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NEW AXIOMATIZATIONS OF VALUES OF TU-GAMES USING REDUCTION PROPERTIES

Abstract. We propose new axiomatizations of values of cooperative games where traditional properties connected with special players (dummy, null or zero) are replaced with weaker properties relating to such participants of the game. We assume that the change of payoff of a player when combining the game with another game where this player is special is constant. Using such axioms with an additional assumption that a value is odd and—if necessary—the fairness axioms holds, one can obtain axiomatizations without additivity where not only classical dummy, null or zero players axioms but even equal treatment can be redundant. These properties are used to construct new axiomatizations of the Shapley, Banzhaf and Deegan–Packel values. Some of them contain a new mirror game axiom.

1. Introduction. Axiomatizations of values of cooperative games, constructed from the beginning of development of this theory by Shapley (1953), are based on various consistent collections of practically useful axioms. They concern games both in their classical forms (cf. Lehrer (1988); Van den Brink (2001)) as well as their variations defined using additional restrictions or special assumptions, often connected with possible preliminary decisions made by some players or with change of rules of coalition creation. Most important examples of the latter class of models are: games with a priori unions (e.g. Owen (1977, 1981); Albizuri (2008); Gómez-Rúa and Vidal-Puga (2010)), games with coalition structure (e.g. Aumann and Drèze (1974)), games in partition function form (e.g. Pham Do and Norde (2007); Cheng-Cheng and Yang (2010)) or games with weights of players (e.g. Radzik (2012); Radzik et al. (1997)).

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Despite a variety of axiomatic characterizations, they exhibit some common features. Firstly, many axiomatizations contain the additivity axiom. However, in a few cases it was perceived as being slightly controversial and some alternatives were proposed. For example, Young (1985) and Nowak (1997) considered the marginal contribution axiom, Van den Brink (2001) used the fairness property, and Malawski (2002) the neutrality of some operators on the space of games. On the other hand, well-known proofs of axiomatizations usually have similar structures. Namely, using some subset of the proposed collection of axioms it is proved that the analyzed value is uniquely determined on games forming a basis of the space of cooperative games; next, by using additivity or its alternatives, it is shown that this uniqueness is also valid for any game (taking into account that every game can be uniquely represented by a linear combination of games in the basis).

The basis consists of basic (also called unity or Dirac) games (as in the case of the Deegan–Packel value, cf. Deegan and Packel (1979), or the least square prenucleolus, cf. Ruiz et al. (1996)) or unanimity games (used to characterize the Shapley and Banzhaf value; cf. Shapley (1953), Lehrer (1988), Khmelnitskaya (2003), Młodak (2003, 2005, 2007) or Albizuri (2008)). In general, the choice of the basis depends on whether the value is assumed to satisfy the zero-player or dummy (null) player property. These axioms state that the payoff for a player which has no or very small importance in the game is also insignificant (i.e. it equals the power of the player itself as a singleton or zero). The dummy and null player properties are effectively used if the proof is based on unanimity games. The zero-player property is applicable mainly if basic (unity) games are considered. Moreover, the dummy player and the zero-player properties cannot practically be considered jointly (in non-trivial games no player can be simultaneously dummy and zero).

Moreover, it is worth noting that concrete, arbitrarily established levels of a value for a special player in these axioms can be redundant: instead, a better (and much more general) solution seems to be to assume that being a special player of some game should not diversify contribution to his payoff when combining this game with another one. We will call this a special (dummy, null or zero) player reduction property. This approach has several advantages. Firstly, it shows that each of these types of special players can have similar importance for a cooperative game value. Secondly, this axiom is independent of the classical dummy player, null player and zero player properties. This independence will be kept even when we assume in addition that a value is odd. These two properties are so useful that one can build valuable axiomatizations using them. We will show that the respective reduction property can effectively replace additivity. Moreover, we use the fairness and reduction properties to show that if a value is odd then the classical symmetry, special player axiom and additivity are no longer necessary.

The second aim of this research is to find some axiomatizations of the Shapley and Banzhaf values which can be proved using the fundamental basis of the game space, i.e. the basic games. All the collections of axioms used in both cases contain the mirror game axiom: if the power of a given player as a singleton in a given game is zero, then the value of this player is opposite to the value for its mirror game (i.e. such that—with respect to the original game—the powers of coalitions with it and without it are interchanged). We also obtain an axiomatization of the Deegan–Packel value containing the fairness axiom.

The paper is organized as follows. First (Section 2) we recall basic assumptions and properties of cooperative games with their values. In Section 3 some basic axioms commonly used in the literature are presented. Next, in Section 4, we introduce new axioms concerning special types of players and study their most important features. Section 5 contains axiomatizations of some values (generalized versions of the Shapley, Banzhaf and Deegan–Packel values) based on the new properties. Some of these theorems (relating to the first two values) contain the mirror game axiom and their proofs are based on the basic games. Finally, the independence of the newly proposed axioms is briefly discussed.

2. Preliminaries. Now, we recall the basic definitions of cooperative game theory. Let n be a fixed natural number ([1]). An n-person *transferable utility game* (briefly, a *TU-game* or a *cooperative game*) is defined by a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players (the *grand coalition*) and $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ is called the *characteristic function* of the game. The value $v(S)$ is called the *worth* (or *power*) of the coalition $S \subset 2^N$. Therefore, if N is fixed, a TU-game (N, v) can be uniquely identified with v. The cardinality of $S \subseteq N$ will be denoted by $|S| = s$. If $s = 1$, i.e. the coalition contains only one player, it is called a *singleton*. Two players $i, j \in N$, $i \neq j$, are said to be *symmetric* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \setminus \{i, j\}$. A player $i \in N$ is *dummy* if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for every $S \subseteq N \setminus \{i\}$, and *a null player* if $v(S \cup \{i\}) = v(S)$ for every $S \subseteq N \setminus \{i\}$. If $v(S) = 0$ whenever $i \in S$, then i is said to be a *zero-player* in v.

Let G_N be the set of all n-person games. The *zero* game $\underline{\mathbf{0}} \in G_N$ is the trivial game with $\underline{\mathbf{0}}(S) = 0$ for any $S \subseteq N$. The sum of $v, w \in G_N$,

([1]) From the practical point of view, we are only interested in games with a non-trivial set of players, i.e. with $n \geq 2$. However, for some formal reasons, we will sometimes also refer to the situation when $n = 1$.

$v + w \in G_N$, is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. The product of these games, $v \cdot w \in G_N$, is defined by $(v \cdot w)(S) = v(S) \cdot w(S)$ for all $S \subseteq N$. Similarly, if $v \in G_N$ and a is a nonzero real number, then we define the games $a \cdot v \in G_N$ and $v^a \in G_N$ by $(a \cdot v)(S) = a \cdot v(S)$ and $(v^a)(S) = (v(S))^a$, for all $S \subseteq N$.

