and can be concatenated with it, because $r_1 \leq p_1 - 1 < m - 1$. So, we are done by considering, for instance, the loop $\lambda_p^{\beta_1-1}\lambda_m$.

If $r_1 \geq p_1$, then Lemma 4.8 gives the elementary loop

$$\eta = I_{p_1-1+n+r} \to I_{p_1-(n+r)} \to \ldots \to I_{p_1-1+n+r}$$

of length $2(n+r)$. Since $p_1-(n+r) < p_1-1 < p_1-1+n+r$, this loop can be concatenated with λ_p. Moreover, we have $m \geq p_1+n+r \geq 2(n+r)$. If we assume that $m = 2(n+r)$, we get $m = p_1+n+r$ and $p_1 = n+r$ and then, by Proposition 2.10, we infer that $x_{p_1-1} < x_{m-1}$ and $x_0 < x_{p_1}$. This contradicts the fact that the arrows are not crossing. Thus, we have $2(n+r) < m = \beta_1(n+r)$. Therefore, $\beta_1 = 3$ and the loop $\eta\lambda_p$ is the one we are looking for.

SUBCASE (b.ii): $I_{p_1-1} = [x_{m-1}, x_{p_1-1}]$. By Proposition 2.9, $m-1-r \leqq p_1-1$. Hence $q(m-1-r) = 0$ and, again by Proposition 2.9 and by Corollary 2.4, $x_{m-1} < x_{m-1-r} \leq x_{p_1-1}$. That is, $p_1 = m-r$. Hence, $r_1 < p_1$ and, by Lemma 4.7, we have the elementary loop λ_m of Subcase (b.i). Moreover, if $p_1 = n+r$, then $r_1 = m-(n+r) = m-p_1 = r$, and we are in Case (a). Therefore, we may assume that $p_1 > n+r$. Thus, by Lemma 4.5, we also have the elementary loop λ_p which, as in Subcase (b.i), is different from λ_m and can be concatenated with it. So, as above, we are done if $m = \beta_1(m-r_1)$.

When $m > \beta_1(m-r_1)$ we still need a suitable loop of length n. If $m-n \geq p_1$, by Lemma 4.7, we have the elementary loop $I_{m-1} \to I_{m-n} \to \ldots \to I_{m-1}$ of length n. If $m-n < p_1$, then $r_1 \leq p_1-r-1$. Therefore, since $p_1 \leq m-1$, by Lemmas 4.7 and 4.3, we have the elementary loop

$$I_{m-1} \to I_{r_1} \to \ldots \to I_{p_1-r-1} \to I_{p_1} \to \ldots \to I_{m-1}$$

of length n ($I_{p_1-r-1} \to I_{p_1}$ is a shortcut of r arrows in the loop λ_m of length $n+r$). These two loops can be concatenated with λ_m. Since λ_m also goes through a nonbranching interval (namely, I_{p_1-1}), we get the required loop as in Subcase (b.i). This ends the proof of the theorem. ∎

We recall that for maps from $\mathcal{Y}$, double orbits must have the coloured arrows of the same colour. In that case, we can define numbers r_0 and r_1 in a similar way to Definition 2.6. However, these two numbers coincide. Namely, $r_q - 1$ is the number n from [1, Definition 4.27]. We have an analogous result for double orbits of maps from $\mathcal{X}_4$ with coloured arrows of the same colour.

COROLLARY 4.10. *Let P be a spiral orbit with $\nu = 2$ and $c(F_0) = c(F_1)$. Then P is primary if and only if $r_0 = r_1 \not\equiv 2$. In that case, $r_0 = (m-n)/2$.*

Proof. By Theorem 4.9 it is enough to show that (2) is equivalent to $r_0 = r_1 \not\equiv 2$.

Set $c = c(F_0) = c(F_1)$. First of all, by Lemma 2.7(a), we have $r = r_0 \equiv c(F_1)$ and $r_1 \equiv c(F_0)$. Hence, $r \equiv c \equiv r_q$ for any $q \in \mathbb{Z}_2$. Thus, for each $q \in \mathbb{Z}_2$, $m \equiv 2r \equiv 2(r_q+n) = 2(m-r_{q\oplus 1})$ by Proposition 2.12.

