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# LINEAR-QUADRATIC DIFFERENTIAL GAMES: FROM FINITE TO INFINITE DIMENSION

*Abstract.* The object of this paper is the generalization of the pioneering work of P. Bernhard [J. Optim. Theory Appl. 27 (1979)] on two-person zero-sum games with a quadratic utility function and linear dynamics. It relaxes the semidefinite positivity assumption on the matrices in front of the state in the utility function and introduces affine feedback strategies that are not necessarily $L^2$-integrable in time. It provides a broad conceptual review of recent results in the finite-dimensional case for which a fairly complete theory is now available under most general assumptions. At the same time, we single out finite-dimensional concepts that do not carry over to evolution equations in infinite-dimensional spaces. We give equivalent notions and concepts. One of them is the *invariant embedding for almost all initial times*. Another one is the *structural closed loop saddle point*. We give complete classifications in terms of open loop values of the game and compare results.

**1. Introduction.** Two-person zero-sum games with a quadratic utility function and linear dynamics have been studied in the pioneering work of P. Bernhard [2] in 1979. He gave necessary and sufficient conditions for the existence of a closed loop saddle point for smooth feedback strategies with possible isolated singularities in time. He restricted his attention to utility functions where the matrices $F$ and $Q(t)$ in front of the state are positve semidefinite, which implies that the utility function is convex with respect to the control of the minimizing player.

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Next to the concept of closed loop saddle point, we have the open loop concepts of lower value, upper value, and value of the game. In 2005 P. Zhang [10] proved that if the upper and lower values of the game are finite, then they are equal and the game has a finite value. In that case we have an open loop saddle point that coincides with the notion of a Nash equilibrium. This work was done without the positivity assumptions on the matrices $F$ and $Q(t)$ and opened the way to other investigations.

In 2007 M. Delfour [5] gave a necessary and sufficient condition for the existence of a lower value of the game in terms of the usual coupled state-adjoint state system and some concavity-convexity conditions on the utility function without the positivity assumptions. By duality, those conditions extend to the upper value of the game, and, by combining the two, to the value of the game.

In a recent paper M. Delfour and O. Dello Sbarba [6] extended the work of [2] to the case of bounded measurable coefficients, symmetric matrices $F$ and $Q(t)$ that are not necessarily positive, and affine feedback strategies that are not necessarily $L^2$-integrable. Connections were made between the fundamental notions from the calculus of variations of *normality* and *normalizability* and *invariant embedding for all and almost all initial times*. The apparently new and equivalent concept of *structural feedback saddle point* was also introduced, but it was already present in the proofs of [2].

In this paper, we provide a broad conceptual review of the above cited literature where detailed proofs can be found. We focus our attention on the finite-dimensional case for which a fairly complete theory is available under the most general assumptions. At the same time we single out finite-dimensional concepts that do not carry over to evolution equations in infinite-dimensional spaces. We give equivalent notions and concepts. One of them is the invariant embedding for almost all initial times that turns out to have been observed for the Helmholtz equation of waveguides (cf. I. Champagne [4]). We give complete classifications in terms of open loop values of the game and compare results in the presence of an $L^2$-integrable and a possibly non-$L^2$-integrable closed loop saddle point.

## 2. Definitions and notation

**2.1.** *System, utility function, values of the game.* Given a finite-dimensional Euclidean space $\mathbb{R}^d$ of dimension $d \geq 1$, the *norm* and *inner product* will be denoted by $|x|$ and $x \cdot y$, irrespective of the dimension $d$ of the space. Given $T > 0$, the norm and inner product in $L^2(0,T;\mathbb{R}^n)$ will be denoted by $\|f\|$ and $(f,g)$. The norm in the Sobolev space $H^1(0,T;\mathbb{R}^n)$ will be written $\|f\|_{H^1}$.

Consider the following two-player zero-sum game over the finite time interval $[0,T]$ characterized by the quadratic *utility function*

$$C_{x_0}(u,v) \stackrel{\text{def}}{=} Fx(T)\cdot x(T) + \int_0^T (Q(t)x(t)\cdot x(t) + |u(t)|^2 - |v(t)|^2)\,dt, \tag{2.1}$$

where $x$ is the solution of the linear differential system

$$x'(t) = A(t)x(t) + B_1(t)u(t) + B_2(t)v(t) \quad \text{a.e. in } [0,T], \qquad x(0) = x_0, \tag{2.2}$$

$x_0 \in \mathbb{R}^n$ is the *initial state* at time $t=0$, $u \in L^2(0,T;\mathbb{R}^m)$, $m \geq 1$, is the strategy of the first player and $v \in L^2(0,T;\mathbb{R}^k)$, $k \geq 1$, is the strategy of the second player. We assume that $F$ is an $n\times n$-matrix and that $A$, $B_1$, $B_2$, and $Q$ are matrix-functions of appropriate orders that are measurable and bounded almost everywhere in $[0,T]$. Moreover $Q(t)$ and $F$ are symmetric. It will be convenient to use the following compact notation and drop the "a.e. in $[0,T]$" wherever no confusion arises:

$$C_{x_0}(u,v) = Fx(T)\cdot x(T) + \int_0^T (Qx\cdot x + |u|^2 - |v|^2)\,dt, \tag{2.3}$$

$$x' = Ax + B_1u + B_2v \quad \text{in } [0,T], \qquad x(0) = x_0. \tag{2.4}$$

The above assumptions on $F$, $A$, $B_1$, $B_2$, and $Q$ will be used throughout this paper. The transpose of a matrix $M$ will be denoted by $M^\top$, the inverse of its transpose by $M^{-\top}$, and $R(t)$ will denote the matrix $B_1(t)B_1(t)^\top - B_2(t)B_2(t)^\top$.

Definition 2.1. Let $x_0$ be an initial state in $\mathbb{R}^n$ at time $t=0$.

