APPLICATIONES MATHEMATICAE
38,4 (2011), pp. 485–494

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A NOTE ON STOCHASTIC ORDERING OF ESTIMATORS OF EXPONENTIAL RELIABILITY

Abstract. Recently Balakrishnan and Iliopoulos [Ann. Inst. Statist. Math. 61 (2009)] gave sufficient conditions under which the maximum likelihood estimator (MLE) is stochastically increasing. In this paper we study test plans which are not considered there and we prove that the MLEs for those plans are also stochastically ordered. We also give some applications to the estimation of reliability.

1. Introduction and preliminaries. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random sample from the exponential distribution $\mathrm{Ex}(\theta)$ with density $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $x > 0$, $\theta > 0$. Let $X_{1:n}, \ldots, X_{n:n}$ denote the corresponding order statistics.

In this paper we consider the problem of estimating the mean θ of the exponential distribution as well as the reliability function $R(t) = e^{-t/\theta}$, $\theta > 0$, $t > 0$ fixed, and then we examine stochastic monotonicity of the estimators obtained.

Let X and Y be two random variables, F and G their respective probability distribution functions and f and g their respective density functions, if they exist. Recall that X is *stochastically smaller* than Y ($X \le_{\mathrm{st}} Y$) if $F(x) \ge G(x)$ for every x. We say that X is *smaller in the likelihood ratio order* ($X \le_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing. It is well known that the likelihood ratio ordering is stronger than the usual stochastic order. For discussions on properties of those stochastic orderings we refer to Shaked and Shanthikumar [11].

Definition 1.1. A family of distributions $\{F(x;\theta), \theta \in \Theta\}$ is said to be *stochastically increasing* in θ if $F(x;\theta_1) \ge F(x;\theta_2)$ whenever $\theta_1 < \theta_2$, i.e.

2010 *Mathematics Subject Classification*: 60E15, 62F10, 62N05.
Key words and phrases: stochastic ordering, stochastic monotonicity, coupling, exponential distribution, estimation, reliability, censored samples.

DOI: 10.4064/am38-4-7

$X(\theta_1) \le_{\text{st}} X(\theta_2)$, where $X(\theta_i)$ denotes a random variable with distribution $F(x;\theta_i)$, $i = 1, 2$.

DEFINITION 1.2. We say that the estimator $\hat{\gamma}$ of the function $\gamma(\theta)$, $\theta \in \Theta$, is *stochastically increasing* in θ if the family of the distributions of $\hat{\gamma}$ is stochastically increasing in θ.

Stochastic orders play an important role in statistics. In particular, stochastic monotonicity of estimators is useful in constructing confidence intervals and testing statistical hypotheses (see Shao [12, p. 475]).

2. Point estimation of the mean based on censored samples. Assume that n identical units are independently put on test at the initial time $t = 0$. Suppose that the failure time distribution is exponential with mean θ.

The estimation of the mean based on the complete sample $\mathbf{X}$ is the simplest one from the theoretical point of view but in practice often impossible because the units in the sample may not all have failed or the exact times to failure are not known. In the usual lifetime data analyses, lifetimes are not always fully observed. For example, some observations will be censored due to the limited time of the experiment or due to cost.

Below we describe the best known censoring schemes (see Gnedenko et al. [7], [8]).

1. *Plan* $[n, U, T]$: n identical units are placed under test, failed units are unrepairable during the testing. The test terminates at time T. This type of censoring is often called *Type I censoring*. The number of observed failures is a random variable $D = \#\{X_i \le T\}$ with the binomial distribution $b(n, p)$, where $p = 1 - e^{-T/\theta}$. The observations obtained are $X_{1:n}, \ldots, X_{D:n}$. The MLE of θ is

$$\hat{\theta} = \frac{1}{D}\Big(\sum_{i=1}^{D} X_{i:n} + (n - D)T\Big), \tag{2.1}$$

provided $D \ge 1$, and it does not exist if $D = 0$.