Fix any $T \subseteq N$, $T \neq \emptyset$. The *basic* (or *unity* or *Dirac*) game $\omega_T \in G_N$ is defined by $\omega_T(S) = 1$ if $S = T$ and $\omega_T(S) = 0$ if $S \neq T$, for every $S \subseteq N$. The *unanimity* game $u_T \in G_N$ is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, for every $S \subseteq N$.

A *value* $\varphi(v) = (\varphi_1(v), \ldots, \varphi_n(v))$ on G_N is thought of as a vector-valued mapping $\varphi : G_N \to \mathbb{R}^n$, which uniquely determines, for each game $v \in G_N$, the distribution of the total wealth available to all the players $1, \ldots, n$, through their participation in the game v. Thus, the number $\varphi_i(v)$ represents the *payoff* (*outcome*) of player i in the game v on N.

We now recall three classical values for cooperative games, the Shapley value (Shapley (1953)), the Banzhaf value (Banzhaf III (1965)) and the Deegan–Packel value (Deegan and Packel (1979)). They will be the main object of our study in the next sections.

Definition 1. The *Shapley value* of player $i \in N$ in a game $v \in G_N$ is defined as

$$\mathrm{Sh}_i(v) = \sum_{S \subseteq N} \frac{s!(n - s - 1)!}{n!}(v(S \cup \{i\}) - v(S)). \tag{2.1}$$

Definition 2. The *Banzhaf value* of player $i \in N$ in a game $v \in G_N$ is defined as

$$B_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} (v(S \cup \{i\}) - v(S)). \tag{2.2}$$

Definition 3. The *Deegan–Packel value* of player $i \in N$ in a game $v \in G_N$ is defined as

$$\mathrm{DV}_i(v) = \sum_{S \subseteq N,\, i \in S} \frac{v(S)}{s}. \tag{2.3}$$

3. Standard axioms and classical results. Below we present most popular axioms used to construct the known axiomatizations of values (such as the Shapley value, Banzhaf value and Deegan–Packel value) which will be useful for us.

Axiom 1 (Efficiency, EF). A value φ on G_N satisfies the *efficiency axiom* if for any game $v \in G_N$,

$$\sum_{i=1}^{n} \varphi_i(v) = v(N).$$

This axiom states that for any $v \in G_N$, $\varphi(v)$ should be an allocation of the power of the grand coalition in v between all players.

Axiom 2 (Quasi-efficiency, QE). A value φ on G_N satisfies *quasi-efficiency* if for any game $v \in G_N$,

$$\sum_{i=1}^{n} \varphi_i(v) = \sum_{S \subseteq N} v(S).$$

This property means that for any $v \in G_N$, $\varphi(v)$ is an allocation of the sum of the powers of all coalitions between all players.

Axiom 3 (Dummy player, DP). A value φ on G_N satisfies the *dummy player axiom* if for any dummy player $i \in N$ of $v \in G_N$ we have $\varphi_i(v) = v(\{i\})$.

Thus, the payoff of a player which contributes nothing to any coalition except for itself as singleton (in this case the contribution is expressed by the worth of this singleton indicated by the characteristic function of the game) coincides with the power of this singleton.

Axiom 4 (Null player, NP). A value φ on G_N satisfies the *null player axiom* if for any null player $i \in N$ of $v \in G_N$ we have $\varphi_i(v) = 0$.

This axiom states that a player contributing nothing to all coalitions gets no payoff.

Axiom 5 (Zero-player, ZP). A value φ on G_N satisfies the *zero-player axiom* if for any zero-player $i \in N$ of $v \in G_N$ we have $\varphi_i(v) = 0$.

That is, if a player is a zero-player, i.e. if the power of any coalition containing it amounts to zero, then the payoff of this player is also zero.

The next axiom is connected with the notion of amalgamation of two players introduced by Lehrer (1988).

Definition 4. *Amalgamation* of two different players $i, j \in N$ in an n-person game v is a transformation of the game v into an $(n-1)$-person game $v_{(ij)}$ with the set of players $(N \setminus \{i,j\}) \cup \{p\}$, where p denotes a player represented by the coalition $\{i,j\}$. The characteristic function of the latter game is defined by

$$v_{(ij)}(S) := \begin{cases} v(S) & \text{if } p \notin S, \\ v((S \setminus \{p\}) \cup \{i,j\}) & \text{if } p \in S, \end{cases}$$

for any set $S \subseteq (N \setminus \{i,j\}) \cup \{p\}$.

For a better description of the next notion, it is necessary to consider the set of all games with grand coalition contained in N,

$$\tilde{G}_N = \bigcup_{T \subseteq N} G_T. \tag{3.1}$$

Of course, in the same way as for G_N, using (3.1) we can define a value $\varphi : \tilde{G}_N \to \bigcup_{k=1}^{n} \mathbb{R}^k$ of a game belonging to any of the aforementioned classes.

Axiom 6 (Amalgamation, AM). A value φ on $\tilde{G}_N$ satisfies the *amalgamation axiom* if for any $i, j \in N$, $i \neq j$, and $v \in \tilde{G}_N$, $\varphi_p(v_{(ij)}) = \varphi_i(v) + \varphi_j(v)$.

This axiom states that if a pair of players merges into a new player then the sum of their payoffs should be equal to the payoff of the new player which represents them.

The next classical axiom is the equal treatment property.

Axiom 7 (Equal Treatment, ET). A value φ on G_N satisfies the *equal treatment axiom* if for any players $i, j \in N$, $i \neq j$, symmetric in v, $\varphi_i(v) = \varphi_j(v)$.

That is, the payoffs of symmetric players should coincide.

We will pay special attention to the well-known notion of additivity of a value (e.g. Shapley (1953)):

Axiom 8 (Additivity, AD). A value φ on G_N satisfies *additivity* if

$$\varphi(v + w) = \varphi(v) + \varphi(w) \tag{3.2}$$

for any two games $v, w \in G_N$.

Because, according to some researchers, the additivity axiom is slightly controversial, our research will focus on some properties which can replace it. Van den Brink (2001) proposed the following fairness property instead of additivity.

Axiom 9 (Fairness, FA). If $i, j \in N$, $i \neq j$, are symmetric in a game $w \in G_N$ then for any game $v \in G_N$, $\varphi_i(v + w) - \varphi_i(v) = \varphi_j(v + w) - \varphi_j(v)$.

That is, if a game $v \in G_N$ is combined with another game where two players are symmetric then their payoffs change by the same amount.

Nowadays, there exist various axiomatizations of the Shapley, Banzhaf and Deegan–Packel values. We now recall some of them. Useful axiomatizations using additivity are as follows:

Proposition 1 (Shapley (1953); Young (1985)). *A value on G_N satisfies EF, ET, DP and AD if and only if it is the Shapley value.*

Proposition 2 (Lehrer (1988)). *A value on $\tilde{G}_N$ satisfies AM, ET, DP and AD if and only if it is the Banzhaf value.*

Proposition 3 (Deegan and Packel (1979)). *A value on G_N satisfies QE, ET, ZP and AD if and only if it is the Deegan–Packel value.*

Below we recall three other axiomatizations obtained recently, using the fairness property instead of additivity:

Proposition 4 (Van den Brink (2001)). *A value on G_N satisfies EF, DP and FA if and only if it is the Shapley value.*

Proposition 5 (Młodak (2005)). *A value on $\tilde{G}_N$ satisfies AM, DP and FA if and only if it is the Banzhaf value.*

Proposition 6 (Młodak (2005)). *A value on G_N satisfies QE, ZP and FA if and only if it is the Deegan–Packel value.*

It is worth noting that in all these theorems, axioms for special players, e.g. DP or ZP, are used. Next we will show how to formulate other, independent properties based on such types of players, and how to use them in some new axiomatizations without AD.