(i) The game is said to achieve its *open loop lower value* (resp. *upper value*) if

$$v^-(x_0) \stackrel{\text{def}}{=} \sup_{v\in L^2(0,T;\mathbb{R}^k)} \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v) \tag{2.5}$$

$$(\text{resp. } v^+(x_0) \stackrel{\text{def}}{=} \inf_{u\in L^2(0,T;\mathbb{R}^m)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v)) \tag{2.6}$$

is finite. By definition $v^-(x_0) \leq v^+(x_0)$.

(ii) The game is said to achieve its *open loop value* if its open loop lower value $v^-(x_0)$ and upper value $v^+(x_0)$ are achieved and $v^-(x_0) = v^+(x_0)$. The *open loop value* of the game will be denoted by $v(x_0)$.

(iii) A pair $(\bar u, \bar v)$ in $L^2(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{R}^k)$ is an *open loop saddle point* of $C_{x_0}(u,v)$ in $L^2(0,T;\mathbb{R}^m)\times L^2(0,T;\mathbb{R}^k)$ if for all $u$ in $L^2(0,T;\mathbb{R}^m)$ and all $v$ in $L^2(0,T;\mathbb{R}^k)$,

$$C_{x_0}(\bar u, v) \leq C_{x_0}(\bar u,\bar v) \leq C_{x_0}(u,\bar v). \tag{2.7}$$

DEFINITION 2.2. Associate with $x_0 \in \mathbb{R}^n$ the sets

$$V(x_0) \stackrel{\text{def}}{=} \left\{ v \in L^2(0,T;\mathbb{R}^k) : \inf_{u \in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v) > -\infty \right\}, \tag{2.8}$$

$$U(x_0) \stackrel{\text{def}}{=} \left\{ u \in L^2(0,T;\mathbb{R}^m) : \sup_{v \in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v) < +\infty \right\}. \tag{2.9}$$

**2.2.** *Properties of the utility function.* Recall from [5] that the utility function $C_{x_0}(u,v)$ is infinitely differentiable and that its Hessian of second order derivatives is independent of $(u,v)$. Indeed (1),

$$\tfrac{1}{2} dC_{x_0}(u,v;\bar{u},\bar{v}) = Fx(T) \cdot \bar{y}(T) + (Qx,\bar{y}) + (u,\bar{u}) - (v,\bar{v}), \tag{2.10}$$

where $x$ is the solution of (2.4) and $\bar{y}$ is the solution of

$$\bar{y}' = A\bar{y} + B_1\bar{u} + B_2\bar{v}, \qquad \bar{y}(0) = 0. \tag{2.11}$$

It is customary to introduce the *adjoint system*

$$p' + A^\top p + Qx = 0, \qquad p(T) = Fx(T), \tag{2.12}$$

and rewrite expression (2.10) for the gradient in the form

$$\tfrac{1}{2} dC_{x_0}(u,v;\bar{u},\bar{v}) = (B_1^\top p + u, \bar{u}) + (B_2^\top p - v, \bar{v}). \tag{2.13}$$

Hence $dC_{x_0}(\hat{u},\hat{v};\bar{u},\bar{v}) = 0$ for all $\bar{u}$ and $\bar{v}$ if and only if the *coupled system*

$$\begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0, \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T). \end{cases} \tag{2.14}$$

has a solution $(\hat{x},\hat{p})$ in $H^1(0,T;\mathbb{R}^n)^2$ with $(\hat{u},\hat{v}) = (-B_1^\top \hat{p}, B_2^\top \hat{p})$.

As expected, the Hessian is independent of $(u,v)$:

$$\tfrac{1}{2} d^2 C_{x_0}(u,v;\bar{u},\bar{v};\tilde{u},\tilde{v}) = F\tilde{y}(T) \cdot \bar{y}(T) + (Q\tilde{y},\bar{y}) + (\tilde{u},\bar{u}) - (\tilde{v},\bar{v}), \tag{2.15}$$

where $\bar{y}$ is the solution of (2.11) and $\tilde{y}$ is the solution of

$$\tilde{y}' = A\tilde{y} + B_1\tilde{u} + B_2\tilde{v}, \qquad \tilde{y}(0) = 0. \tag{2.16}$$

In particular, for all $x_0$, $u$, $v$, $\bar{u}$, and $\bar{v}$, $d^2C_{x_0}(u,v;\bar{u},\bar{v};\bar{u},\bar{v}) = 2C_0(\bar{u},\bar{v})$.

**3. Games with finite open loop lower value, upper value, or value.** We recall and sharpen the results of [5, Thms. 2.2–2.4] and [6, Thms. 2.1 and 2.2] when the open loop lower or upper value of the game is finite for a given initial states $x_0$. In each case, the global assumption

(1) Given a real function $f$ defined on a Banach space $B$, the *first directional semiderivative* at $x$ in the direction $v$ (when it exists) is defined as $df(x;v) = \lim_{t \searrow 0}(f(x+tv) - f(x))/t$. When the map $v \mapsto df(x;v) : B \to \mathbb{R}$ is linear and continuous, it defines the *gradient* $\nabla f(x)$ as an element of the dual $B^*$ of $B$. The *second order bidirectional derivative* at $x$ in the directions $(v,w)$ (when it exists) is defined as $d^2f(x;v,w) = \lim_{t \searrow 0}(df(x+tw;v) - df(x;v))/t$. When the map $(v,w) \mapsto d^2f(x;v,w) : B \times B \to \mathbb{R}$ is bilinear and continuous, it defines the *Hessian operator* $Hf(x)$ as a continuous linear operator from $B$ to $B^*$.

of finiteness for *all* initial states $x_0 \in \mathbb{R}^n$ yields the *uniqueness* of solution $(x,p)$ of the coupled system (2.14) (cf. [5, Thms. 2.6–2.8]).