2. *Plan* $[n, U, r]$: n identical units are placed under test, failed units are unrepairable during the testing. The test terminates at the time of the rth failure. This type of censoring is often called *Type II censoring*. In this case, the MLE of θ is given by

$$\begin{aligned}\hat{\theta} &= \frac{1}{r}\bigg(\sum_{i=1}^{r} X_{i:n} + (n - r)X_{r:n}\bigg) \\ &= \frac{1}{r}\sum_{i=1}^{r}(n - i + 1)(X_{i:n} - X_{i-1:n}) = \frac{1}{r}\sum_{i=1}^{r} W_i,\end{aligned} \tag{2.2}$$

where $W_i = (n-i+1)(X_{i:n} - X_{i-1:n})$, $i = 1, \ldots, r$, are independent and identically distributed exponential random variables with mean θ (we put $X_{0:n} = 0$). The estimator $\hat{\theta}$ is also the uniformly minimum variance unbiased estimator (UMVUE) of θ. When $r = n$ we have the experiment based on the complete sample, and the estimator $\hat{\theta}$ given by (2.2) reduces to the sample mean of $\mathbf{X}$.

3. *Plan* $[n, R, T]$: n identical units are placed under test, failed units are immediately renewed after failure. The test terminates at time T. This type of censoring is often called *Type I censoring with replacement*. In this case the times of failure are signals of a Poisson process with intensity n/θ. The MLE of θ has the form

$$\hat{\theta} = \frac{nT}{D}, \tag{2.3}$$

provided $D \geq 1$, and it does not exist if $D = 0$. The number D of failures has the Poisson distribution with parameter nT/θ, and the intervals between successive failures are independent and identically distributed exponential random variables with mean θ/n. This estimator $\hat{\theta}$ is a function of the sufficient and complete statistic D.

4. *Plan* $[n, R, r]$: n identical units are placed under test, failed units are immediately renewed after failure. The test terminates at the time $X_{(r)}$ of the rth failure. This type of censoring is often called *Type II censoring with replacement*. In this case, the MLE of θ is given by

$$\hat{\theta} = \frac{n}{r} X_{(r)} = \frac{n}{r} \sum_{i=1}^{r} (X_{(i)} - X_{(i-1)}) = \frac{n}{r} \sum_{i=1}^{r} \tilde{W}_i, \tag{2.4}$$

where $X_{(1)}, \ldots, X_{(r)}$ is the sequence of failure times and $\tilde{W}_i = X_{(i)} - X_{(i-1)}$, $i = 1, \ldots, n$, are independent and identically distributed exponential random variables with mean θ/n (we put $X_{(0)} = 0$). The estimator $\hat{\theta}$ is also the UMVUE of θ.

5. *Plan* $[n, U, (r, T)]$: n identical units are placed under test, failed units are unrepairable during the testing. The test terminates at time $\min\{X_{r:n}, T\}$, where $X_{r:n}$ is the time of occurrence of the rth failure. This type of censoring is often called *hybrid censoring*. In this case, the MLE of θ is of the form

$$\hat{\theta} = \begin{cases} \dfrac{1}{D}\Big(\displaystyle\sum_{i=1}^{D} X_{i:n} + (n-D)T\Big) & \text{if } D = 1, \ldots, r-1, \\[2ex] \dfrac{1}{r}\Big(\displaystyle\sum_{i=1}^{r} X_{i:n} + (n-r)T\Big) & \text{if } D \geq r. \end{cases} \tag{2.5}$$

The random variable D has the binomial distribution $b(n, p)$, where $p = 1 - e^{-T/\theta}$.

6. *Plan* $[n, R, (r, T)]$: n identical units are placed under test, failed units are immediately renewed after failure. The test terminates at time $\min\{X_{(r)}, T\}$, where $X_{(r)}$ denotes the time of occurrence of the rth failure. This type of censoring is often called *hybrid censoring with replacement.* In this case, the MLE of θ is of the form

$$\hat{\theta} = \begin{cases} nT/D & \text{if } D = 1, \ldots, r-1, \\ nX_{(r)}/r & \text{if } D \geq r. \end{cases} \tag{2.6}$$

The number D of failures has the Poisson distribution with parameter nT/θ.