Assume that (2) holds. Then $m < 2(r_q + n)$ for each $q \in \mathbb{Z}_2$. That is, $m \lesseqgtr 2r_q + n$. From this and Proposition 2.12 it follows that $r_{q\oplus 1} \leq r_q$ for each $q \in \mathbb{Z}_2$. Therefore, $r_0 = r_1$. Moreover, $r \not\equiv 2$ since (2) implies that $m \not\equiv 0$.

Assume now that $r_0 = r_1 = r \not\equiv 2$. Since $r \equiv c \neq 0$ and $r \not\equiv 2$, we see that $r \equiv c \in \{1, 3\}$. Hence $m \equiv 2r \equiv 2$. Then, since $m - r = r + n \equiv r$, clearly $m \not\equiv \beta(m-r)$ if $\beta \in \{0, 1, 3\}$. On the other hand, $m \equiv 2(m-r)$ but, by Proposition 2.12, $2(m-r) = 2r + 2n = m + n > m$. Thus, (2) is satisfied.

The last statement of the corollary follows immediately from Proposition 2.12. ∎

5. Triple orbits. In this section we study the last class of orbits we need to consider to end our characterization of strongly directed primary orbits. Namely, the spiral orbits P with three coloured arrows. We keep, of course, the notation of the previous sections and assume that f is EP-adjusted. We note that if we want P to be primary then it must be colour compatible by [2, Theorem B]. That is, all three coloured arrows must have the same colour $c \in \{1, 3\}$ (see [2, Definition 4.11]). We also know, by [2, Theorem 5.10], that m cannot be a multiple of 3. Moreover, by [2, Lemma 2.6], m must be congruent to $3c$.

Theorem 3.11 gives a sufficient condition for P to be primary. We want to show that this condition is also necessary. In fact we are going to show that, under the hypotheses $c(F_0) = c(F_1) = c(F_2) = c \in \{1, 3\}$ and m not a multiple of 3, there is a simpler condition which is equivalent to primarity and to the Primarity Condition. To state this new condition we need the following definition. Note that if $m \in \mathbb{N}$ is not a multiple of 3 and $m \equiv 3c$, we can write $m = (3l + k)n + 3c$ with $l \geq 0$ and $k \in \mathbb{Z}_3 \setminus \{0\}$.

Definition 5.1. For $m = (3l + k)n + 3c$ with $l \geq 0$ and $k \in \{1, 2\}$, we set $\mu = (m - kn)/3$.

The properties of μ summarized in the next lemma are obvious (see Lemma 2.7(a) and Definition 2.6(a)).

Lemma 5.2. *Let P be a spiral orbit of period m not a multiple of three, with three coloured arrows of the same colour c. Then $\mu \equiv c \lesseqgtr r_q$ for each $q \in \mathbb{Z}_3$ and $m - 2n \lesseqgtr 3\mu \lesseqgtr m - n$.*

We shall show that the primary spiral orbits with three coloured arrows are precisely those defined as follows.

Definition 5.3. A spiral orbit P of period m will be called *triple green* (resp. *blue*) if it satisfies:

(i) m is not a multiple of 3.

(ii) P has exactly three coloured arrows, and all arrows are green (resp. blue).

(iii) $r_0 = r_2 = \mu \geqslant r_1$.

(iv) If $d(2,1)$ is defined or $d(1,0)$ or $d(0,2)$ are not defined, then $r_1 = \mu$.

When the colour of the arrows is irrelevant, we shall call these orbits simply *triple*.

We begin by showing that triple orbits are indeed primary.