THEOREM 3.1. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbb{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbb{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,\hat{v}) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v). \tag{3.1}$$

(ii) *The open loop lower value $v^-(x_0)$ of the game is finite.*

(iii) *There exists a solution in $H^1(0,T;\mathbb{R}^n)^2$ of the* coupled system

$$\begin{cases} x' = Ax - Rp, & x(0) = x_0, \\ p' + A^\top p + Qx = 0, & p(T) = Fx(T), \end{cases} \tag{3.2}$$

*and the following identities hold*:

$$\sup_{v\in V(0)} \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0) = C_0(0,0). \tag{3.3}$$

*Under such conditions*, *the optimal controls and the open loop lower value are given by the following expressions*:

$$\hat{u} = -B_1^\top p, \qquad \hat{v} = B_2^\top p \quad \text{and} \quad C_{x_0}(\hat{u},\hat{v}) = p(0)\cdot x_0. \tag{3.4}$$

We have a dual theorem.

THEOREM 3.2. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbb{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbb{R}^k)$ such that*

$$C_{x_0}(\hat{u},\hat{v}) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(\hat{u},v) = \inf_{u\in L^2(0,T;\mathbb{R}^m)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v). \tag{3.5}$$

(ii) *The open loop upper value $v^+(x_0)$ of the game is finite.*

(iii) *There exists a solution $(x,p) \in H^1(0,T;\mathbb{R}^n)^2$ of the* coupled system (3.2) *and the following identities hold*:

$$\inf_{u\in U(0)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(u,v) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v) = C_0(0,0). \tag{3.6}$$

*Under such conditions*, *the optimal controls and the open loop upper value are given by expressions* (3.4).

Finally, by combining the previous two theorems and a result of P. Zhang [10], we get the following equivalences.

THEOREM 3.3. *The following conditions are equivalent.*

(i) *There exist $\hat{u}$ in $L^2(0,T;\mathbb{R}^m)$ and $\hat{v}$ in $L^2(0,T;\mathbb{R}^k)$ such that*

$$\begin{aligned} C_{x_0}(\hat{u},\hat{v}) &= \inf_{u\in L^2(0,T;\mathbb{R}^m)} \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_{x_0}(u,v) \\ &= \sup_{v\in L^2(0,T;\mathbb{R}^k)} \inf_{u\in L^2(0,T;\mathbb{R}^m)} C_{x_0}(u,v) \end{aligned} \tag{3.7}$$

(*that is*, *$C_{x_0}(u,v)$ has a saddle point*).

(ii) *The open loop value* $v(x_0)$ *of the game is finite.*

(iii) *There exists a solution* $(x,p) \in H^1(0,T;\mathbb{R}^n)^2$ *of the* coupled system (3.2) *and the following identities hold*:

$$\inf_{u\in L^2(0,T;\mathbb{R}^m)} C_0(u,0) = C_0(0,0) = \sup_{v\in L^2(0,T;\mathbb{R}^k)} C_0(0,v). \tag{3.8}$$

(iv) *The lower and upper open loop values*, $v^-(x_0)$ *and* $v^+(x_0)$, *of the game are finite* (cf. P. Zhang [10]).

*Under such conditions, the optimal controls and the value are given by expressions* (3.4) *and* $v(x_0) = v^-(x_0) = v^+(x_0)$.

The common condition in part (iii) of the above three theorems is the existence of a solution $(x,p) \in H^1(0,T;\mathbb{R}^n)^2$ to the *coupled system* (3.2). Below we give a classification of the six possible cases that can occur when (3.2) has a solution:

(a) $\left|\begin{array}{l} v^-(x_0) \text{ finite} \\ v^+(x_0) \text{ finite} \end{array}\right.$ $\left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0) \\ \sup_v C_0(0,v) = C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ convex} \\ v \mapsto C_0(0,v) \text{ concave} \end{array}\right.$

(b) $\left|\begin{array}{l} v^-(x_0) \text{ finite} \\ v^+(x_0) = +\infty \end{array}\right.$ $\left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0) \\ \sup_v \inf_u C_0(u,v) = C_0(0,0) \\ \sup_v C_0(0,v) > C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ convex} \\ v \mapsto \inf_u C_0(u,v) \text{ concave} \\ v \mapsto C_0(0,v) \text{ not concave} \end{array}\right.$

(c) $\left|\begin{array}{l} v^-(x_0) = -\infty \\ v^+(x_0) \text{ finite} \end{array}\right.$ $\left|\begin{array}{l} \inf_u C_0(u,0) < C_0(0,0) \\ \inf_u \sup_v C_0(u,v) = C_0(0,0) \\ \sup_v C_0(0,v) = C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ not convex} \\ u \mapsto \sup_v C_0(u,v) \text{ convex} \\ v \mapsto C_0(0,v) \text{ concave} \end{array}\right.$

(d) $\left|\begin{array}{l} v^-(x_0) = -\infty \\ v^+(x_0) = +\infty \end{array}\right.$ $\left|\begin{array}{l} \inf_u C_0(u,0) < C_0(0,0) \\ \sup_v C_0(0,v) > C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ not convex} \\ v \mapsto C_0(0,v) \text{ not concave} \end{array}\right.$

(e) $\left|\begin{array}{l} v^-(x_0) = +\infty \\ v^+(x_0) = +\infty \end{array}\right.$ $\left|\begin{array}{l} \inf_u C_0(u,0) = C_0(0,0) \\ \sup_v \inf_u C_0(u,v) > C_0(0,0) \\ \sup_v C_0(0,v) > C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ convex} \\ v \mapsto \inf_u C_0(u,v) \text{ not concave} \\ v \mapsto C_0(0,v) \text{ not concave} \end{array}\right.$

(f) $\left|\begin{array}{l} v^-(x_0) = -\infty \\ v^+(x_0) = -\infty \end{array}\right.$ $\left|\begin{array}{l} \inf_u \sup_v C_0(u,v) < C_0(0,0) \\ \inf_u C_0(u,0) < C_0(0,0) \\ \sup_v C_0(0,v) = C_0(0,0) \end{array}\right.$ $\left|\begin{array}{l} u \mapsto C_0(u,0) \text{ not convex} \\ u \mapsto \sup_v C_0(u,v) \text{ not convex} \\ v \mapsto C_0(0,v) \text{ concave} \end{array}\right.$

**4. $L^2$-integrable closed loop strategies and saddle point.** For specific results and detailed proofs the reader is referred to [6, §3].

