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GENERALISATIONS OF SOME PROPERTIES OF INVARIANT POLYNOMIALS WITH MATRIX ARGUMENTS

Abstract. Some extensions of the properties of invariant polynomials proved by Davis (1980), Chikuse (1980), Chikuse and Davis (1986) and Ratnarajah et al. (2005) are given for symmetric and Hermitian matrices.

1. Introduction. A number of integrals in distribution theory of random matrices are expanded in terms of zonal polynomials; see Constantine (1963) and Díaz-García and Gutiérrez-Jáimez (2001) among many others. However, important distributional problems cannot be solved via zonal polynomials, such as the distribution of the eigenvalues of a noncentral Wishart distribution or the doubly noncentral Beta type I and II distributions. Solutions of such problems have been provided via invariant polynomials with matrix arguments; see for example Davis (1980), Chikuse (1980), Chikuse (1981), Chikuse and Davis (1986), Díaz-García and Gutiérrez-Jáimez (2001) and Ratnarajah et al. (2005), among many others.

The fundamental property of the real invariant polynomials is the following:

$$(1)\qquad \int\limits_{\mathcal{O}(m)} C_\kappa(\mathbf{A}\mathbf{H}'\mathbf{X}\mathbf{H})C_\lambda(\mathbf{B}\mathbf{H}'\mathbf{Y}\mathbf{H})(d\mathbf{H}) = \sum_{\phi\in\kappa\cdot\lambda} \frac{C_\phi^{\kappa,\lambda}(\mathbf{A},\mathbf{B})C_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{C_\phi(\mathbf{I})}$$

(see Davis (1980) or its generalisation for r matrices in Chikuse (1980, eq. (2.2)) and Chikuse and Davis (1986, eq. (2.2))), where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$ and $\mathbf{Y}$ are $m\times m$ symmetric matrices, $(d\mathbf{H})$ is the invariant Haar measure over the group $\mathcal{O}(m)$ of $m\times m$ orthogonal matrices, $C_\phi^{\kappa,\lambda}$ is the invariant polynomial and C_ϕ denotes the zonal polynomial (see Davis (1980), Chikuse (1980) and Chikuse and Davis (1986)).

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An analogous property for invariant polynomials of Hermitian matrix arguments can be found in Ratnarajah et al. (2005, eq. (7)).

Many other characteristics of the invariant polynomials are a consequence of the following basic property:

$$\int_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(\mathbf{AH}'\mathbf{XH}, \mathbf{AH}'\mathbf{HY})(d\mathbf{H}) = \frac{C_\phi(\mathbf{A})C_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{C_\phi(\mathbf{I})}; \tag{2}$$

see Davis (1980), Chikuse and Davis (1986) for the real case, and Ratnarajah et al. (2005) for the complex case.

In the present work we generalise the property (1) when the product of the zonal polynomials is replaced by the product of invariant polynomials. To obtain this fundamental property we need to generalise (2) by placing a new matrix $\mathbf{B} \neq \mathbf{A}$ in the second argument of the invariant polynomial. We emphasise that the generalisation of (2) provides alternative derivations for most of the invariant polynomial properties given independently by Davis (1980), Chikuse and Davis (1986) and Chikuse (1980); see Díaz-García (2006). The same generalisations and alternatives apply to the complex invariant polynomial expressions of Ratnarajah et al. (2005).

2. New properties of invariant polynomials. A detailed discussion of invariant polynomials may be found in Davis (1980), Chikuse (1980) and Chikuse and Davis (1986). In addition, the zonal polynomials are studied in detail in James (1960), James (1961), Saw (1977), Muirhead (1982) and Takemura (1984). For convenience, we shall introduce their definitions and some notations, although in general we adhere to standard notations.