Proposition 5.4. *Triple orbits are primary.*

Proof. By Theorem 3.11, we only have to prove that if P is a triple orbit of colour c, then P satisfies the Primarity Condition. To do it we use the notation of Definition 5.3 and Table I. By (iii) and (iv) of Definition 5.3, the weights of the fourteen elementary nontrivial loops different from the fundamental one which we can find in the qP-graph of f satisfy:

$$\begin{aligned}
&\Psi(\lambda_i) \geqslant m-\mu && \text{for } i=1,2,4,\\
&\Psi(\lambda_i) \geqslant m-2\mu && \text{for } i=3,5,6,\\
&\Psi(\lambda_7) \geqslant m-3\mu, &&\\
&\Psi(\lambda_8) = 3(\mu+n)-m, &&\\
&\Psi(\lambda_i) = \mu+n && \text{for } i=9,11,13,\\
&\Psi(\lambda_i) = n && \text{for } i=10,12,14
\end{aligned}$$

(where the first three inequalities are equalities except, maybe, when $d(2,1)$ is not defined but $d(1,0)$ and $d(0,2)$ are).

If $k \in \mathbb{Z}_3 \setminus \{0\}$ is such that $m = 3\mu + kn$ then, for each choice of the numbers $\beta_\lambda \in \mathbb{Z}_n$ for $\lambda \in \Lambda$, we can write

$$\sum_{\lambda\in\Lambda} \beta_\lambda \Psi(\lambda) \geqslant \alpha(2\mu+kn) + \beta(\mu+n) = S$$

for some $\alpha, \beta \in \mathbb{Z}_n$. It is enough to see that $m \not\geqslant S$.

When $\alpha\beta > 0$, since $S \geq 3\mu + (k+1)n = m+n > m$, we get $m \not\geqslant S$.

If $\alpha > 0$ and $\beta = 0$, we see that $m \not\geqslant S$ because $m \not\equiv \alpha(2\mu+kn)$ for each $\alpha \in \mathbb{Z}_n$. Indeed, since $m \equiv 3c$, $\mu \equiv c$ and $c \in \{1,3\}$, it follows that $m \equiv 3$ or $m \equiv 1$, $2\mu \equiv 2$ and $\alpha(2\mu+kn) \equiv l \in \{0,2\}$.

If $\alpha = 0$, from $m \equiv 3c$, $\mu \equiv c$ and $c \in \{1,3\}$, it follows that $m \not\equiv \beta(\mu+n)$ for $\beta \in \mathbb{Z}_n \setminus \{3\}$. When $\beta = 3$, we have $3(\mu+n) = m+(3-k)n > m$. Hence, $m \not\geqslant S$. ■

Now we want to prove the converse of this proposition. Therefore, until otherwise stated, we assume that P is a primary spiral orbit with three coloured arrows.

First we have the following result which is already well known.

Lemma 5.5. *If P is a primary spiral orbit with three coloured arrows, then P satisfies* (i) *and* (ii) *of Definition* 5.3.

Proof. This follows from [2, Theorem 5.10] and [2, Theorem B] (see also [2, Definition 4.11]). ∎

In what follows we will denote by c the colour of the three coloured arrows of P. Now it remains to show that P also satisfies (iii) and (iv) of Definition 5.3. To do this we will use the following technical lemmas.

Lemma 5.6. $\min(r_0, r_1, r_2) \lesseqqgtr \mu$.

Proof. By Lemma 5.2, $\mu \equiv \min(r_0, r_1, r_2)$. So, if $\mu < \min(r_0, r_1, r_2)$, then $\mu + n \lesseqqgtr \min(r_0, r_1, r_2)$. Hence $3(\mu + n) \lesseqqgtr r_0 + r_1 + r_2 \lesseqqgtr m - n$ by Proposition 2.13. However, by Lemma 5.2, we have $3\mu + 3n \gtreqqless m + n$. ∎

Recall that $r_q + n < l(S_q) + l(S_{q\oplus 1})$ is the condition we need to define $d(q \oplus 1, q)$, for each $q \in \mathbb{Z}_3$ (see Remark 2.16 and Proposition 2.10). The following lemmas give useful relationships when the numbers $d(q \oplus 1, q)$ are defined.