**4.1.** *Definitions and main results.* We restrict our attention to the class of affine state feedback strategies and firstly replace the smoothness assumption with respect to time in [2] by a global $L^2$-integrability assumption. This will be later relaxed to deal with closed loop strategies that are not necessarily $L^2$-integrable.

DEFINITION 4.1 ($L^2$-integrable affine closed loop strategies).

$$\varPhi \overset{\text{def}}{=} \left\{ \phi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^m \,\middle|\, \begin{array}{l} x \mapsto \phi(t,x) \text{ is affine and} \\ t \mapsto \phi(t,x) \text{ belongs to } L^2(0,T;\mathbb{R}^m) \end{array} \right\},$$

$$\varPsi \overset{\text{def}}{=} \left\{ \psi : [0,T]\times\mathbb{R}^n \to \mathbb{R}^k \,\middle|\, \begin{array}{l} x \mapsto \psi(t,x) \text{ is affine and} \\ t \mapsto \psi(t,x) \text{ belongs to } L^2(0,T;\mathbb{R}^k) \end{array} \right\}.$$

For all pairs $(\phi,\psi)\in\varPhi\times\varPsi$ and all initial conditions $x_0\in\mathbb{R}^n$, the *closed loop system* has a unique solution in $H^1(0,T;\mathbb{R}^n)$.

$$\begin{aligned} x' &= Ax + B_1\phi(x) + B_2\psi(x), & x(0) &= x_0,\\ x' &= (A+B_1U+B_2V)x + B_1u + B_2v, & x(0) &= x_0. \end{aligned}$$

DEFINITION 4.2 ($L^2$-integrable closed loop saddle point).

$$\left.\begin{array}{l} \exists(\phi^*,\psi^*)\in\varPhi\times\varPsi \text{ such that} \\ \phi^*(t,x) = U_*(t)x + u_*(t) \\ \psi^*(t,x) = V_*(t)x + v_*(t) \end{array}\right| \begin{array}{l} \forall x_0\in\mathbb{R}^n, \\ \forall\phi\in\varPhi,\ \forall\psi\in\varPsi, \\ C_{x_0}(\phi^*,\psi) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi,\psi^*). \end{array}$$

We now have a series of equivalent necessary and sufficient conditions.

THEOREM 4.1. *Each of the following conditions is equivalent to the existence of a* $L^2$-integrable closed loop saddle point.

(i) $L^2$-linear feedback:

$$\left.\begin{array}{ll} \forall x_0\in\mathbb{R}^n,\ \exists!(\hat x,\hat p)\in H^1(0,T)^2 & \\ \hat x' = A\hat x - R\hat p, & \hat x(0) = x_0 \\ \hat p' + A^\top\hat p + Q\hat x = 0, & \hat p(T) = F\hat x(T) \end{array}\right| \begin{array}{l} \exists(U_*,V_*)\in L^2(0,T)^2,\ \forall x_0\in\mathbb{R}^n, \\ \hat u = -B_1^\top\hat p = U_*\hat x \\ \hat v = B_2^\top\hat p = V_*\hat x. \end{array}$$

(ii) Matrix Riccati differential equation:

$$\left.\begin{array}{l} \exists P\in H^1(0,T;\mathbb{R}^{n\times n}) \\ P^\top(t) = P(t) \end{array}\right| \begin{array}{l} P' + PA + A^*P - PRP + Q = 0 \\ P(T) = F. \end{array}$$

(iii) Normality:

$$\left\{\begin{array}{ll} X' = AX - R\varLambda, & X(T) = I \\ \varLambda' + A^\top\varLambda + QX = 0, & \varLambda(T) = F \end{array}\right. \Rightarrow \begin{array}{l} \det X(t)\neq 0 \textit{ everywhere in } [0,T] \\ P(t) = \varLambda(t)X^{-1}(t)\in H^1(0,T). \end{array}$$

(iv) Invariant embedding:

$$\left.\begin{array}{l}\forall s \in [0,T[,\\ \textit{there exists a unique}\\ \textit{matrix solution}\end{array}\right| \begin{array}{ll}\hat{X}_s \in H^1(s,T), & \hat{\Lambda}_s \in H^1(s,T),\\ \hat{X}_s' = A\hat{X}_s - R\hat{\Lambda}_s, & \hat{X}_s(s) = I\\ \hat{\Lambda}_s' + A^\top \hat{\Lambda}_s + Q\hat{X}_s = 0, & \hat{\Lambda}_s(T) = F\hat{X}_s(T).\end{array}$$

(v) Structural $L^2$-integrable closed loop saddle point:

$$\left.\begin{array}{l}\exists(\phi^*,\psi^*) \in \Phi \times \Psi\\ \phi^*(t,x) = U_*(t)x + u_*(t)\\ \psi^*(t,x) = V_*(t)x + v_*(t)\end{array}\right| \begin{array}{l}\forall x_0 \in \mathbb{R}^n,\\ \forall u \in L^2(0,T;\mathbb{R}^m),\ \forall v \in L^2(0,T;\mathbb{R}^k),\\ C_{x_0}(\phi^*,\psi^*+v) \le C_{x_0}(\phi^*,\psi^*) \le C_{x_0}(\phi^*+u,\psi^*).\end{array}$$

It is important to observe that in the necessary and sufficient condition (ii) the solution of the matrix Riccati equation is continuous and that no singularity occurs.

The structural necessary and sufficient condition (v) is particularly interesting. It says that we can change the system by affine feedback in such a way that the new system has an open loop saddle point for the controls $(\hat{u},\hat{v}) = (0,0)$.

**4.2.** *Classification of $L^2$-integrable closed loop saddle points.* Since we have introduced a six-case classification in terms of open loop values, we can now use that classification to be more specific about an $L^2$-integrable saddle point. The global picture is as shown below.