Let $\mathbf{X}$ be a symmetric $m \times m$ matrix, and $\varphi(\mathbf{X})$ a polynomial in the $m(m+1)/2$ different elements of $\mathbf{X}$. Then the transformation

$$\varphi(\mathbf{X}) \to \varphi(\mathbf{L}^{-1}\mathbf{X}\mathbf{L}'^{-1}), \quad \mathbf{L} \in GL(m), \tag{3}$$

defines a representation of the real linear group $GL(m)$ in the vector space of all polynomials in $\mathbf{X}$. The space V_k of homogeneous polynomials of degree k is invariant under the transformation (3) and decomposes into the direct sum of irreducible subspaces, $V_k = \bigoplus_\kappa V_{k,\kappa}$, where $\kappa = (k_1, \dots, k_m)$, $k_1 \geq \cdots \geq k_m \geq 0$, runs over all partitions of k into not more than m parts. In each $V_{k,\kappa}$, the irreducible representation $\{2\kappa\} = (2k_1, \dots, 2k_m)$ of $GL(m)$ acts, each of these representations occurring exactly once in the decomposition. Each $V_{k,\kappa}$ contains a unique one-dimensional subspace invariant under the orthogonal group $\mathcal{O}(m)$. These subspaces are generated by the *zonal polynomials*, $Z_\kappa(\mathbf{X})$. Being invariant under the orthogonal group, i.e.,

$$Z_\kappa(\mathbf{H}'\mathbf{XH}) = Z_\kappa(\mathbf{X}), \quad \mathbf{H} \in \mathcal{O}(m), \tag{4}$$

they are homogeneous symmetric polynomials in the eigenvalues of $\mathbf{X}$.

The normalised $Z_\kappa(\mathbf{X})$ are obtained by assuming that the coefficient of s_1^k in $Z_\kappa(\mathbf{X})$ is 1, where $s_1 = \mathrm{tr}(\mathbf{X})$ denotes the sum of the eigenvalues of $\mathbf{X}$. This way the normalised zonal polynomials are defined as

$$C_\kappa(\mathbf{X}) = \left(\frac{\chi_{[2\kappa]}(1)2^k k!}{(2k)!}\right) Z_\kappa(\mathbf{X})$$

where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols, and furthermore

$$\chi_{[2\kappa]}(1) = \frac{k! \prod_{i<j}^m (k_i - k_j - i - j)}{\prod_{i=1}^m (k_i + m - i)!}.$$

The zonal polynomials were defined above only for symmetric matrices $\mathbf{X}$. However, since they are polynomials in the eigenvalues of $\mathbf{X}$, their definition may be extended to arbitrary complex symmetric matrices. Alternative definitions are given in Saw (1977), Muirhead (1982) and Takemura (1984).

Invariant polynomials with two (or more) matrix arguments are a direct extension of the zonal polynomials. Davis (1980) has defined this class of polynomials $C_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})$ in the elements of the $m \times m$ symmetric complex matrices $\mathbf{X}$ and $\mathbf{Y}$, having the property of being invariant under the simultaneous transformations

$$\mathbf{X} \mapsto \mathbf{H}'\mathbf{X}\mathbf{H}, \quad \mathbf{Y} \mapsto \mathbf{H}'\mathbf{Y}\mathbf{H}, \quad \mathbf{H} \in \mathcal{O}(m).$$

These invariant polynomials satisfy the basic relationship (1), where C_κ, C_λ and C_ϕ are zonal polynomials, indexed by the ordered partitions κ, λ and ϕ of the nonnegative integers k, l and $f = k + l$ respectively into not more than m parts and "$\phi \in \kappa \cdot \lambda$" signifies that the irreducible representation of $GL(m)$ indexed by 2ϕ occurs in the decomposition of the Kronecker product $2\kappa \otimes 2\lambda$ of the irreducible representation indexed by 2κ and 2λ.

Now, the following theorem generalises eq. (5.4) of Davis (1980).