Lemma 5.7. *If $r_q + n < l(S_q) + l(S_{q\oplus 1})$ for some $q \in \mathbb{Z}_3$, then there are $r_q + n - 1 = \varrho$ black arrows, $B_0, B_1, \ldots, B_{\varrho-1}$, such that $B_0, B_1, \ldots, B_{\varrho-1}, F_{q\oplus 1}$ are overlapping.*

Proof. By Proposition 2.10 we deduce that $x_{p_{q\oplus 1}\ominus(r_q+n)} < x_{p_{q\oplus 1}}$ if $r_q + n \le l(S_q)$ and $x_{p_q} < x_{p_q\oplus(r_q+n)}$ if $r_q + n > l(S_q)$. Hence the arrows $A_{p_{q\oplus 1}\ominus(r_q+n)}, \ldots, A_{p_{q\oplus 1}\ominus 1}$ in the first case and $A_{p_{q\oplus 1}}, \ldots, A_{p_q\oplus(r_q+n-1)}, A_{p_q}, \ldots, A_{p_{q\oplus 1}\ominus 1}$ in the second are overlapping. ∎

Lemma 5.8. *If $r_q + n < l(S_q) + l(S_{q\oplus 1})$ for some $q \in \mathbb{Z}_3$, then $r_q \gtreqqless \mu$.*

Proof. By Lemma 5.2 we have $r_q \equiv \mu$. Assume that $r_q < \mu$. Then $r_q + n \lesseqqgtr \mu$. That is, $\mu = r_q + n + k'n$ for some $k' \ge 0$. Let $k \ge 0$ be such that $m = 3\mu + kn$. By Lemma 5.7 and [2, Lemma 4.5], f has a single orbit of period $r_q + n$. Then, by [2, Lemma 3.7(b)], and the Adjusting Lemma ([1, Lemma 1.18]), f has a periodic orbit of period $3(r_q + n) + (3k' + k)n = m$, with span strictly included in $\langle P \rangle$. Since f is EP-adjusted, this contradicts the primarity of P by the First Theorem of [1]. ∎

Lemma 5.9. *If $r_q + n < l(S_q) + l(S_{q\oplus 1})$ for some $q \in \mathbb{Z}_3$, then $r_q = \max(r_0, r_1, r_2)$.*

Proof. We will show that for each $k \in \mathbb{Z}_3 \setminus \{q\}$ we have $r_k \le r_q$. If $c \ge l(S_k) + l(S_{k\oplus 1})$, then by Definition 2.6(a) and Lemma 5.2 we have $r_k = c \lesseqqgtr r_q$. So we assume that $c < l(S_k) + l(S_{k\oplus 1})$. By Definition 2.6(b), the $m - r_k$ arrows

$$A_{j_k}, \ldots, A_{i_k\ominus 1}$$

are overlapping. Note that we get this sequence of arrows by eliminating the arrows $A_{i_k}, \dots, A_{j_k \ominus 1}$ from the whole sequence of arrows of P. Hence we have eliminated exactly one coloured arrow, namely $F_{k \oplus 1}$, which separates the strings S_k and $S_{k \oplus 1}$. That is, in the remaining sequence all the arrows are black except $A_{p_k \ominus 1} = F_k$ and $A_{p_{k \ominus 1} \ominus 1} = F_{k \ominus 1}$.

Assume that $r_k > r_q$. By Lemma 5.2 we have $r_k = r_q + ln$ for some $l \geq 1$. We note that since $r_q \gtreqqless c \geq 1$, it follows that $\varrho = r_q + n - 1 \geq n$. We claim that the n black arrows $B_0, B_1, \dots, B_{n-1}$ from Lemma 5.7 are overlapping. To prove the claim we consider several cases.

From the proof of Lemma 5.7 it follows that if $r_q + n \leq l(S_q)$, then the arrows $B_0, B_1, \dots, B_{n-1}$ are $A_{p_{q \oplus 1} \ominus (r_q + n)}, \dots, A_{p_{q \oplus 1} \ominus (r_q + 1)}$. They are overlapping because, by Corollary 2.4, $x_{p_{q \oplus 1} \ominus (r_q + n)} < x_{p_{q \oplus 1} \ominus r_q}$.