4. New reduction properties. The classical additivity axiom is often perceived as controversial because it imposes practically no possibility of benefit from summing two games, which can be interpreted as bundling some different goods. Therefore several attempts to replace it with some other, more 'natural' properties, have been made. Young (1985) and Nowak (1997) have used in this context the marginal contribution axiom: whenever we have two games $v, w \in G_N$ such that the marginal contribution (with respect to any given coalition) of a given player is the same under v and w then the payoffs of this player for both games are the same. Later, Van den Brink (2001) introduced fairness, and Malawski (2002) used some other axioms such as the representation property stating that the sum of the payoffs of two players is equal to the payoff of their representative in a certain representation game.

To propose new axioms which could be used instead of additivity, we can look more closely at the axioms concerning special types of players: dummy, null and zero, i.e. DP, NP and ZP. All of them assume that for such specific players a value should also establish a special, strictly defined, payoff. However, it also seems of interest to consider how combining any game with another game, where a player is dummy, zero or null, affects the payoff of this player. More precisely, we should consider the difference between the payoff of this player when combining both these games and before this action.

We assume here that this difference is constant for every game in G_N, i.e. if a player $i \in N$ is special in $w \in G_N$ then

$$\varphi_i(u+w) - \varphi_i(u) = \varphi_i(z+w) - \varphi_i(z) \quad \text{for any } u, z \in G_N. \tag{4.1}$$

In other words, for any $v \in G_N$ the difference $\varphi_i(v+w) - \varphi_i(v)$ does not depend on v. Formally, our new axioms can be formulated as follows.

Axiom 10 (Dummy player reduction property, DR). A value φ on G_N satisfies the *dummy player reduction property* if for any dummy player $i \in N$ of a game $w \in G_N$, (4.1) holds.

Axiom 11 (Null player reduction property, NR). A value φ on G_N satisfies the *null player reduction property* if for any null player $i \in N$ of a game $w \in G_N$, (4.1) holds.

Axiom 12 (Zero-player reduction property, ZR). A value φ on G_N satisfies the *zero-player reduction property* if for any zero-player $i \in N$ of a game $w \in G_N$, (4.1) holds.

These axioms reflect the natural intuition that combining two games can provide extra contribution to the payoff of a given player only if he is neither dummy, null nor zero in both games. It is worth underlining that DR, NR and ZR axioms are independent of DP, NP and ZP, respectively. To see this, we consider the following examples.

Example 1. (a) The value φ on G_N given by

$$\varphi_i(v) = \left(\frac{\sum_{S\subseteq N}\left(\frac{v(S\cup\{i\})-v(S)}{s+1}\right)^3}{\sum_{s=0}^{n-1}\frac{\binom{n-1}{s}}{(s+1)^3}}\right)^{1/3} \quad \text{for } i \in N \text{ and } v \in G_N$$

satisfies DP but not DR.

(b) DV satisfies DR but not DP.

Example 2. (a) The value φ on G_N given by

$$\varphi_i(v) = \left(\sum_{S\subseteq N}\left(\frac{(v(S\cup\{i\})-v(S))v(S\cup\{i\})}{s}\right)^3\right)^{1/3} \quad \text{for } i\in N \text{ and } v\in G_N$$

satisfies NP but not NR.

(b) DV satisfies NR but not NP.

Example 3. (a) the value φ on G_N given as

$$\varphi_i(v) = \sum_{S\subseteq N:\, i\in S}\frac{v(S)}{s}\sum_{S\subseteq N:\, S\neq\emptyset,\, i\notin S}\frac{v(S)}{s} \quad \text{for } i \in N \text{ and } v \in G_N$$

satisfies ZP but not ZR.

(b) Sh satisfies ZR but not ZP.

The property DR (respectively NR and ZR) is also weaker than additivity. It is clear that any value on G_N which satisfies AD must also satisfy DR, NR and ZR, but there exist values satisfying these axioms which are not additive. For example, the value

$$\varphi_i(v) = \sum_{S\subseteq N}\left(\frac{v(S\cup\{i\})-v(S)-v(\{i\})}{s}\right)^2 \quad \text{for } i \in N \text{ and } v \in G_N$$

satisfies DR but it is not additive. Similarly

$$\varphi_i(v) = \sum_{S \subseteq N} \left(\frac{v(S \cup \{i\}) - v(S)}{s} \right)^2 \quad \text{for } i \in N \text{ and } v \in G_N$$

satisfies NR but not AD, and

$$\varphi_i(v) = \sum_{S \subseteq N} \frac{v^2(S)}{s} \quad \text{for } i \in N \text{ and } v \in G_N$$

satisfies ZR but not AD.

We will now introduce one more interesting axiom.

Axiom 13 (Oddity, OD). A value φ on G_N satisfies the *oddity axiom* (or is *odd*) if $\varphi(-v) = -\varphi(v)$ for any $v \in G_N$.

It is a classical postulate to require a value to be antisymmetric, which is motivated by obvious antisymmetry of the contribution of a player to a coalition which he enters to. In this context, the following question is straightforward: does there exist a connection between these new axioms and those well-known properties? The following lemma gives a specific answer.

Lemma 1. *Let φ be a value on G_N. If φ satisfies FA, OD and DR then it satisfies ET. Moreover, DR can be here replaced with NR or ZR.*

Proof. Assume that φ satisfies FA, OD and DR. Let $i, j \in N$, $i \neq j$, be two symmetric players in a game $v \in G_N$. Of course, there exists a non-zero game $w \in G_N$ such that i, j are dummy players in w. Therefore, by DR, we have

$$\varphi_i(v+w) - \varphi_i(v) = \varphi_i(u+w) - \varphi_i(u), \tag{4.2}$$

$$\varphi_j(v+w) - \varphi_j(v) = \varphi_j(u+w) - \varphi_j(u) \tag{4.3}$$

for every $u \in G_N$. Taking $u := v - w$ in (4.2) and (4.3) we obtain

$$\varphi_i(v+w) + \varphi_i(v-w) = 2\varphi_i(v),$$
$$\varphi_j(v+w) + \varphi_j(v-w) = 2\varphi_j(v).$$

Hence, by FA and OD, taking into account that if i, j are symmetric in v then they are also symmetric in $2v$, we have

$$\begin{aligned} 2\varphi_i(v) &= \varphi_i(v+w) + \varphi_i(v-w) = \varphi_i(w-v+2v) - \varphi_i(w-v) \\ &= \varphi_j(w-v+2v) - \varphi_j(w-v) = \varphi_j(v+w) + \varphi_j(v-w) = 2\varphi_j(v). \end{aligned}$$

Therefore $\varphi_i(v) = \varphi_j(v)$ and hence φ satisfies ET. If DR is replaced with NR or ZR then the proof is analogous. ■

Remark 1. It is worth noting that we have not used here, in fact, any special features of dummy, null and zero players, respectively. This means that one can assume a more general reduction axiom: if $i \in N$ then there exists a family of games $\Gamma_i \subseteq G_N$, $\Gamma_i \neq \emptyset$, such that for every $w \in \Gamma_i$ and all $u, v \in G_N$ the equality (4.1) is satisfied.