THEOREM 2.1. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{X}$ *and* $\mathbf{Y}$ *be* $m \times m$ *symmetric matrices. Then*

$$\int_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(\mathbf{A}\mathbf{H}'\mathbf{X}\mathbf{H}, \mathbf{B}\mathbf{H}'\mathbf{Y}\mathbf{H})(d\mathbf{H}) = \frac{C_\phi^{\kappa,\lambda}(\mathbf{A},\mathbf{B})C_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{\theta_\phi^{\kappa,\lambda} C_\phi(\mathbf{I})} \tag{5}$$

with

$$\theta_\phi^{\kappa,\lambda} = \frac{C_\phi^{\kappa,\lambda}(\mathbf{I},\mathbf{I})}{C_\phi(\mathbf{I})}.$$

Proof. We give a straightforward proof by extending the procedures of James (1961) and by using the results of Davis (1980), From Muirhead (1982, eq. (3), p. 259) we have

$$\begin{aligned}(6)\qquad &\int_{\mathcal{O}(m)} \operatorname{etr}(\mathbf{AH'XH}+\mathbf{BH'YH})(d\mathbf{H})\\ &\qquad = \sum_{k}^{\infty}\sum_{\kappa}\sum_{l}^{\infty}\sum_{\lambda}\frac{1}{k!\,l!}\int_{\mathcal{O}(m)} C_{\kappa}(\mathbf{AH'XH})C_{\lambda}(\mathbf{BH'YH})(d\mathbf{H}).\end{aligned}$$

Then from Davis (1980, eq. (5.8)) and by using the notation for the summations given in Davis (1980), we get

$$\begin{aligned}&\int_{\mathcal{O}(m)} \operatorname{etr}(\mathbf{AH'XH}+\mathbf{BH'YH})(d\mathbf{H})\\ &\qquad = \sum_{\kappa,\lambda;\phi}^{\infty}\frac{\theta_{\phi}^{\kappa,\lambda}}{k!\,l!}\int_{O(m)} C_{\phi}^{\kappa,\lambda}(\mathbf{AH'XH},\mathbf{BY'YH})(d\mathbf{H});\end{aligned}$$

but from Davis (1980, eq. (5.12)),

$$\int_{\mathcal{O}(m)} \operatorname{etr}(\mathbf{AH'Q'XQH}+\mathbf{BH'Q'YQH})(d\mathbf{H}) = \sum_{\kappa,\lambda;\phi}^{\infty}\frac{C_{\phi}^{\kappa,\lambda}(\mathbf{A},\mathbf{B})C_{\phi}^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{k!\,l!C_{\phi}(\mathbf{I})},$$

so the required result follows. ∎

Remark 2.1. Observe that no recurrence is used in the proof of Theorem 2.1, i.e., we do not require (5) to demonstrate (5.12) in Davis (1980), because the last expression is obtained by integrating (6) via (4.13) of Davis (1980).

Remark 2.2. Different approaches can be implemented to prove this result, for instance: we can assume that there exists a differential operator for the invariant polynomials, like the operator given in equation (3.41) of Chikuse (1980), and then we can generalise the proof of Theorem 7.2.5 in Muirhead (1982); we can proceed as in the proof of Theorem 1, pp. 27–28 of Takemura (1984); or, we can follow the ideas of James (1961) and James (1960).

For r matrices we have the following generalisation of Chikuse (1980, eq. (3.14)):

Corollary 2.1. *Let $\mathbf{A}_1,\dots,\mathbf{A}_r$ be $m\times m$ symmetric matrices. Then*

$$\begin{aligned}(7)\qquad &\int_{\mathcal{O}(m)} C_{\phi}^{\kappa[r]}(\mathbf{A}_1\mathbf{H'X}_1\mathbf{H},\dots,\mathbf{A}_r\mathbf{H'X}_r\mathbf{H})(d\mathbf{H})\\ &\qquad = \frac{C_{\phi}^{\kappa[r]}(\mathbf{A}_1,\dots,\mathbf{A}_r)C_{\phi}^{\kappa[r]}(\mathbf{X}_1,\dots,\mathbf{X}_r)}{\theta_{\phi}^{\kappa[r]}C_{\phi}(\mathbf{I})},\end{aligned}$$

where we keep the notations of Chikuse and Davis (1986)*, i.e.,* $\alpha[s,r] = (\alpha_s,\ldots,\alpha_r)$*, in particular* $\alpha[1,r] \equiv \alpha[r] = (\alpha_1,\ldots,\alpha_r)$*.*

It is important to note that from Theorem 2.1 and Corollary 2.1 most of the results in Davis (1980), Chikuse (1980) and Chikuse and Davis (1986) can be derived in an alternative way. For example, by taking $\mathbf{A}=\mathbf{B}$ and by using equation (5.1) in Davis (1980), we find equation (5.4) of Davis (1980).