Assume now that $r_q + n > l(S_q)$. We note that $p_q \oplus (r_q + n) \geq p_{q \oplus 1} + n$ if and only if $r_q \geq l(S_q)$. So, from the proof of Lemma 5.7, we see that the arrows $B_0, B_1, \dots, B_{n-1}$ are $A_{p_{q \oplus 1}}, \dots, A_{p_q \oplus (r_q + n - 1)}, A_{p_q}, \dots, A_{p_{q \oplus 1} \ominus (r_q + 1)}$ when $r_q < l(S_q)$ and $A_{p_{q \oplus 1}}, \dots, A_{p_{q \oplus 1} + n - 1}$ when $r_q \geq l(S_q)$. In the first case they are overlapping because $x_{p_{q \oplus 1}} < x_{p_{q \oplus 1} \ominus r_q}$ by Proposition 2.9. In the second case we have $x_{p_{q \oplus 1}} < x_{p_{q \oplus 1} + n}$ by Corollary 2.4 and the claim is proved.

From the claim and Lemma 5.7, we obtain the r_k overlapping arrows

$$(B_0, B_1, \dots, B_{n-1})^{l-1},\ B_0, B_1, \dots, B_{\varrho - 1}, F_{q \oplus 1}, \tag{3}$$

where $(B_0, \dots, B_{n-1})^{l-1}$ means that the sequence $B_0, B_1, \dots, B_{n-1}$ is repeated $l - 1$ times. We note that all of them are black except the last one, $F_{q \oplus 1}$.

Since $q \neq k$, we have $q \oplus 1 \in \{k, k \ominus 1\}$. So we can connect this sequence of arrows with the one obtained above. In this way we get a sequence of m overlapping arrows, $C_0, C_1, \dots, C_{m-1}$. We will use it to obtain a contradiction with the primarity of P.

Assume that for some $p \in \{p_k \ominus 1, p_{k \ominus 1} \ominus 1\}$, $I_p = [a, x_p]$ is such that a is the beginning of a black arrow. Let us label the arrows $C_0, C_1, \dots, C_{m-1}$ in such a way that $C_0 = A_p$. Then we define the intervals $J_t = [0, b(C_t)]$ for $t = 1, \dots, m-1$ and $J_0 = I_p$. In this way, since x_p is the beginning of a coloured arrow, $J_0 \to J_1 \to \dots \to J_{m-1} \to J_0$. By [1, Lemma 1.12], since J_0 is a basic interval, we get a loop

$$I_{s_0} \to I_{s_1} \to \dots \to I_{s_{m-1}} \to I_{s_0}$$

of basic intervals, of length m, with $I_{s_t} \subset J_t$ for each $t \in \mathbb{Z}_m$ (in particular $I_{s_0} = I_p$). Moreover, for each $t \in \mathbb{Z}_m$ the step $I_{s_t} \to I_{s_{t \oplus 1}}$ is of the same colour as C_t. This loop is nonrepetitive since it has three coloured steps and m is not a multiple of 3. Also it goes through I_p which does not contain 0. Then, by [3, Lemma 2.2], f has a periodic point $y \in I_{s_0}$ of period m such that $f^t(y) \in I_{s_t}$ for each $t \in \mathbb{Z}_m$. Let Q be the orbit of this point.

We claim that $Q \neq P$. Otherwise, $y = x_p$, because the step $I_{s_0} \to I_{s_1}$ is of the same colour as C_0 and, hence, different from the colour of $(a, f(a))$. Then we can see inductively that $b(C_t) = x_{p\oplus t}$ for all $t \in \mathbb{Z}_m$. Indeed, since $f^t(y) \in I_{s_t}$, $I_{s_t} \subset J_t$ and the arrows C_t are overlapping, for $t \geq 1$ we have

$$x_{p\oplus t} = f^t(y) \leq \max I_{s_t} \leq b(C_t) \leq e(C_{t-1}) = f(x_{p\oplus(t-1)}) = x_{p\oplus t}.$$

Therefore, $C_t = A_{p\oplus t}$ and, hence, $b(C_t) = e(C_{t\ominus 1})$ for all $t \in \mathbb{Z}_m$. But this contradicts the definition of the arrows C_t. Indeed, since there exists $j \in \mathbb{Z}_m$ such that $C_j = A_{j_k}$ and $C_{j\ominus 1} = A_{i_k\ominus 1}$, for such j we have $b(C_j) \neq e(C_{j\ominus 1})$. This ends the proof of the claim. Then, since f is EP-adjusted, the First Theorem of [1] gives a contradiction with the primarity of P. Up to now we have proved that $r_k \leq r_q$ when some of the intervals $I_{p_k\ominus 1}$ or $I_{p_{k\ominus 1}\ominus 1}$ has the beginning of a black arrow as its lower endpoint.