7. *Plan* $[n, U, (r, HS_0)]$: n identical units are placed under test, failed units are unrepairable during the testing. Recall the definition of $S(t)$ calculated as the total testing time accumulated until time t,

$$S(t) = nX_{1:n} + (n-1)(X_{2:n} - X_{1:n}) + \cdots + (n-D)(t - X_{D:n}), \tag{2.7}$$

where D is the number of failures that occur before time t and $X_{1:n}, \ldots, X_{D:n}$ is the sequence of failure times. The test terminates at time t_0 if $S(t_0) = S_0$ or at the time of the rth failure if $S(X_{r:n}) < S_0$. The MLE of θ is given by

$$\hat{\theta} = \begin{cases} S_0/D & \text{if } D = 1, \ldots, r-1, \\ S(X_{r:n})/r & \text{if } D \geq r. \end{cases} \tag{2.8}$$

Here, D has the Poisson distribution with parameter T/θ.

It is easy to verify that estimators (2.3), (2.2) and (2.4) are stochastically increasing in θ, since the families of the distributions of the estimators have monotone likelihood ratio.

3. Methods of proving stochastic monotonicity. We consider two methods of proving stochastic monotonicity. One is based on the coupling method, and the other on the so-called Three Monotonicities Lemma.

Coupling method. Note that if the random variables X and Y satisfy $P(X \leq Y) = 1$, then $X \leq_{\text{st}} Y$. The following theorem gives an important characterization of the usual stochastic order (see Shaked [11] or Lindvall [9]).

THEOREM 3.1. *$X \leq_{\text{st}} Y$ if and only if there exist random variables X' and Y' defined on the same probability space such that $X' =_{\text{st}} X$, $Y' =_{\text{st}} Y$ and $P(X' \leq Y') = 1$.*

We often use the following version of coupling.

THEOREM 3.2. *If $X \leq_{\text{st}} Y$ then there exists a random variable $Z =_{\text{st}} X$ such that $P(Z \leq Y) = 1$.*

Three Monotonicities Lemma. Balakrishnan and Iliopoulos [3] considered the case when the MLE of θ has the survival function of the form

$$P(\hat{\theta} > x) = \sum_{d\in\mathcal{D}} P(D = d)P(\hat{\theta} > x \mid D = d), \tag{3.1}$$

where $\mathcal{D}$ is some finite set of natural numbers whereas D is a random variable denoting the number of failures. They established a lemma concerning the stochastic monotonicity of such estimators, which proves the required monotonicity of the MLEs in all the above mentioned censoring schemes.

The following lemma gives conditions under which the family of the distributions of the estimator $\hat{\theta}$ is stochastically increasing in θ.

LEMMA 3.3 ([3]). *Suppose that the estimator $\hat{\theta}$ has the survival function* (3.1) *and the following assumptions are satisfied:*

(M1) $P_\theta(\hat{\theta} > x \mid D = d)$ *is increasing in θ for all x and $d \in \mathcal{D}$,*
(M2) $P_\theta(\hat{\theta} > x \mid D = d)$ *is decreasing in $d \in \mathcal{D}$ for all x and θ,*
(M3) *D is stochastically decreasing in θ.*

Then the estimator $\hat{\theta}$ is stochastically increasing in θ.

Balakrishnan et al. [2] proved via the coupling method that the estimator $\hat{\theta}$ given by (2.1) is stochastically increasing in θ. In the next section, we prove this property for the estimator obtained from a more general version of the test plan $[n, U, T]$ studied by Bartholomew [4].

Using the Three Monotonicities Lemma Balakrishnan and Iliopoulos [3] proved that the estimator given by (2.5) is stochastically increasing in θ. In the next section we prove analogous theorems for the MLEs obtained from the plans $[n, R, (r, T)]$ and $[n, U, (r, HS_0)]$. Proving this property for the plans $[n, U, r]$, $[n, R, T]$ and $[n, R, r]$ is easy, because the MLEs have distributions with monotone likelihood ratio.

4. Main results

4.1. Stochastic monotonicity of estimators of the mean. In this section we consider test plans which are not considered in [2] and [3]. We start from a more general version of the test plan $[n, B, T]$ (see Bartholomew [4]).