*$L^2$-integrable closed loop saddle point*

$$\text{(a)}\ \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto C_0(0,v)\text{ concave}\end{array}\right|\begin{array}{l} v^-(x_0)\text{ finite}\\ v^+(x_0)\text{ finite}\end{array}\left|\ C_{x_0}(\phi^*,\psi^*) = v(x_0)\right.$$

$$\text{(b)}\ \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto \inf\limits_u C_0(u,v)\text{ concave}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right|\begin{array}{l} v^-(x_0)\text{ finite}\\ v^+(x_0) = +\infty\end{array}\left|\ C_{x_0}(\phi^*,\psi^*) = v^-(x_0)\right.$$

$$\text{(c)}\ \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ not convex}\\ u\mapsto \sup\limits_v C_0(u,v)\text{ convex}\\ v\mapsto C_0(0,v)\text{ concave}\end{array}\right|\begin{array}{l} v^-(x_0) = -\infty\\ v^+(x_0) = \text{ finite}\end{array}\left|\ C_{x_0}(\phi^*,\psi^*) = v^+(x_0)\right.$$

$$\text{(d)}\ \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ not convex}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right|\begin{array}{l} v^-(x_0) = -\infty\\ v^+(x_0) = +\infty\end{array}\left|\ C_{x_0}(\phi^*,\psi^*)\text{ finite}\right.$$

$$\text{(e)}\ \left|\begin{array}{l} u\mapsto C_0(u,0)\text{ convex}\\ v\mapsto \inf\limits_u C_0(u,v)\text{ not concave}\\ v\mapsto C_0(0,v)\text{ not concave}\end{array}\right|\begin{array}{l} v^-(x_0) = +\infty\\ v^+(x_0) = +\infty\end{array}\left|\ \begin{array}{l}C_{x_0}(\phi^*,\psi^*) = +\infty\\ \text{cannot occur}\end{array}\right.$$

| | | | |
|---|---|---|---|
| (f) | $u \mapsto C_0(u,0)$ not convex<br>$u \mapsto \sup_v C_0(u,v)$ not convex<br>$v \mapsto C_0(0,v)$ concave | $v^-(x_0) = -\infty$<br>$v^+(x_0) = -\infty$ | $C_{x_0}(\phi^*,\psi^*) = -\infty$<br>cannot occur |

REMARK 4.1. Case (d) can occur. An example can be constructed by using a first system of the type (b) and a second system of the type (c) without interconnection with utility function equal to the sum of the two utility functions.

REMARK 4.2. In the case of P. Bernhard [2], the utility function was convex in $u$ since $F \geq 0$ and $Q(t) \geq 0$. Hence, only cases (a) and (b) could occur and case (e) was a degenerate one.

**5. The curse of singularities.** We now extend the definitions and results of the previous section to Lebesgue measurable feedback strategies with singularities that are not necessarily $L^2$-integrable in any of their neighborhoods. We shall consider the families $\tilde{\Phi}$ and $\tilde{\Psi}$ of affine feedback strategies that are only measurable in time.

DEFINITION 5.1 (class of affine closed loop strategies).

$$\tilde{\Phi} \stackrel{\text{def}}{=} \left\{ \phi : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m \;\middle|\; \begin{array}{l} x \mapsto \phi(t,x) \text{ is affine,} \\ t \mapsto \phi(t,x) \text{ is Lebesgue measurable, and} \\ t \mapsto \phi(t,0) \text{ belongs to } L^2(0,T;\mathbb{R}^m) \end{array} \right\},$$

$$\tilde{\Psi} \stackrel{\text{def}}{=} \left\{ \psi : [0,T] \times \mathbb{R}^n \to \mathbb{R}^k \;\middle|\; \begin{array}{l} x \mapsto \psi(t,x) \text{ is affine,} \\ t \mapsto \psi(t,x) \text{ is Lebesgue measurable, and} \\ t \mapsto \psi(t,0) \text{ belongs to } L^2(0,T;\mathbb{R}^k) \end{array} \right\}.$$

This will require the concept of an *admissible pair of strategies* (cf. Definition 5.3) in order to make sense of a solution to the closed loop differential equation and of a non-$H^1(0,T)$-solution $P$ to the matrix Riccati differential equation. It is clear that the choice of the space of solutions of the matrix Riccati differential equation and the specification of the *admissibility* in the family $\tilde{\Phi} \times \tilde{\Psi}$ are closely related.

It has been known that the solution of the scalar Riccati differential equation can exhibit singularities that are not *movable branch points* at least when the coefficients are smooth functions of $t$ (cf. Ince [7, §12.51, p. 293]). Another interesting property is that "the general solution of the Riccati equation is expressible rationally in terms of any three distinct particular solutions, and also that the anharmonic ratio of any four solutions is constant. It also shows that the general solution is a rational function of the constant of integration" (cf. Ince [7, §2.15, p. 23; also §12.51, p. 294]).

This result was extended to the $n \times n$ ($n \geq 2$) solution of the matrix Riccati differential equation by Sorine and Winternitz [9] but with five particular solutions in the general case and four in the *symplectic case* that corresponds to our assumptions on the data matrices. They also give some thoughts to the space of solutions: "For smooth coefficients $A$, $B$, $C$, and $D$ in the MRE (6) (2) the solution space consists of meromorphic matrices: the matrix elements may have first-order poles, the positions of which depend on the initial conditions. In other words, the MRE (6) has the Painlevé property [7]: the solutions have no moving critical points, i.e., no branch points or essential singularities, the positions of which depend on the initial conditions" (cf. [9, pp. 271–272]).