Assume now that none of the intervals $I_{p_k\ominus 1}$ and $I_{p_{k\ominus 1}\ominus 1}$ has an endpoint which is the beginning of a black arrow. Since there are only three coloured arrows, these two intervals must have a common point. In particular, x_{m-1} must be the smallest of the three endpoints of these intervals since, otherwise, A_{m-1} would cross another coloured arrow. Hence, since the upper endpoints of $I_{p_k\ominus 1}$ and $I_{p_{k\ominus 1}\ominus 1}$ are the beginnings of the arrows $A_{p_k\ominus 1} = F_k$ and $A_{p_{k\ominus 1}\ominus 1} = F_{k\ominus 1}$ respectively, we have $F_{k\oplus 1} = A_{m-1} = F_0$, that is, $k = 2$. Furthermore, $I_{m-1} = [x_p, x_{m-1}]$, where x_p is the beginning of a black arrow, because x_{m-1} is not one of the smallest points. Moreover, since x_{p_1-1} is the beginning of the coloured arrow F_1, we have $x_{m-1} < x_{p_1-1}$. So, by Proposition 2.10, $r_2+n \lesseqqgtr p_1-1-(m-1)+m$; that is, $r_2 \lesseqqgtr p_1-n = l(S_0)-n$. Therefore, since $1 \leq c \lesseqqgtr r_2 < l(S_0)$, it follows that $q(r_2-1) = q(r_2) = 0$. Hence, A_{r_2-1} is also black. Furthermore, $r_2-1-(m-1)+m \lesseqqgtr r_2$. Hence, by Proposition 2.9, $x_{r_2-1} < x_{m-1}$. Therefore, $x_{r_2-1} \leq x_p$ and so $x_{r_2} \leq f(x_p)$. We then have the sequence of $m - r_2$ overlapping arrows

$$A_{r_2}, \ldots, A_{p_1-1}, \ldots, A_{p_2-1}, \ldots, A_{m-2}, A_p.$$

Since $q \neq k = 2$, $F_{q\oplus 1}$ is either A_{p_1-1} or A_{p_2-1}. So, with this sequence and the sequence (3) we can construct a new sequence of m overlapping arrows which will also be called $C_0, C_1, \ldots, C_{m-1}$.

Now the rest of the proof follows as in the previous case upon replacing I_p by I_{m-1} and taking $C_j = A_p$ and $C_{j\ominus 1} = A_{m-2}$. ∎

In the following lemmas we will see that P satisfies (iii) and (iv) of Definition 5.3.

Lemma 5.10. *If $r_q + n < l(S_q) + l(S_{q\oplus 1})$ for all $q \in \mathbb{Z}_3$, then $r_0 = r_1 = r_2 = \mu$.*

Proof. By Lemma 5.9 we see that $\max(r_0, r_1, r_2) = r_0 = r_1 = r_2 = \min(r_0, r_1, r_2)$. On the other hand, by Lemmas 5.6 and 5.8, we have $\mu = \min(r_0, r_1, r_2)$. ∎

LEMMA 5.11. *If* $r_0 + n \geq l(S_0) + l(S_1)$, *then* $r_0 = r_1 = r_2 = \mu = c$ *and* $m = 3c + n$.

Proof. From $r_0+n \geq l(S_0)+l(S_1)$ and $r_2+n > l(S_2)$ (see Lemma 2.11(c)), it follows that $r_0+r_2+2n > m$. That is, $m-2n < r_0+r_2 < r_0+r_1+r_2$. Hence, $r_0 + r_1 + r_2 = m - n$ by Proposition 2.13. Then $r_1 = m - n - (r_0 + r_2) < n$ and, hence, $r_1 = c = \min(r_0, r_1, r_2)$ by Lemma 5.2.