Let T be a time when we stop the experiment and let $T_i \le T$ denote the time that has elapsed since the ith element was installed. Define a random variable C_i as follows:

$$C_i = \begin{cases} 1 & \text{if the } i\text{th element has failed before time } T, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we are observing $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{c} = (c_1, \ldots, c_n)$. If the lifetime of each element is a random variable with cumulative distribution function

$F(x;\theta) = 1 - e^{-x/\theta}$ then the likelihood function for this plan is

$$(4.1)\qquad L(\theta;\mathbf{x},\mathbf{c},T_1,\ldots,T_n) = \prod_{i=1}^{n} \frac{1}{\theta^{c_i}} \exp\biggl(-\frac{c_i x_i}{\theta} - \frac{(1-c_i)T_i}{\theta}\biggr).$$

Note that C_i has the binomial distribution $b(1, F(T_i))$, and $D = \sum_{i=1}^{n} C_i$ is the number of failures.

From (4.1), we get the MLE of θ

$$(4.2)\qquad \hat{\theta} = \frac{1}{D}\sum_{i=1}^{n}(C_i X_i + (1-C_i)T_i) \qquad \text{if } D > 0.$$

Observe that if $T_i = T$, $i = 1,\ldots,n$, i.e. all elements start to work simultaneously, then we have the estimator (2.1).

We now prove the following theorem.

THEOREM 4.1. *The MLE of θ given by* (4.2) *is stochastically increasing in θ.*

Proof. We use the coupling method. Let $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be random samples from the exponential distributions $\mathrm{Ex}(\theta_1)$ and $\mathrm{Ex}(\theta_2)$ respectively, where $\theta_1 < \theta_2$. Suppose that $\hat{\theta}_1$ is the MLE of θ_1, and similarly $\hat{\theta}_2$ is the MLE of θ_2.

Let $D_1 = \sum_{i=1}^{n} C_i^X$, where $C_i^X = \mathbf{1}_{[0,T_i]}(X_i)$, be a random variable denoting the number of failures on the sample $\mathbf{X}$. Similarly, $D_2 = \sum_{i=1}^{n} C_i^Y$, $C_i^Y = \mathbf{1}_{[0,T_i]}(Y_i)$.

We know that

$$\hat{\theta}_1 = \frac{1}{D_1}\sum_{i=1}^{n}(C_i^X X_i + (1-C_i^X)T_i),$$

$$\hat{\theta}_2 = \frac{1}{D_2}\sum_{i=1}^{n}(C_i^Y Y_i + (1-C_i^Y)T_i).$$

By the coupling method there exist independent random variables $Z_1,\ldots,Z_n$ such that $Z_i =_{\mathrm{st}} X_i$ and $Z_i \le Y_i$ pointwise for $i = 1,\ldots,n$. Hence

$$\hat{\theta}_1 =_{\mathrm{st}} \hat{\theta}_1^{\star} = \frac{1}{D_1^{\star}}\sum_{i=1}^{n}(C_i^Z Z_i + (1-C_i^Z)T_i),$$

where $D_1^{\star} = \sum_{i=1}^{n} C_i^Z = \sum_{i=1}^{n} \mathbf{1}_{[0,T_i]}(Z_i)$. Since $Z_i \le Y_i$ pointwise, we have $\{Y_i \le T_i\} \subseteq \{Z_i \le T_i\}$ and therefore the following inequalities hold pointwise:

$$C_i^Z \ge C_i^Y, \qquad C_i^Z Z_i + (1-C_i^Z)T_i \le C_i^Y Y_i + (1-C_i^Y)T_i, \qquad i = 1,\ldots,n,$$
$$D_1^{\star} \ge D_2.$$

It is obvious that $\hat{\theta}_1^\star \le \hat{\theta}_2$ pointwise. Finally $\hat{\theta}_1 \le_{\rm st} \hat{\theta}_2$, since

$$P(\hat{\theta}_1 \ge x) = P(\hat{\theta}_1^\star \ge x) \le P(\hat{\theta}_2 \ge x).$$

THEOREM 4.2. *The MLE of θ given by* (2.6) *is stochastically increasing in θ.*

Proof. The proof is by verifying the assumptions of Lemma 3.3.

(M1) We show that the distribution of $[\hat{\theta} \mid D = d]$ does not depend on θ. If $d \in \{1, \ldots, r-1\}$ this follows immediately from (2.6). Otherwise, we know that $[X_{(r)} \mid D = d]$ has the same distribution as $U_{r:d}$, i.e. the rth order statistic from a sample of size d from the uniform distribution on $(0, T)$ (see, for example, Bain and Engelhardt [1, p. 99]).