**5.1.** *Bernhard's conditions in the free end case.* In the free end case with $F \geq 0$ and $Q(t) \geq 0$, the necessary and sufficient condition of P. Bernhard [2, Thm. 3.1] for the existence of a non-degenerate closed loop saddle point in the sense of [2, Definition 2.3 and Remark 5.1] reduces to the following three properties:

(ii) $X(t)$ is invertible except possibly at isolated points in $[0, T]$, where $(X, \Lambda)$ is the unique $H^1(0, T)$ matrix solution of

$$\left\{\begin{array}{ll} X' = AX - R\Lambda, & X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F, \end{array}\right. \tag{5.1}$$

(iii) $x_0 \in \operatorname{Im} X(0)$,

(iv) for all $t \in [0, T]$, $P(t) \geq 0$,

where $P$ is defined in terms of $\Lambda$ and the pseudo inverse of $X$ as follows:

$$P(t) = \Lambda(t)X(t)^\dagger, \quad X(t)^\dagger \stackrel{\text{def}}{=} \left\{\begin{array}{ll} [X(t)^\top X(t)]^{-1} X(t)^\top & \text{if } X(t) \neq 0, \\ \text{arbitrary} & \text{if } X(t) = 0, \end{array}\right. \tag{5.2}$$

and $[X(t)^\top X(t)]^{-1}$ is the inverse of $X(t)^\top X(t)$ as a matrix from $\operatorname{Im} X(t)^\top$ onto itself.

Condition (ii) defines the matrix function $P(t)$ a.e. in $[0, T]$ and gives a meaning to a solution of the Riccati differential equation via the solution $(\Lambda, X)$ of system (5.1). The positivity of $F$ and $Q(t)$ makes the utility function $C_{x_0}(u, v)$ convex in $u$ and this leads to the positivity of $P(t)$. It also says that only cases (a), (b) and (e) can occur. Finally, as we have seen in the previous section, conditions (iii) and (iv) are redundant.

The relaxation of those positivity assumptions generates the two dual cases (c) and (f) of (a) and (e), and a new case (d) that can occur in the presence of an $L^2$-integrable closed loop saddle point. In relaxing the

(2) Sorine and Winternitz [9] consider solutions $W$ of the general matrix Riccati differential equation $W' = A + WB + CW + WDW$, $W(T) = W_0$.

positivity conditions, the main difficulty is to make sense of the definition of a closed loop saddle point since some of the competitive terms in the utility function may simultaneously blow up making it difficult to set the utility function equal to $\pm\infty$ (cf. [2, p. 68 and Remark 5.1]). So the very definition of a closed loop saddle point has to be properly rewritten and the family of pairs of admissible strategies is no longer $\Phi \times \Psi$ but a subspace $S$ of an enlarged product space $\tilde{\Phi} \times \tilde{\Psi}$ containing $\Phi \times \Psi$ (Definition 5.3).

**5.2.** *Normalizability and its consequences.* We start with *normalizability* that seems to be the fundamental underlying property and try to generate equivalent necessary and sufficient conditions along the lines of the previous section. It will turn out that, at this stage, it is difficult to extend the definition of an $L^2$-integrable closed loop saddle point. Yet, from the equivalences that we shall establish, the *structural closed loop saddle point* will turn out to be a naturally extendable definition.

DEFINITION 5.2 (normalizability a.e.).

$$\left.\begin{array}{l}(X,\Lambda) \text{ the } H^1\text{-matrix solution of system}\\ \begin{cases} X' = AX - R\Lambda, \quad X(T) = I \\ \Lambda' + A^\top \Lambda + QX = 0, \quad \Lambda(T) = F \end{cases}\end{array}\right\} \Rightarrow \det X(t) \neq 0 \text{ a.e. in } [0,T].$$

In particular $P(t) = \Lambda(t)X^{-1}(t)$ a.e. in $[0,T]$.

The first observation is that the subset $Z \overset{\text{def}}{=} \{t \in [0,T] : \det X(t) = 0\}$ of zero measure does not contain non-trivial intervals. If the points of $Z$ are isolated as in condition (ii) of [2], then $Z$ is finite. If the number of points of $Z$ is infinite, then $Z$ has accumulation points that are not isolated. An example where $Z$ is countable is given in [6]. At this stage it is not completely clear if there are examples where $Z$ is uncountable.

Secondly, linear partial differential equations cannot in general be run backward, as required by the first equation in the definition of normalizability. Yet, it is equivalent to the capacity to do invariant embedding for almost all initial times.

THEOREM 5.1. *The following conditions are equivalent.*

(i) (normalizability a.e.) *Given the system*

$$\begin{cases} X' = AX - R\Lambda, & X(T) = I, \\ \Lambda' + A^\top \Lambda + QX = 0, & \Lambda(T) = F, \end{cases}$$

*the set* $Z \overset{\text{def}}{=} \{t \in [0,T] : \det X(t) = 0\}$ *has zero measure.*

(ii) (invariant embedding a.e.) *There exists a subset $Z$ of $[0,T[$ with zero measure such that*

$$\begin{array}{l|ll} \forall s \in [0,T[\setminus Z, & \hat{X}_s \in H^1(s,T), \quad \hat{\Lambda}_s \in H^1(s,T), & \\ \textit{there exists a unique} & \hat{X}_s' = A\hat{X}_s - R\hat{\Lambda}_s, & \hat{X}_s(s) = I, \\ \textit{matrix solution} & \hat{\Lambda}_s' + A^\top \hat{\Lambda}_s + Q\hat{X}_s = 0, & \hat{\Lambda}_s(T) = F\hat{X}_s(T). \end{array}$$

Part (i) of the theorem means that we have a candidate for the solution of the matrix Riccati differential equation except on a closed set of zero measure:

*Matrix Riccati differential equation*

$$\begin{array}{l|l} \exists P & P' + PA + A^*P - PRP + Q = 0 \quad \text{in } [0,T]\setminus Z, \\ P(t) = P^\top(t) & P(T) = F. \end{array}$$

The next result sheds light on the choice of a definition of the closed loop saddle point in the presence of closed loop strategies with non-$L^2$-integrable singularities.