If $r_1+n < l(S_1)+l(S_2)$, then, by Lemma 5.9, $r_1 = \max(r_0, r_1, r_2)$. Hence, $r_0 = r_1 = r_2 = c$.

If $r_1+n \geq l(S_1)+l(S_2)$ we have $r_0+r_1+2n \geq m+l(S_1) = r_0+r_1+r_2+n+l(S_1)$. That is, $r_2 \leq n-l(S_1)$ and, hence, $r_2 = c = r_1$. If we now assume that $r_2 + n < l(S_2) + l(S_0)$, then again by Lemma 5.9, $r_2 = \max(r_0, r_1, r_2)$. Hence $r_0 = r_1 = r_2 = c$. If, on the contrary, $r_2 + n \geq l(S_2) + l(S_0)$, then adding this inequality to $r_0 + n \geq l(S_0) + l(S_1)$ and $r_1 + n \geq l(S_1) + l(S_2)$, we get $r_0 + r_1 + r_2 + 3n \geq 2m$. Since $r_0 + r_1 + r_2 = m - n$, we find that $n \geq r_0 + r_1 + r_2$ and, then, $r_0 = c$ (in fact $r_0 = r_1 = r_2 = c = 1$).

Since $r_0 = r_1 = r_2 = c$, we have $m = 3c + n$, and so $\mu = c$. ∎

LEMMA 5.12. *If* $r_2 + n \geq l(S_2) + l(S_0)$, *then* $r_0 = r_1 = r_2 = \mu$.

Proof. We may assume that $r_0 + n < l(S_0) + l(S_1)$ since otherwise, by Lemma 5.11, the conclusion is true. Then by Lemma 5.9 we have $r_0 = \max(r_0, r_1, r_2)$ and, by Lemma 5.8, $r_0 \gtreqless \mu$.

From the hypothesis it follows that $r_2 + n > l(S_0)$. Since $r_0 < l(S_0)$ by Lemma 2.11(b), we have $r_2 + n > r_0$. Thus, $r_2 \gtreqless r_0$ since $r_2 \equiv r_0$ by Lemma 5.2. Therefore, $r_2 = r_0 = \max(r_0, r_1, r_2)$.

If $r_1 + n < l(S_1) + l(S_2)$, then also $r_1 = \max(r_0, r_1, r_2)$ by Lemma 5.9 and $r_0 = r_1 = r_2$. If, on the contrary, $r_1 + n \geq l(S_1) + l(S_2)$, then from Proposition 2.13 it follows that $r_0+r_2 \lesseqgtr m-(r_1+n) \leq m-(l(S_1)+l(S_2)) = l(S_0) < r_2 + n$. Hence, $r_0 < n$ and, therefore, $r_0 = c = \min(r_0, r_1, r_2)$. Then also $r_0 = r_1 = r_2$.

Finally, the inequality $\mu \lesseqgtr r_0 = r_1 = r_2$ cannot happen to be strict. Otherwise, again by Proposition 2.13 and Lemma 5.2, we would obtain $m - n \gtreqless r_0 + r_1 + r_2 \gtreqless 3(\mu + n) \gtreqless m + n$; a contradiction. ∎

LEMMA 5.13. *If* $r_1 + n \geq l(S_1) + l(S_2)$, *then* $r_0 = r_2 = \mu \gtreqless r_1$.

Proof. We may assume that $r_q+n < l(S_q)+l(S_{q\oplus 1})$ for $q = 0$ and $q = 2$. Otherwise we already know that $r_0 = r_1 = r_2 = \mu$ by Lemmas 5.11 and 5.12. Then, by Lemmas 5.9, 5.8 and 5.6, we deduce that $\max(r_0, r_1, r_2) = r_0 = r_2 \gtreqless \mu \gtreqless \min(r_0, r_1, r_2) = r_1$.