(M2) We must show that $[\hat{\theta} \mid D = d]$ is decreasing in d. If $d \le r-2$ then

$$[\hat{\theta} \mid D = d] - [\hat{\theta} \mid D = d+1] = \frac{nT}{d} - \frac{nT}{d+1} > 0.$$

Let $d = r-1$. Then

$$[\hat{\theta} \mid D = d] - [\hat{\theta} \mid D = d+1] = \frac{nT}{r-1} - \frac{nX_{(r)}}{r} = \frac{n}{r(r-1)}[rT - (r-1)X_{(r)}] > 0.$$

The last inequality holds with probability one, hence the condition (M2) is satisfied. Finally, if $d \ge r$ then

$$[\hat{\theta} \mid D = d] =_{\rm st} \frac{nU_{r:d}}{r}, \qquad [\hat{\theta} \mid D = d+1] =_{\rm st} \frac{nU_{r:d+1}}{r}.$$

Since $U_{r:d} \ge_{\rm st} U_{r:d+1}$, we conclude that here (M2) also holds.

(M3) It is obvious that $D_1 \ge_{\rm st} D_2$ if $\theta_1 \le \theta_2$. This completes the proof.

THEOREM 4.3. *The MLE of θ given by* (2.8) *is stochastically increasing in θ.*

Proof. It is well known that the quantities $nX_{1:n}$, $(n-1)(X_{2:n} - X_{1:n})$, $\ldots, (n-r+1)(X_{r:n} - X_{r-1:n})$ are independent and identically distributed exponential random variables with mean θ, which actually means that we are observing a Poisson process with intensity $1/\theta$. This problem was considered above for the plan $[n, R, (r, T)]$, where $n := 1$, $X_{(r)} := S(X_{r:n})$, $T := S_0$. So the theorem follows from Theorem 4.2.

Recall that the estimator $\hat{\theta}$ given by (2.3) is not defined if $D = 0$. Consider now the following modification of that estimator:

$$\tilde{\theta} = \begin{cases} \hat{\theta} & \text{if } D \ge 1, \\ nT & \text{if } D = 0. \end{cases} \tag{4.3}$$

We now prove the following theorem.

THEOREM 4.4. *The estimator of θ given by* (4.3) *is stochastically increasing in θ.*

Proof. Note the following facts:

(1) the random variable $Z = I\{D \geq 1\}$ is stochastically decreasing in θ,
(2) $[\tilde{\theta} \,|\, Z = 1]$ is stochastically increasing in θ (Balakrishnan et al. [2]) and the same is true for $[\tilde{\theta} \,|\, Z = 0]$,
(3) $[\tilde{\theta} \,|\, Z = 1] \leq_{\mathrm{st}} [\tilde{\theta} \,|\, Z = 0]$, since $\hat{\theta} \leq nT$ with probability one.

Applying Lemma 3.3 and the above facts ends the proof.

Similarly, we can put $\hat{\theta} = nT$ if $D = 0$ for the estimators (2.4), (2.6). Analogously, we put $\hat{\theta} = S_0$ for the estimator given by (2.8) and $\hat{\theta} = \sum_{i=1}^{n} T_i$ for the estimator given by (4.1) if $D = 0$. These new estimators are also stochastically increasing in θ. We omit the proof because it is very similar to the proof of Theorem 4.4.

4.2. Stochastic monotonicity of estimators of reliability. We now consider the estimation of reliability $R(t) = e^{-t/\theta}$, i.e. the probability that the lifetime of an element is not less than a given time t. It is obvious that the MLE of $R(t)$ is $\hat{R}(t) = e^{-t/\hat{\theta}}$, where $\hat{\theta}$ is the MLE of θ. We observe that the function $R(t)$ is increasing in θ for any fixed $t > 0$ and the usual stochastic order is closed under any increasing operation. So we have the following theorem.

THEOREM 4.5. *The MLEs of reliability obtained on the basis of one of the following plans:* $[n, U, T]$, $[n, U, r]$, $[n, R, T]$, $[n, R, r]$, $[n, U, (r, T)]$, $[n, R, (r, T)]$, $[n, U, (r, S_0)]$ *are stochastically increasing in* θ.