THEOREM 5.2. *Assume that problem* (2.1)–(2.2) *is normalizable. Let* $(X,\Lambda)$ *be the solution of system* (5.1) *and* $P$ *be defined by* (5.2). *Consider the linear closed loop strategies*

$$\phi^*(t,x) = -B_1^\top(t)P(t)x \quad \textit{and} \quad \psi^*(t,x) = B_2^\top(t)P(t)x. \tag{5.3}$$

*Then, for all* $x_0 \in \operatorname{Im} X(0)$, $u \in L^2(0,T;\mathbb{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbb{R}^m)$, *and* $v \in L^2(0,T;\mathbb{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbb{R}^k)$, *there exist* (*not necessarily unique*) *solutions* $x$ *and* $\hat{x}$ *in*

$$H^1_X(0,T;\mathbb{R}^n) \stackrel{\text{def}}{=} \{x \in H^1(0,T;\mathbb{R}^n) : \exists y \in H^1(0,T;\mathbb{R}^n) \textit{ such that } x = Xy\}$$

*to the state equations*

$$x' = Ax + B_1(-B_1^\top Px + u) + B_2(B_2^\top Px + v), \quad x(0) = x_0, \tag{5.4}$$

$$\hat{x}' = (A + B_1U_* + B_2V_*)\hat{x} + B_1u_* + B_2v_*, \quad \hat{x}(0) = x_0. \tag{5.5}$$

*For all solutions* $\hat{x} \in H^1_X(0,T;\mathbb{R}^n)$ *of* (5.5), *all* $u \in L^2(0,T;\mathbb{R}^m)$ *such that* $|X^{-1}|u \in L^2(0,T;\mathbb{R}^m)$, *all* $v \in L^2(0,T;\mathbb{R}^k)$ *such that* $|X^{-1}|v \in L^2(0,T;\mathbb{R}^k)$, *and all solutions* $x_u, x_v \in H^1_X(0,T;\mathbb{R}^n)$ *of the state equations*

$$x_u' = (A + B_1U_* + B_2V_*)x_u + B_1(u_* + u) + B_2v_*, \quad x_u(0) = x_0, \tag{5.6}$$

$$x_v' = (A + B_1U_* + B_2V_*)x_v + B_1u_* + B_2(v_* + v), \quad x_v(0) = x_0, \tag{5.7}$$

*the following inequalities hold*:

$$C_{x_0}(\phi^*(x_v), \psi^*(x_v)+v) \le C_{x_0}(\phi^*(\hat{x}), \psi^*(\hat{x})) \le C_{x_0}(\phi^*(x_u)+u, \psi^*(x_u)). \tag{5.8}$$

*In particular, for all* $x_0 \in \operatorname{Im} X(0)$, $C_{x_0}(\phi^*, \psi^*) = \Lambda(0)y_0 \cdot x_0$ *for some* $y_0 \in \mathbb{R}^n$ *such that* $x_0 = X(0)y_0$, *and this value is independent of the choice of* $y_0$ *such that* $x_0 = X(0)y_0$.

REMARK 5.1. In general equations (5.5), (5.6), and (5.7) will not have a unique solution. The use of the new space $H^1_X(0,T;\mathbb{R}^n)$ does not ensure uniqueness, but it filters out meaningless solutions. To our best knowledge, this space is introduced in this context for the first time and should probably play a key role in the theory of linear differential equations with non-$L^2$-integrable coefficients, thus generalizing the classical case of isolated singularities.

**5.3.** *Admissible closed loop strategies and saddle points in the presence of non-$L^2$-integrable singularities.* Choosing $(\phi, \psi) \in \tilde{\Phi} \times \tilde{\Psi}$ is not sufficient to ensure that the resulting closed loop system will have a nice $H^1(0,T;\mathbb{R}^n)$ solution. Furthermore, the strategies $(\phi, \psi)$ cannot be independently chosen. They occur in pairs. In contrast to the $L^2$-integrable case, the elements of two admissible pairs $(\phi_1, \psi_1)$ and $(\phi_2, \psi_2)$ cannot be mixed: $(\phi_1, \psi_2)$ and $(\phi_2, \psi_1)$ are not necessarily admissible pairs.

With the motivation of Theorem 5.2, we now introduce the admissible pairs of affine closed loop strategies of Definition 5.1.

DEFINITION 5.3 (admissible strategies $(\phi, \psi)$ or $(\phi, \psi) \in S$).

$$\left.\begin{array}{l} (\phi, \psi) \in \tilde{\Phi} \times \tilde{\Psi} \\ \phi(t,x) = U(t)x + u(t) \\ \psi(t,x) = V(t)x + v(t) \end{array}\right| \begin{array}{l} \exists X \in H^1(0,T;\mathbb{R}^n)^n \text{ solution of} \\ X' = (A + B_1 U + B_2 V)X, \quad X(T) = I \\ UX, VX \in L^2(0,T) \\ \exists Z \subset [0,T] \text{ of zero measure such that} \\ \det X(t) \neq 0 \text{ on } [0,T]\backslash Z \\ |X^{-1}|u,\ |X^{-1}|v \in L^2(0,T). \end{array}$$

This definition does not extend to infinite-dimensional systems. Yet, there is an equivalent definition of the invariant embedding a.e. type.

DEFINITION 5.4 (admissible strategies $(\phi, \psi) \in S$: alternative definition).

$$\left.\begin{array}{l} (\phi, \psi) \in \tilde{\Phi} \times \tilde{\Psi} \\ \phi(t,x) = U(t)x + u(t) \\ \psi(t,x) = V(t)x + v(t) \end{array}\right| \begin{array}{l} \exists Z \subset [0,T] \text{ of zero measure such that} \\ \text{for } s \in [0,T[\,\backslash Z, \quad \exists X_s \in H^1(s,T), \\ X_s' = (A + B_1 U + B_2 V)X_s, \quad X_s(s) = I \\ UX_s, VX_s \in L^2(0,T) \\ |X_s^{-1}|u, |X_s^{-1}|v \in L^2(s,T). \end{array}$$

With the above definitions we are now ready to introduce the following more interesting equivalent conditions to normalizability a.e.

THEOREM 5.3. *Each of the following conditions is equivalent to* normalizability a.e.