Analogously, from (5) we have

$$\int\limits_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(\mathbf{A}'\mathbf{H}'\mathbf{X}\mathbf{H}\mathbf{A},\mathbf{B}'\mathbf{H}'\mathbf{H}\mathbf{Y}\mathbf{B})(d\mathbf{H}) = \frac{C_\phi^{\kappa,\lambda}(\mathbf{A}\mathbf{A}',\mathbf{B}\mathbf{B}')C_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{\theta_\phi^{\kappa,\lambda}C_\phi(\mathbf{I})}. \tag{8}$$

Thus, by replacing $\mathbf{X}=\mathbf{I}$, $\mathbf{A}'\mathbf{A}=\mathbf{R}$ or $\mathbf{Y}=\mathbf{I}$, $\mathbf{B}'\mathbf{B}=\mathbf{S}$ in (8) and by using Davis (1980, eq. (5.2)), we obtain

$$\int\limits_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(\mathbf{R},\mathbf{B}'\mathbf{H}'\mathbf{H}\mathbf{Y}\mathbf{B})(d\mathbf{H}) = \frac{C_\phi^{\kappa,\lambda}(\mathbf{R},\mathbf{B}\mathbf{B}')C_\lambda(\mathbf{Y})}{C_\lambda(\mathbf{I})}$$

and

$$\int\limits_{\mathcal{O}(m)} C_\phi^{\kappa,\lambda}(\mathbf{A}'\mathbf{H}'\mathbf{H}\mathbf{Y}\mathbf{A},\mathbf{S})(d\mathbf{H}) = \frac{C_\phi^{\kappa,\lambda}(\mathbf{A}\mathbf{A}',\mathbf{S})C_\kappa(\mathbf{Y})}{C_\kappa(\mathbf{I})},$$

respectively (see Davis (1980, eq. 5.13)).

Of course, a number of new properties can be established from Theorem 2.1 and Corollary 2.1. For instance, when (7) is mixed with Chikuse and Davis (1986, eq. (2.7)), then the following generalisation of expression (1.1) in Davis (1980) is obtained, with the product of zonal polynomials is substituted by a product of invariant polynomials:

Corollary 2.2.

$$\begin{aligned}\int\limits_{\mathcal{O}(m)} & C_\sigma^{\kappa[q]}(\mathbf{A}_1\mathbf{H}'\mathbf{X}_1\mathbf{H},\ldots,\mathbf{A}_q\mathbf{H}'\mathbf{X}_q\mathbf{H})C_\tau^{\kappa[q+1,r]} \\ &\times(\mathbf{A}_{q+1}\mathbf{H}'\mathbf{X}_{q+1}\mathbf{H},\ldots,\mathbf{A}_r\mathbf{H}'\mathbf{X}_r\mathbf{H})(d\mathbf{H}) \\ &= \sum_{\phi\in\sigma^*\cdot\,\tau^*} \pi_{\sigma,\tau}^{\kappa[r]:\phi}\frac{C_\phi^{\kappa[r]}(\mathbf{A}_1,\ldots,\mathbf{A}_r)C_\phi^{\kappa[r]}(\mathbf{X}_1,\ldots,\mathbf{X}_r)}{\theta^{\kappa[r]}C_\phi(\mathbf{I})},\end{aligned} \tag{9}$$

where $\pi_{\sigma,\tau}^{\kappa[r]:\phi}$ *is defined in Chikuse and Davis* (1986, Lemma 2.2(iii)) *and* σ^* *denotes the partition* σ *ignoring multiplicity.*

3. Complex case. Now, we can apply the above procedures to similar expressions of complex invariant polynomials for deriving extensions of Ratnarajah et al. (2005).