REMARK 2. Reduction properties with OD do not imply special player properties. That is, DR with OD does not imply DP (counterexample: DV), NR with OD does not imply NP (counterexample: DV), and ZR with OD does not imply ZP (counterexample: Sh).

5. Main results. In this section we give our six main results (Theorems 1–6). They describe new axiomatizations of the Shapley, Banzhaf and Deegan–Packel values. They are based on three versions of the reduction property (Axioms 10–12), without using the additivity property.

THEOREM 1. *The only value φ on G_N satisfying EF, ET, DP and DR is the Shapley value.*

Proof. First notice that the Shapley value satisfies the four properties mentioned in the theorem (EF, ET and DP by Proposition 1, and DR by the additivity of Sh). So, to complete the proof it suffices to show that for any $v \in G_N$ and $j \in N$ the value $\varphi_j(v)$ is uniquely determined.

Let φ be a value on G_N satisfying EF, ET, DP and DR. The axioms EF, ET and DP uniquely determine $\varphi(v)$ for any game of the form $v = c \cdot u_S$ where $c \in \mathbb{R}$ and u_S is the unanimity game of $S \subseteq N$ (see Shapley (1953, Lemma 2)). Using this, we will show that $\varphi(v)$ is uniquely determined for any $v \in G_N$. To this end, we will use the known fact that any $v \in G_N$ can be uniquely represented as a linear combination of unanimity games in the form

$$v = \sum_{S \subseteq N} \eta_S u_S,$$

where $\eta_S \in \mathbb{R}$, $S \subseteq N$.

Let $\Xi(v) := \{S : \eta_S \neq 0\}$. We use induction on the cardinality $|\Xi(v)|$. If $|\Xi(v)| = 0$ then v is the zero game $\underline{\mathbf{0}}$ and by DP, $\varphi_i(v) = 0$ for any $i \in N$. If $|\Xi(v)| = 1$ then $v = \eta_S u_S$ for some $S \subseteq N$, $S \neq \emptyset$, and $\eta_S \neq 0$. Consequently, by the above, $\varphi(v)$ is uniquely determined for any game $v \in G_N$ with $|\Xi(v)| = 1$. Assume that $\varphi(v)$ is uniquely determined for any game $v \in G_N$ with $|\Xi(v)| \leq k-1$ for some natural number k satisfying $2 \leq k < 2^n - 1$ (the induction assumption). Fix a player $j \in N$ and a game v with $|\Xi(v)| = k$ and define $\Delta(v) := \{i \in N : \exists S \subseteq N, i \notin S, \eta_S \neq 0\}$. Since $k > 1$, $\Delta(v) \neq \emptyset$. We consider two cases.

CASE 1.1: $j \in \Delta(v)$. Then there are $T \subseteq N$, $T \neq \emptyset$, such that $j \notin T$ and $\eta_T \neq 0$. Obviously, player j is dummy in the game $\eta_T u_T$. Hence, by DR,

$$\varphi_j(v) - \varphi_j(v - \eta_T u_T) = \varphi_j(\eta_T u_T) \tag{5.1}$$

(put $z = \underline{\mathbf{0}}$, $w = \eta_T u_T$ and $u = v - \eta_T u_T$ in (4.1)). The value $\varphi_j(\eta_T u_T)$ is uniquely determined, as stated earlier. On the other hand, since we have $|\varXi(v - \eta_T u_T)| = k - 1$, the induction assumption implies that $\varphi_j(v - \eta_T u_T)$ is uniquely determined. Hence, by (5.1), $\varphi_j(v)$ is uniquely determined.

Case 1.2: $j \notin \varDelta(v)$. Then any two players in $N \setminus \varDelta(v)$ are symmetric, whence, by ET, $\varphi_p(v) = \varphi_j(v)$ for every $p \in N \setminus \varDelta(v)$. Therefore, the EF property can be written in the form $\sum_{i\in\varDelta(v)} \varphi_i(v) + |N \setminus \varDelta(v)|\varphi_j(v) = v(N)$. But this, by Case 1.1, immediately implies that $\varphi_j(v)$ is also uniquely determined. ■

Corollary 1. *The Shapley value is uniquely determined by the axioms EF, ET, NP and NR.*

Proof. Indeed, the Shapley value satisfies EF, ET (by Proposition 1), NP (easily seen from (2.1)) and NR (by the additivity of Sh).

Conversely, assume that φ is a value on G_N satisfying EF, ET, NP and NR. It is easy to observe that any player $j \notin S$ is a null player in the unanimity game u_S and that the axioms EF, ET and NP uniquely determine $\varphi(v)$ for any game of the form $v = c \cdot u_S$, where $c \in \mathbb{R}$. Moreover, if $v = \underline{\mathbf{0}}$ then, by NP, $\varphi_i(v) = 0$ for any $i \in N$. Now, we easily check that the "$\Leftarrow$" part of the proof of Theorem 1 can be repeated, replacing there "dummy player", DP and DR by "null player", NP and NR, respectively. ■

Corollary 2. *The Shapley value is uniquely determined by the axioms EF, FA, OD and DR.*

Proof. Of course, the Shapley value satisfies the four properties: EF, FA (by Proposition 4), OD (easily seen from (2.1)) and DR (by the additivity of Sh).

Conversely, let φ be a value on G_N satisfying EF, FA, OD and DR. We will show that in this case FA, OD and DR imply DP for unanimity games. Let u_S be the unanimity game of $S \subseteq N$, $S \neq \emptyset$ and $i \in N$. Define the game $u_S^{(i)} \in G_N$ in the following way: $u_S^{(i)}(K) = u_S(K)$ if $i \notin K$ and $u_S^{(i)}(K) = u_S(K \setminus \{i\})$ if $i \in K$ for every $K \subseteq N$. Note that if $S \subseteq N \setminus \{i\}$ then $u_S^{(i)}$ and u_S are identical. Of course, only players belonging to $N \setminus S$ are dummy in the game u_S. Let then $S \subseteq N$ and $j \in N \setminus S$. Then j is dummy in u_S. Therefore, by OD, $\varphi_j(\underline{\mathbf{0}}) = \varphi_j(-\underline{\mathbf{0}}) = -\varphi_j(\underline{\mathbf{0}})$, which implies that $\varphi_j(\underline{\mathbf{0}}) = 0 = \varphi_j(u_S - u_S^{(j)})$. Hence, by FA and DR (putting $u = -u_S^{(j)}$, $w = u_S$ and $z = \underline{\mathbf{0}}$ in (4.1)) we obtain

$$\begin{aligned}\varphi_j(u_S - u_S^{(j)}) - \varphi_j(u_S) = \varphi_j(-u_S^{(j)} + u_S) - \varphi_j(-u_S^{(j)}) &= \varphi_j(u_S) - \varphi_j(\underline{\mathbf{0}}) \\ &= \varphi_j(u_S).\end{aligned}$$