(i) (Linear feedback)

$$\exists (U_*, V_*) \in S \text{ such that } \forall x_0 \in \operatorname{Im} X(0),$$

$$\begin{array}{l|l} \exists (\hat{x}, \hat{p}) \in H^1_X(0,T;\mathbb{R}^n) \times H^1(0,T;\mathbb{R}^n) & \text{and} \\ \begin{cases} \hat{x}' = A\hat{x} - R\hat{p}, & \hat{x}(0) = x_0 \\ \hat{p}' + A^\top \hat{p} + Q\hat{x} = 0, & \hat{p}(T) = F\hat{x}(T) \end{cases} & \begin{array}{l} \hat{u} = -B_1^\top \hat{p} = U_* \hat{x} \\ \hat{v} = B_2^\top \hat{p} = V_* \hat{x}. \end{array} \end{array}$$

(ii) ($X(0)$-structural closed loop saddle point)

$$\begin{array}{l|l} \exists (\phi^*, \psi^*) \in S & \forall x_0 \in \operatorname{Im} X(0), \\ \phi^*(t,x) = U_*(t)x + u_*(t) & \forall u \in L^2(0,T;\mathbb{R}^m) \text{ with } |X^{-1}|u \in L^2(0,T;\mathbb{R}^m) \\ \psi^*(t,x) = V_*(t)x + v_*(t) & \forall v \in L^2(0,T;\mathbb{R}^k) \text{ with } |X^{-1}|v \in L^2(0,T;\mathbb{R}^k), \\ & C_{x_0}(\phi^*, \psi^* + v) \le C_{x_0}(\phi^*, \psi^*) \le C_{x_0}(\phi^* + u, \psi^*). \end{array}$$

The conclusion is that we adopt as a definition of a *closed loop saddle point under possibly non-$L^2$-integrable strategies* the characterization of the *$X(0)$-structural closed loop saddle point*. With that definition, an *$L^2$-integrable closed loop saddle point* is an $X(0)$-structural closed loop saddle point which makes the new definition coherent with the result of the previous section. Another way to look at this issue is to notice that Berkovitz's [1] equivalence lemma does not hold any more.

**5.4.** *Classification.* We finally give the classification with respect to open loop concepts and $L^2$-integrable feedback concepts (cf. [6, §4]).

*Closed loop saddle point with possible non-$L^2$-integrable singularities*

| | | | |
|---|---|---|---|
| (a) | $u \mapsto C_0(u,0)$ convex<br>$v \mapsto C_0(0,v)$ concave | $v^-(x_0)$ finite<br>$v^+(x_0)$ finite | $C_{x_0}(\phi^*,\psi^*) = v(x_0)$<br>$\inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi)$<br>$= \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = v(x_0)$ |
| (b) | $u \mapsto C_0(u,0)$ convex<br>$v \mapsto \inf_u C_0(u,v)$ concave<br>$v \mapsto C_0(0,v)$ not concave | $v^-(x_0)$ finite<br>$v^+(x_0) = +\infty$ | $C_{x_0}(\phi^*,\psi^*) = v^-(x_0)$ |
| (e) | $u \mapsto C_0(u,0)$ convex<br>$v \mapsto \inf_u C_0(u,v)$ not concave<br>$v \mapsto C_0(0,v)$ not concave | $v^-(x_0) = +\infty$<br>$v^+(x_0) = +\infty$ | $\inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi)$<br>$= \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = +\infty$<br>degenerate case |

| | | | |
|---|---|---|---|
| (c) | $u \mapsto C_0(u,0)$ not convex<br>$u \mapsto \sup_v C_0(u,v)$ convex<br>$v \mapsto C_0(0,v)$ concave | $v^-(x_0) = -\infty$<br>$v^+(x_0) =$ finite | $C_{x_0}(\phi^*, \psi^*) = v^+(x_0)$ |
| (f) | $u \mapsto C_0(u,0)$ not convex<br>$u \mapsto \sup_v C_0(u,v)$ not convex<br>$v \mapsto C_0(0,v)$ concave | $v^-(x_0) = -\infty$<br>$v^+(x_0) = -\infty$ | $\inf_{\phi\in\Phi} \sup_{\psi\in\Psi} C_{x_0}(\phi,\psi)$<br>$= \sup_{\psi\in\Psi} \inf_{\phi\in\Phi} C_{x_0}(\phi,\psi) = -\infty$<br>degenerate case |
| (d) | $u \mapsto C_0(u,0)$ not convex<br>$v \mapsto C_0(0,v)$ not concave | $v^-(x_0) = -\infty$<br>$v^+(x_0) = +\infty$ | |

Remark 5.2. Case (d) can definitely occur. In contrast to the $L^2$-integrable closed loop saddle point, case (e) can occur as can be seen from the following system and utility function:

$$
\begin{gathered}
x'(t) = tu(t) + t^3 v(t) \quad \text{in } [0,2], \qquad x(0) = x_0, \\
C_{x_0}(u,v) = \frac{3}{8}\, x(2)\cdot x(2) + \int_0^2 (|u(t)|^2 - |v(t)|^2)\, dt.
\end{gathered}
\tag{5.9}
$$

By duality, case (f) can also occur.

Remark 5.3. Again, in the case of P. Bernhard [2], the utility function was convex in $u$ since $F \geq 0$ and $Q(t) \geq 0$. Hence, only cases (a) and (b) could occur and case (e) remains a degenerate one in the sense of his definition.

**6. Conclusions and additional comments.** For the finite-dimensional case we now have a fairly complete theory that extends the pioneering work of P. Bernhard [2] to matrices $F$ and $Q(t)$ that are only symmetrical. We have also extended the concept of isolated singularities to the more general concept of non-$L^2$-integrable singularities. $L^2$-integrable feedback strategies can exhibit isolated singularities, but they do not yield singularities in the solution of the matrix Riccati differential equation. For more details the reader is referred to [6].

In this paper we indicated where finite-dimensional concepts such as normality and normalizability a.e. could and should be modified to deal with evolution equations with states in an infinite-dimensional space. The concept of invariant embedding for almost all initial times seems to be a natural one. During the IFIP meeting in Kraków, Jacques Henry pointed out to me that he came across this concept while studying the Riccati differential equation associated with the Helmholtz equation of waveguides. Due to resonance the invariant embedding could not be done at an at most countable number of initial times. This material can be found in the Ph.D. thesis of Isabelle Champagne [4].

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