Let $\mathbf{Y}, \mathbf{B}, \mathbf{A}_j$, $\mathbf{X}_j$, $j = 1, \dots, r$, be $m \times m$ Hermitian matrices. Then

$$\int_{\mathcal{U}(m)} \widetilde{C}_\phi^{\kappa,\lambda}(\mathbf{A}\mathbf{U}^H\mathbf{X}\mathbf{U}, \mathbf{B}\mathbf{U}^H\mathbf{H}\mathbf{Y}\mathbf{U})[d\mathbf{U}] = \frac{\widetilde{C}_\phi^{\kappa,\lambda}(\mathbf{A},\mathbf{B})\widetilde{C}_\phi^{\kappa,\lambda}(\mathbf{X},\mathbf{Y})}{\widetilde{\theta}_\phi^{\kappa,\lambda}\widetilde{C}_\phi(\mathbf{I})}, \tag{10}$$

$$\int_{\mathcal{U}(m)} \widetilde{C}_\phi^{\kappa[r]}(\mathbf{A}_1\mathbf{U}^H\mathbf{X}_1\mathbf{U}, \mathbf{A}_2\mathbf{U}^H\mathbf{X}_2\mathbf{U}, \dots, \mathbf{A}_r\mathbf{U}^H\mathbf{X}_r\mathbf{U})[d\mathbf{U}] = \frac{\widetilde{C}_\phi^{\kappa[r]}(\mathbf{A}_1, \dots, \mathbf{A}_r)\widetilde{C}_\phi^{\kappa[r]}(\mathbf{X}_1, \dots, \mathbf{X}_r)}{\widetilde{\theta}_\phi^{\kappa[r]}\widetilde{C}_\phi(\mathbf{I})}, \tag{11}$$

with

$$\widetilde{\theta}_\phi^{\kappa[r]} = \frac{\widetilde{C}_\phi^{\kappa[r]}(\mathbf{I}, \cdots, \mathbf{I})}{\widetilde{C}_\phi(\mathbf{I})},$$

$$\begin{aligned} \int_{\mathcal{U}(m)} \widetilde{C}_\sigma^{\kappa[q]}(\mathbf{A}_1\mathbf{U}^H\mathbf{X}_1\mathbf{U}, \dots, \mathbf{A}_q\mathbf{U}^H\mathbf{X}_q\mathbf{U})\widetilde{C}_\tau^{\kappa[q+1,r]} \\ \times (\mathbf{A}_{q+1}\mathbf{U}^H\mathbf{X}_{q+1}\mathbf{U}, \dots, \mathbf{A}_r\mathbf{U}^H\mathbf{X}_r\mathbf{U})[d\mathbf{U}] \\ = \sum_{\phi \in \sigma^* \cdot \tau^*} \widetilde{\pi}_{\sigma,\tau}^{\kappa[r]:\phi} \ \frac{\widetilde{C}_\phi^{\kappa[r]}(\mathbf{A}_1, \dots, \mathbf{A}_r)\widetilde{C}_\phi^{\kappa[r]}(\mathbf{X}_1, \dots, \mathbf{X}_r)}{\theta_\phi^{\kappa[r]}\widetilde{C}_\phi(\mathbf{I})}, \end{aligned} \tag{12}$$

where $\widetilde{\pi}_{\sigma,\tau}^{\kappa[r]:\phi}$ are similar extensions of Chikuse and Davis (1986) to the complex case, $[d\mathbf{U}]$ denotes the unit invariant Haar measure on the unitary group $\mathcal{U}(m)$ and $\widetilde{C}_\phi^{\kappa[r]}$ denotes the invariant polynomial with Hermitian matrix arguments (see Ratnarajah et al. (2005)).

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