Thus, $2\varphi_j(u_S) = 0$ and $2\varphi_j(u_S) = 0 = u_S(\{j\})$. Note also that, because φ satisfies FA, OD and DR, by Lemma 1, φ satisfies ET. Thus, taking the aforementioned statements into account and repeating the reasoning in the proof of Theorem 1 (where the EF property of φ is also used), we conclude that $\varphi_j(v)$ is uniquely determined for any $j \in N$. ∎

THEOREM 2. *The only value on $\tilde{G}_N$ satisfying AM, ET, DP and DR is the Banzhaf value.*

Proof. First note that the Banzhaf value satisfies the axioms AM, ET and DP (by Proposition 2) and DR (by the additivity of B).

Conversely, let φ be a value on $\tilde{G}_N$ satisfying AM, ET, DP and DR. The axioms AM, ET and DP uniquely determine $\varphi(v)$ for any game of the form $v = c \cdot u_S$, where $c \in \mathbb{R}$ and $S \subseteq N$. This is proved in Nowak (1997, Step 1 in the proof of Theorem).

We keep the notation of the proof of Theorem 1. We will apply induction on the number of players, $|N|$. If $|N| = 1$, i.e. if $N = \{1\}$, then $v = c \cdot u_{\{1\}}$ for some $c \in \mathbb{R}$ and hence, as stated earlier, $\varphi(v)$ is uniquely determined. Assume that $\varphi(v)$ is uniquely determined for any game $v \in G_N$ with $|N| \leq n-1$ for some $n \geq 2$ (the first induction assumption). Let $|N| = n$ and $v \in G_N$. Now we will use induction on $|\Xi(v)|$. If $|\Xi(v)| = 0$ then $v = \underline{\mathbf{0}}$ and by DP, $\varphi_i(v) = 0$ for any $i \in N$. If $|\Xi(v)| = 1$ then $v = \eta_S u_S$ for some $S \subseteq N$, $S \neq \emptyset$, and $\eta_S \neq 0$. Consequently, by the above, $\varphi(v)$ is uniquely determined for any $v \in G_N$ with $|\Xi(v)| = 1$.

Now assume that $\varphi_i(v)$ is uniquely determined for any game $v \in G_N$ with $|\Xi(v)| \leq k-1$ for some natural number k satisfying $2 \leq k < 2^n - 1$ (the second induction assumption). Fix $j \in N$ and v with $|\Xi(v)| = k$. Since $k > 1$, $\Delta(v) \neq \emptyset$. We consider two cases.

CASE 2.1: $j \in \Delta(v)$. Repeating the reasoning in the proof of Theorem 1, Case 1.1, and using the second induction assumption we conclude by (5.1) that $\varphi_j(v)$ is uniquely determined.

CASE 2.2: $j \notin \Delta(v)$. By ET, any two players belonging to $N \setminus \Delta(v)$ are symmetric, i.e. $\varphi_l(v) = \varphi_j(v)$ for every $l \in N \setminus \Delta(v)$. Amalgamate then player j with a player $i \in \Delta(v)$. By AM we have

$$\varphi_i(v) + \varphi_j(v) = \varphi_p(v_{(ij)}).$$

Then, by Case 2.1, $\varphi_i(v)$ is uniquely determined. On the other hand, the first induction assumption (concerning $|N|$) implies that $\varphi_p(v_{(ij)})$ is also uniquely determined. Therefore $\varphi_j(v)$ is uniquely determined as well. ∎

COROLLARY 3. *The Banzhaf value is uniquely determined by the axioms AM, ET, NP and NR.*

Proof. Indeed, the Banzhaf value satisfies these four axioms: AM and ET by Proposition 2, NR by the additivity of B, and NP immediately follows from (2.2).

Conversely, let φ be a value on $\tilde{G}_N$ satisfying AM, ET, NP and NR. It is easy to observe that any player $j \notin S$ is a null player in u_S and (using the same approach as in Nowak (1997) indicated in the proof of Theorem 2) we conclude that the axioms AM, ET and NP uniquely determine $\varphi(v)$ for any game of the form $v = c \cdot u_S$, where $c \in \mathbb{R}$ and $S \subseteq N$. Moreover, if $v = \underline{\mathbf{0}}$ then, by NP, $\varphi_i(v) = 0$ for any $i \in N$. Now, we can repeat the reasoning in the proof of Theorem 2 and using the fact that in Case 2.1 player j is a null player in the game $\eta_T u_T$ (which, by NR, also implies (5.1)), we conclude that $\varphi_j(v)$ is uniquely determined for any $j \in N$. ■

COROLLARY 4. *The Banzhaf value is uniquely determined by the axioms AM, FA, OD and DR.*

Proof. Of course, the Banzhaf value satisfies the four axioms: AM (by Proposition 2), OD (easily seen from (2.2)), FA (by Proposition 5), and DR (by the additivity of B).

Conversely, we know (cf. the proof of Corollary 2) that OD and DR imply DP for unanimity games. Moreover, by Lemma 1, because $\varphi(v)$ satisfies FA, OD and DR, it also satisfies ET. Thus, taking the aforementioned statements into account and repeating the reasoning in the proof of Theorem 2 (where the AM property of φ is also used), we conclude that $\varphi_j(v)$ is uniquely determined for any $j \in N$. ■

In the next two theorems other axiomatizations of the Shapley and Banzhaf values are given. They do not include additivity or the dummy player axiom. Both axiomatizations use a new mirror game axiom.

DEFINITION 5. Let $i \in N$ and $v \in G_N$. The *mirror game* $v_i \in G_N$ is defined by

$$v_i(S) := \begin{cases} v(S \setminus \{i\}) & \text{if } i \in S, \\ v(S \cup \{i\}) & \text{if } i \notin S, \end{cases}$$

for any $S \subseteq N$.

That is, the construction of the mirror game is based on an exchange of power between coalitions containing and not containing player i for which the power of i as a singleton amounts to zero. We can now formulate the following axiom:

AXIOM 14 (Mirror game, MG). A value φ on G_N satisfies the *mirror game axiom* if $\varphi_i(v_i) = -\varphi_i(v)$ for any $v \in G_N$ and $i \in N$ with $v(\{i\}) = 0$.

The above axiom says that the sum of a player's payoffs in the original game and in the corresponding mirror game is equal to 0 when his power (as a singleton) in the original game is 0.

Now we have the following theorem.