Finally, we discuss the stochastic monotonicity of the UMVUEs of reliability. The minimum variance unbiased estimation of reliability was investigated by many authors (see, for example, Basu [6], Pugh [10]). In the cases described below this estimator is an increasing function of $\hat{\theta}$.

EXAMPLE 4.6. The UMVUE of reliability on the basis of the plan $[n, U, r]$ is

$$\hat{R}(t) = \left(1 - \frac{t}{r\hat{\theta}}\right)_{+}^{r-1}, \tag{4.4}$$

where $\hat{\theta}$ is given by (2.2) and $(a)_{+} = \max\{a, 0\}$. Hence the estimator (4.4) is stochastically increasing in θ.

EXAMPLE 4.7. The UMVUE of reliability on the basis of the plan $[n, R, T]$ is

$$\hat{R}(t) = \left(1 - \frac{t}{nT}\right)_{+}^{D}, \tag{4.5}$$

where D has the Poisson distribution with parameter nT/θ. Hence the estimator (4.5) is stochastically increasing in θ.

EXAMPLE 4.8. The UMVUE of reliability on the basis of the plan $[n, R, r]$ is

$$\hat{R}(t) = \left(1 - \frac{t}{r\hat{\theta}}\right)_+^{r-1}, \tag{4.6}$$

where $\hat{\theta}$ is given by (2.4). Hence the estimator (4.6) is stochastically increasing in θ.

Now, let us consider the test plan $[n, U, T]$. The sufficient statistic for θ is $(D(T), S(T))$, where $D(T)$ denotes the number of failures until time T and $S(T)$ is the accumulated observed lifetime of all items. We cannot construct the UMVUE of reliability, since the sufficient statistic is not complete (see Bartoszewicz [5]). However, we can give an unbiased estimator of reliability based on the empirical distribution function, for example

$$\hat{R}(t) = \begin{cases} 1 - \dfrac{D(t)}{n} & \text{if } t \in (0, T], \\ \left(1 - \dfrac{D(T)}{n}\right) \cdots \left(1 - \dfrac{D(t - pT)}{n - p}\right) & \text{if } t \in (pT, (p+1)T], \end{cases} \tag{4.7}$$

where $p = 1, \ldots, n-1$ and $D(t)$ denotes the number of failures until time t, $0 < t \le T$.

THEOREM 4.9. *The unbiased estimator of reliability at $t \in (0, nT]$ given by (4.7) is stochastically increasing in θ.*

Proof. First, we prove that the vector $(D(T), D(t))$ is stochastically decreasing in θ, i.e. $(D_{\theta_2}(T), D_{\theta_2}(t)) \le_{\rm st} (D_{\theta_1}(T), D_{\theta_1}(t))$ if $\theta_1 < \theta_2$. It is sufficient to verify the following conditions (see Veinott [13]):

(a) $D_{\theta_2}(T) \le_{\rm st} D_{\theta_1}(T)$,
(b) $[D_{\theta_2}(t) \mid D_{\theta_2}(T) = d_2] \le_{\rm st} [D_{\theta_1}(t) \mid D_{\theta_1}(T) = d_1]$,

whenever $d_2 < d_1$ and $\theta_1 < \theta_2$.

These follow immediately, since

(1) $D_\theta(t)$ has the binomial distribution $b(n, p_0)$, where $p_0 = 1 - e^{-t/\theta}$,
(2) $[D_\theta(t) \mid D_\theta(T) = d]$ has the binomial distribution $b(d, p_1)$, where $p_1 = (1 - e^{-t/\theta})/(1 - e^{-T/\theta})$.

The functions p_0 and p_1 are decreasing in θ if $0 < t \le T$, so conditions (a) and (b) are satisfied. Finally, note that the estimator (4.7) is a decreasing function of $D(t)$, $t \in (0, T]$, and also a decreasing function of $D(T)$ and $D(t - pT)$, $t \in (pT, (p+1)T]$, $p = 1, \ldots, n-1$. This completes the proof of the theorem.

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Received on 7.2.2011 (2075)