From the hypothesis and from $r_0+r_1+r_2 \lesseqgtr m-n$ (see Proposition 2.13) it follows that $r_0 + r_2 \leq l(S_0)$. Let us see that $\mu = r_0 = r_2$. Otherwise, by Lemma 5.2, $\mu + n \lesseqgtr r_0 = r_2$ and $m - r_2 - r_0 \lesseqgtr m - 2\mu - 2n \lesseqgtr \mu$. Then $l(S_2) < m-l(S_0) \leq m-r_2-r_0 \leq \mu < r_2$. So, $x_{p_2} > x_{p_2 \oplus r_2}$ by Lemma 2.7(c) and Definition 2.6(b). Since $r_2 > l(S_2)$, we have $p_2 \oplus r_2 = p_2 + r_2 - m = l(S_0) + l(S_1) + r_2 - m = r_2 - l(S_2)$. So, $x_{p_2} > x_{r_2 - l(S_2)}$. Since $x_{p_1 - r_0} > x_{p_1}$ by Lemma 2.11(b) and $r_2 - l(S_2) < r_2 \le l(S_0) - r_0 = p_1 - r_0$, the $m - r_2 - r_0$ arrows

$$A_{r_2 - l(S_2)}, \ldots, A_{p_1 - r_0 - 1}, A_{p_1}, \ldots, A_{p_2 - 1}$$

are overlapping and are all black except A_{p_2-1}. Then [2, Lemma 4.5] gives a single orbit of period $m - r_2 - r_0$. Let $k \in \{1, 2\}$ and $k' \ge 0$ be such that $m = 3\mu + kn$ and $\mu = m - r_2 - r_0 + k'n$. By [2, Lemma 3.7(b)] and the Adjusting Lemma ([1, Lemma 1.18]), we obtain an orbit Q of period $3(m - r_2 - r_0) + (3k' + k)n = m$ with three coloured arrows. This orbit is different from P since the strings of Q have lengths at least $m - r_2 - r_0 > l(S_2)$. By the First Theorem of [1], this contradicts the primarity of P. Hence, $\mu = r_0 = r_2$. This ends the proof of the lemma. ∎

Lastly, we can state the main result of this section. It characterizes the primary spiral orbits with three coloured arrows and, hence, it ends the characterization of the primary strongly directed orbits of maps from $\mathcal{X}_4$.

Theorem 5.14. *A spiral orbit with three coloured arrows is primary if and only if it is triple.*

Proof. Triple orbits are primary by Proposition 5.4. Hence, we only have to prove the converse. Conditions (i) and (ii) of Definition 5.3 are satisfied by Lemma 5.5. Conditions (iii) and (iv) of Definition 5.3 are satisfied by virtue of Definition 2.15, Remark 2.16 and Lemmas 5.10–5.13. ∎

6. Conclusions. With the study of triple orbits, we have finished the characterization of the strongly directed primary orbits for self maps of the 4-star with the branching point fixed. In this section we summarize the main results of the two papers where this characterization is carried out. The first of the statements below is [2, Theorem A] about primary directed orbits of maps from $\mathcal{X}_n$ having at most one coloured arrow. The second one puts together [2, Theorems B and C], and Theorems 4.9 and 5.14 about primary strongly directed orbits of maps from $\mathcal{X}_4$ with at least two coloured arrows.

Theorem A. *Let P be a directed orbit of a map $f \in \mathcal{X}_n$ with $\nu \le 1$ coloured arrows.*

(a) *If P has only black arrows*, *then P is primary if and only if it is twist.*

(b) *If P has a coloured arrow A*, *then P is primary if and only if it is single of colour $c(A)$.*

Theorem B. *Let P be a strongly directed orbit of a map $f \in \mathcal{X}_4$ with $\nu \ge 2$ coloured arrows.*

(a) *If P is primary, then $\nu \leq 3$ and it is colour compatible.*

(b) *If P has crossing arrows, then P is primary if and only if it is ν-box.*

(c) *If P has no crossing arrows and $\nu = 2$, then P is primary if and only if it is double.*

(d) *If P has no crossing arrows and $\nu = 3$, then P is primary if and only if it is triple.*

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