THEOREM 3. *There exists a unique value on G_N satisfying EF, ET, MG and ZR. It is the Shapley value.*

Proof. It is clear that the Shapley value satisfies all the axioms mentioned in the theorem (EF and ET by Proposition 1, ZR by the additivity of Sh and MG is easily seen from (2.1)).

Conversely, let φ be a value on G_N satisfying EF, ET, MG and ZR. We will prove that φ is uniquely determined for any game $a \cdot \omega_S$ where $a \in \mathbb{R}$ and of $S \subseteq N$. Fix $S \subseteq N$. Note that any two players belonging to S, or to $N \setminus S$, are symmetric. Thus, by ET, we have $\varphi_i(a \cdot \omega_S) = \varphi_j(a \cdot \omega_S) := c_S$ for any $i, j \in S \subseteq N$ and $\varphi_i(a \cdot \omega_S) = \varphi_j(a \cdot \omega_S) := c_{N \setminus S}$ for any $i, j \in N$, $i, j \notin S \subseteq N$, where $c_S, c_{N\setminus S} \in \mathbb{R}$ are constants depending only on S. Therefore the EF property can be written in the form

$$|S| \cdot c_S + (n - |S|) \cdot c_{N\setminus S} = \omega_S(N). \tag{5.2}$$

We will apply backward induction on $|S|$. If $|S| = n$ (i.e. $S = N$) then $n - |S| = 0$, $\omega_N(N) = 1$ and, by (5.2), $\varphi_i(a \cdot \omega_N) = c_N = a/n$ for any $i \in N$. Therefore $\varphi(v)$ is uniquely determined for $v = a \cdot \omega_S$ with $|S| = n$. Let $s \in \mathbb{N}$, $1 \le s \le n - 1$. Assume that $\varphi(v)$ is uniquely determined for any game of the form $a \cdot \omega_S$ with $|S| \ge s + 1$ (the backward induction assumption). Let $|S| = s$. Then, by MG and ET, for $i \notin S$ we have $c_{N\setminus S} = \varphi_i(a \cdot \omega_S) = -\varphi_i((a \cdot \omega_S)_i) = -\varphi_i(a \cdot \omega_{S\cup\{i\}})$. Therefore, because $|S \cup \{i\}| = s + 1$, the induction hypothesis implies that $c_{N\setminus S}$ is uniquely determined. Hence, by (5.2), so is c_S. Thus, we have shown that $c_S = \varphi_j(a \cdot \omega_S)$ for $j \in N$ is uniquely determined.

Now we will show that $\varphi(v)$ is uniquely determined for any game $v \in G_N$. To this end, we use the well-known fact that any game $v \in G_N$ is uniquely represented as a linear combination of basic games of the form

$$v = \sum_{S\subseteq N} \lambda_S \omega_S, \quad \text{where} \quad \lambda_S = v(S) \text{ for } S \subseteq N.$$

Let $\tilde{\Xi}(v) := \{S \subseteq N : \lambda_S \neq 0\}$. We will apply induction on $|\tilde{\Xi}(v)|$. If $|\tilde{\Xi}(v)| = 0$ then $v = \underline{\mathbf{0}}$ and by EF and ET, $\varphi_i(v) = 0$ for any $i \in N$. If $|\tilde{\Xi}(v)| = 1$ then $v = \lambda_S \omega_S$ for some $S \subseteq N$, $S \neq \emptyset$, and $\lambda_S \neq 0$. Consequently, by the above, $\varphi(v)$ is uniquely determined for any $v \in G_N$ with $|\tilde{\Xi}(v)| = 1$.

Assume that φ is uniquely determined for any $v \in G_N$ with $|\tilde{\Xi}(v)| \le k-1$ for some k satisfying $2 \le k < 2^n - 1$ (the induction assumption). Fix $j \in N$ and v with $|\tilde{\Xi}(v)| = k$. Let $\tilde{\Delta}(v) := \{i \in N : \exists S \subseteq N,\, i \notin S,\, \lambda_S \neq 0\}$. Since $k > 1$, $\tilde{\Delta}(v) \neq \emptyset$. We consider two cases.

Case 3.1: $j \in \tilde{\Delta}$. Then there are $T \subseteq N$, $T \neq \emptyset$ such that $j \notin T$ and $\lambda_T \neq 0$. Obviously, j is the zero-player in the game $\lambda_T \omega_T$. Hence, by ZR,

$$\varphi_j(v) - \varphi_j(v - \lambda_T \omega_T) = \varphi_j(\lambda_T \omega_T) \tag{5.3}$$

(put $z = \underline{\mathbf{0}}$, $w = \lambda_T \omega_T$ and $u = v - \lambda_T \omega_T$ in (4.1)).

The value $\varphi_j(\lambda_T \omega_T)$ is uniquely determined, as stated earlier. On the other hand, since $|\tilde{\Xi}(v - \lambda_T \omega_T)| \leq k - 1$, the induction hypothesis implies that $\varphi_j(v - \lambda_T \omega_T)$ is also uniquely determined. Hence, by (5.3), $\varphi_j(v)$ is uniquely determined.

Case 3.2: $j \notin \tilde{\Delta}$. Then any two players in $N \setminus \tilde{\Delta}(v)$ are symmetric, whence, by ET, $\varphi_j(v) = \varphi_p(v)$ for every $p \in N \setminus \tilde{\Delta}(v)$. Therefore, the EF property can be written in the form

$$\sum_{i \in \tilde{\Delta}(v)} \varphi_i(v) + |N \setminus \tilde{\Delta}(v)| \varphi_j(v) = v(N). \tag{5.4}$$

But this, by Case 3.1, immediately implies that $\varphi_j(v)$ is uniquely determined. ∎

The next theorem is a "modification" of Theorem 3, where axiom EF is replaced by AM and the following new, natural axiom TR.

Axiom 15 (Triviality, TR). A value φ on $\tilde{G}_N$ satisfies the *triviality axiom* if the following two conditions are satisfied:

(1) $\varphi_i(\underline{\mathbf{0}}) = 0$ for every player i in the zero game $\underline{\mathbf{0}} \in \tilde{G}_N$;
(2) $\varphi_i(v) = v(\{i\})$ for every one-person game v, that is, with $N = \{i\}$.

Theorem 4. *A value on $\tilde{G}_N$ satisfies AM, ET, MG, TR and ZR if and only if it is the Banzhaf value.*

Proof. First, the Banzhaf value satisfies AM, ET (by Proposition 2), ZR (by the additivity of B), MG and TR (easily seen from (2.2)).

Conversely, let φ be a value on $\tilde{G}_N$ satisfying AM, ET, MG, TR and ZR. We will prove that φ is uniquely determined for any game of the form $a \cdot \omega_S$ where $a \in \mathbb{R}$ and $S \subseteq N$, $S \neq \emptyset$. We will apply forward induction on $|N|$, and backward induction on $|S|$. If $|N| = 1$ then, by TR, $\varphi(v)$ is uniquely determined. Therefore we can assume that $\varphi(v)$ is uniquely determined for any game v of the form $v = a \cdot \omega_S$, where $a \in \mathbb{R}$ and $S \subseteq N$ with $|N| \leq q - 1$ for some $q \geq 2$ (the forward induction assumption). Fix a game v with $|N| = q$. Let now $|S| = q$. Then $S = N$ and, by ET, $\varphi_i(a \cdot \omega_N) = \varphi_j(a \cdot \omega_N)$ for any $i, j \in N$. Amalgamate two players $i, j \in N, i \neq j$. Hence, by AM, we have

$$2\varphi_i(a \cdot \omega_N) = \varphi_i(a \cdot \omega_N) + \varphi_j(a \cdot \omega_N) = \varphi_p((a \cdot \omega_N)_{(ij)}).$$

Since $(a \cdot \omega_N)_{(ij)}$ is a $q-1$-person game, the forward induction hypothesis implies that $\varphi_p((a \cdot \omega_N)_{(ij)})$ is uniquely determined. Therefore, $\varphi_i(a \cdot \omega_N)$ is uniquely determined for any $i \in N$. Let $S \subseteq N$, $1 \le s \le q-1$. Suppose now that $\varphi(v)$ is uniquely determined for any game of the form $v = a \cdot \omega_S$ with $|S| \ge s+1$ (the backward induction assumption). Now let $|S| = s$. Thus, by ET, we have $\varphi_i(a \cdot \omega_S) = \varphi_j(a \cdot \omega_S) := c_S$ for any $i, j \in S \subseteq N$ and $\varphi_i(a \cdot \omega_S) = \varphi_j(a \cdot \omega_S) := c_{N\setminus S}$ for any $i, j \notin S \subseteq N$ where $c_S, c_{N\setminus S} \in \mathbb{R}$ are constants depending only on S. Because $|S| < q$, $N \setminus S \neq \emptyset$. Amalgamate any two players i, j such that $i \in S$ and $j \notin S$. Hence, by AM, we have

$$c_S + c_{N\setminus S} = \varphi_p(a \cdot (\omega_S)_{(ij)}). \tag{5.5}$$

Because, by TR, $\varphi_p((a \cdot \omega_S)_{(ij)}) = \varphi_p(\underline{\mathbf{0}}) = 0$, the right-hand side of (5.5) is uniquely determined. On the other hand, by MG, we have $c_{N\setminus S} = \varphi_j(a \cdot \omega_S) = -\varphi_j((a \cdot \omega_S)_j) = -\varphi_j(a \cdot \omega_{S\cup\{j\}})$. The backward induction hypothesis implies that $c_{N\setminus S}$ is uniquely determined. Hence, by (5.5), so is c_S. Thus, we have shown that $\varphi_j(a \cdot \omega_S)$ is uniquely determined for any $j \in N$.

Let $\tilde{\varXi}(v)$ and $\tilde{\varDelta}(v)$ be as in the proof of Theorem 3. We will apply induction on $|N|$. If $|N| = 1$, i.e. if $N = \{1\}$, then $v = c \cdot \omega_{\{1\}}$ for some $c \in \mathbb{R}$ and $\varphi(v)$ is uniquely determined. Let $q \ge 2$. Assume that $\varphi(v)$ is uniquely determined for any game $v \in G_N$ with $|N| \le q-1$ for some $q \ge 2$ (the first induction assumption). Now fix a game v with $|N| = q$.

Next, we will apply induction on $|\tilde{\varXi}(v)|$. If $|\tilde{\varXi}(v)| = 0$ then $v = \underline{\mathbf{0}}$ and, by TR, $\varphi_i(v) = 0$ for any $i \in N$. If $|\tilde{\varXi}(v)| = 1$ then $v = \lambda_S \omega_S$ for some $S \subseteq N$, $S \neq \emptyset$ and $\lambda_S \neq 0$. Consequently, by the above, $\varphi_i(v)$ is uniquely determined for any v with $|\tilde{\varXi}(v)| = 1$.

Assume that $\varphi_i(v)$ is uniquely determined for any game v with $|\tilde{\varXi}(v)| \le k-1$ for some natural number k satisfying $2 \le k < 2^q - 1$ (the second induction assumption). Fix a game v with $|\tilde{\varXi}(v)| = k$. Since $k > 1$, $\tilde{\varDelta}(v) \neq \emptyset$. Now fix $j \in N$. We consider two cases.

CASE 4.1: $j \in \tilde{\varDelta}(v)$. This is analogous to Case 3.1 in the proof of Theorem 3. Taking (5.3) and the second induction assumption into account and repeating the relevant implications we conclude that $\varphi_j(v)$ is uniquely determined.

CASE 4.2: $j \in N \setminus \tilde{\varDelta}(v)$. Amalgamate player j with a player $i \in \tilde{\varDelta}(v)$. Hence, by AM, we have

$$\varphi_i(v) + \varphi_j(v) = \varphi_p(v_{(ij)}).$$

Then, by Case 4.1, $\varphi_i(v)$ is uniquely determined. On the other hand, the first induction assumption (concerning $|N|$) implies that $\varphi_p(v_{(ij)})$ is also uniquely determined. Therefore $\varphi_j(v)$ is uniquely determined for all $j \in N$. ∎

One can also formulate an axiomatization for the Deegan–Packel value using ZR.

THEOREM 5. *There exists a unique value on G_N satisfying QE (Axiom 2), ET, ZP and ZR. It is the Deegan–Packel value.*

Proof. First, the Deegan–Packel value satisfies QE, ET, ZP (by Proposition 3) and ZR (by the additivity of DV).

Assume that φ is a value on G_N satisfying QE, ET, ZP and ZR. The axioms QE, ET and ZP uniquely determine $\varphi(v)$ for any game of the form $a \cdot \omega_S$ where $a \in \mathbb{R}$ and $S \subseteq N$; this was proved by Deegan and Packel (1979, proof of Theorem 1).

We keep the notation of the proof of Theorem 3. We will apply induction on $|\tilde{\Xi}(v)|$. If $|\tilde{\Xi}(v)| = 0$ then $v = \underline{\mathbf{0}}$ and by QE and ET, $\varphi_i(v) = 0$ for any $i \in N$. If $|\tilde{\Xi}(v)| = 1$ then $v = \lambda_S \omega_S$ for some $S \subseteq N$, $S \neq \emptyset$, and $\lambda_S \neq 0$. Consequently, $\varphi(v)$ is uniquely determined for any $v \in G_N$ with $|\tilde{\Xi}(v)| = 1$.

Assume that φ is uniquely determined for any game $v \in G_N$ with $|\tilde{\Xi}(v)| \le k-1$ for some k satisfying $2 \le k < 2^n - 1$ (the induction assumption). Fix $j \in N$ and v with $|\tilde{\Xi}(v)| = k$. Since $k > 1$, $\tilde{\Delta}(v) \neq \emptyset$. Thus, if $j \in \tilde{\Delta}(v)$ then, repeating the relevant reasoning of Case 3.1 we conclude that $\varphi_j(v)$ is uniquely determined. Let now $j \in N \setminus \tilde{\Delta}(v)$. This is similar to Case 3.2, except for (5.4). That is, instead of EF, $\varphi(v)$ now satisfies QE, which can be written in the form

$$\sum_{i \in \tilde{\Delta}(v)} \varphi_i(v) + |N \setminus \tilde{\Delta}(v)| \varphi_j(v) = \sum_{S \subseteq N} v(S) \tag{5.6}$$

for $j \in N \setminus \tilde{\Delta}(v)$. Hence, $\varphi_j(v)$ is uniquely determined for all $j \in N$. ∎

In our last theorem we give another axiomatization of the Deegan–Packel value, showing that Theorem 5 remains true when the axioms ET and ZP are replaced with FA and OD.

THEOREM 6. *A value on G_N satisfies QE, ZR, OD and FA if and only if it is the Deegan–Packel value.*

Proof. First, the Deegan–Packel value satisfies QE (by Proposition 3), FA (by Proposition 6), ZR (by the additivity of DV) and OD (an immediate consequence of (2.3)).

Conversely, let φ be a value on G_N satisfying QE, ZR, OD and FA. By Lemma 1, ZR, OD and FA imply ET. But QE and ET immediately imply that $\varphi_j(\underline{\mathbf{0}}) = 0$ for every $j \in N$. Fix $i \in N$ and $S \subseteq N \setminus \{i\}$. Define $\omega_S^{(i)} \in G_N$ such that $\omega_S^{(i)}(K) = \omega_S(K)$ if $i \in K$ and $\omega_S^{(i)}(K) = \omega_S(K \cup \{i\})$ if $i \notin K$. Since $\omega_S^{(i)}$ is the zero game, $\omega_S^{(i)} - \omega_S = -\omega_S$. Moreover, i is the zero-player in ω_S. Hence, by QE, ZR and OD, putting $u = -\omega_S^{(i)}$, $w = \omega_S$ and $z = \underline{\mathbf{0}}$ in (4.1), we have

$$\varphi_i(\omega_S - \omega_S^{(i)}) - \varphi_i(\omega_S) = \varphi_i(-\omega_S^{(i)} + \omega_S) - \varphi_i(\omega_S) = \varphi_i(-\omega_S) - \varphi_i(\underline{\mathbf{0}}) = 0.$$

Hence $\varphi_i(\omega_S) = 0$. Thus, φ satisfies ZP for basic games. Therefore, as in the proof of Theorem 5, $\varphi(v)$ is uniquely determined for any game of the form $v = c \cdot \omega_S$, where $c \in \mathbb{R}$ and $S \subseteq N$.

Let $\tilde{\Xi}(v)$ and $\tilde{\Delta}(v)$ be as in the proof of Theorem 3. We will use induction on $|\tilde{\Xi}(v)|$. If $|\tilde{\Xi}(v)| = 0$ then $v = \mathbf{\underline{0}}$ and $\varphi_i(v) = 0$ for any $i \in N$. If $|\tilde{\Xi}(v)| = 1$ then $v = \lambda_S \omega_S$ where $\lambda_S \in \mathbb{R}, \eta_S \neq 0$ and $S \subseteq N, S \neq \emptyset$. In this case $\varphi(\lambda_S \omega_S)$ is, of course, uniquely determined.

Assume that $\varphi(v)$ is uniquely determined for any game $v \in G_N$ such that $|\tilde{\Xi}(v)| \leq k-1$ for some k satisfying $2 \leq k < 2^n - 1$ (the induction assumption). Let $v \in G_N$ with $|\tilde{\Xi}(v)| = k$. Since $k > 1$, $\tilde{\Delta}(v) \neq \emptyset$. Now fix $j \in N$. We consider two cases.

Case 6.1: $j \in \tilde{\Delta}(v)$. This is similar to Case 3.1. There exists a subset $S \subseteq N$ such that $j \notin S$ and $\lambda_S \neq 0$. Then j is a zero-player in the game $\lambda_S \omega_S$. By ZR we again obtain (5.3). Of course, $\varphi_j(\lambda_S \omega_S)$ is uniquely determined. On the other hand, since $|\tilde{\Xi}(v - \lambda_S \omega_S)| \leq k-1$, the induction hypothesis implies that $\varphi_j(v - \lambda_S \omega_S)$ is uniquely determined. Hence, by (5.3), $\varphi_j(v)$ is uniquely determined.

Case 6.2: $j \notin \tilde{\Delta}(v)$. Any two players in $N \setminus \tilde{\Delta}(v)$ are symmetric and φ satisfies ET. Therefore $\varphi_l(v) = \varphi_j(v)$ for every $l \in N \setminus \tilde{\Delta}(v)$. Thus, the QE property implies (5.6). But this, by Case 6.1, immediately implies that $\varphi_j(v)$ is also uniquely determined for all $j \in N$ and $v \in G_N$, completing the proof. ∎

6. Concluding remarks. We conclude this paper with several examples of values in the context of different groups of axioms and their possible independence.

- The value
$$\varphi_i(v) = v(N)/n, \qquad i = 1, \ldots, n,$$
satisfies EF, ET, OD, ZR, NR and DR but not MG.
- The value
$$\varphi_i(v) = v(N)/2^{n-1} + B_i(v), \qquad i = 1, \ldots, n,$$
satisfies AM, ET, OD, ZR, NR and DR but not MG.
- The value
$$\varphi_i(v) = \begin{cases} v(\{i\}) & \text{if } i \text{ is dummy in } v, \\ \dfrac{\sum_{S \subseteq N, i \in S} v(S)}{\sum_{S \subseteq N} s v(S)} - \displaystyle\sum_{j \in \mathcal{D}(v)} v(\{j\}) & \text{otherwise,} \end{cases}$$
$i = 1, \ldots, n$, where $\mathcal{D}(v)$ denotes the set of dummy players in v, satisfies EF, ET, DP but not DR.

- The value
$$\varphi_i(v) = \begin{cases} 0 & \text{if } N = \{a, i\} \text{ and } v = \omega_{\{a\}}, \\ B_i(v) & \text{otherwise,} \end{cases} \qquad i = 1, \ldots, n,$$
satisfies AM, ET, DP but not DR.
- The value
$$\varphi_i(v) = \frac{\left(\sum_{S \subseteq N, i \in S} \frac{\sqrt{v(S)+1}}{s} - \sum_{k=1}^{n-1} \frac{\binom{n-1}{k}}{k}\right) \sum_{S \subseteq N} v(S)}{\sum_{S \subseteq N} \sqrt{v(S)+1} - 2^n \sum_{k=1}^{n-1} \frac{\binom{n-1}{k}}{k}},$$
$i = 1, \ldots, n$, satisfies QE, ET, ZP but not ZR.
- The value
$$\varphi_i(v) = \begin{cases} \mathrm{DV}_i(v) + v(\{1\}) & \text{if } i = 1, \\ \mathrm{DV}_i(v) - \dfrac{v(\{1\})}{n-1} & \text{if } i \neq 1, \end{cases} \qquad i = 1, \ldots, n,$$
satisfies QE, ZR, OD but not FA.

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