GOODNESS-OF-FIT TESTS USING CHARACTERIZATIONS OF CONTINUOUS DISTRIBUTIONS

Abstract. Using characterization conditions of continuous distributions in terms of moments of order statistics and moments of record values we present new goodness-of-fit techniques.

1. Introduction and preliminaries. Let \((X_1, \ldots, X_n)\) be a sample from a continuous distribution \(F(x) = P[X \leq x], x \in \mathbb{R},\) and let \(X_{k:n}\) denote the \(k\)th smallest order statistic of the sample. We construct goodness-of-fit tests for continuous distributions using characterizations of distributions via moments of order statistics and moments of record values (cf. [2]–[5], [10]). The results presented extend the tests for uniformity and exponentiality discussed in [6] and [7]. Moreover, we give the proof of statements on tests for exponentiality announced in [7]. We include a theorem on the asymptotic effect of substituting estimators for parameters in the tests proposed here. It can be used, among other things, to construct a test for normality.

(O) Characterizations in terms of moments of order statistics. We use the characterization conditions contained in the following theorems.

THEOREM 1 (cf. [10], [3]). Let \(n, k, l\) be given integers such that \(n \geq k \geq l \geq 1\). Assume that \(G\) is a nondecreasing right-continuous function from \(\mathbb{R}\) to \(\mathbb{R}\). Then \(F(x) = G(x)\) on \(I(F)\) (the minimal interval containing the support of \(F\)) and \(F\) is continuous on \(\mathbb{R}\) iff

\[
\frac{(k - l)!}{(n - l + 1)!} E G^2 l(X_{k+1-l:n+1-l}) - \frac{2k!}{(n+1)!} E G^l(X_{k+1:n+1}) + \frac{(k + l)!}{(n + l + 1)!} = 0.
\]

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**Theorem 2** (cf. [5]). **Under the assumptions of Theorem 1,** \( F(x) = G(x) \) on \( I(F) \) and \( F \) is continuous on \( \mathbb{R} \) iff

\[
EG^l(X_{k+1:n+1}) = \frac{(k + l)!(n + 1)!}{k!(n + l + 1)!},
\]

\[
EG^{l2}(X_{k+1-l:n+1-l}) = \frac{(k + l)!(n - l + 1)!}{(k - l)!(n + l + 1)!}.
\]

Note that Theorem 2 is a consequence of Theorem 1, since (1.1) implies

\( F = G \) implies (1.2) implies (1.1).

**Corollary 1.** \( X \sim F \) and \( F \) is continuous iff

\[
EF(X_{2:2}) - EF^2(X) = \frac{1}{3}
\]

or

\[
EF(X_{2:2}) = \frac{2}{3}, \quad EF^2(X) = \frac{1}{3}.
\]

In particular:

(a) \( X \sim U(\alpha, \beta) \) (uniform distribution), i.e. \( F(x) = (x - \alpha)/(\beta - \alpha) \), \( \alpha < x < \beta \), iff

\[
E[(X_{2:2} - \alpha)/(\beta - \alpha)] - E[(X - \alpha)/(\beta - \alpha)]^2 = \frac{1}{3}
\]

or

\[
E[(X_{2:2} - \alpha)/(\beta - \alpha)] = \frac{2}{3}, \quad E[(X - \alpha)/(\beta - \alpha)]^2 = \frac{1}{3},
\]

(b) \( X \sim \text{Exp}(\alpha) \) (exponential distribution), i.e. \( F(x) = 1 - \exp(-\alpha x) \), \( x > 0, \alpha > 0 \), iff

\[
E(1 - \exp(-\alpha X_{2:2})) - E(1 - \exp(-\alpha X))^2 = \frac{1}{3}
\]

or

\[
E(1 - \exp(-\alpha X_{2:2})) = \frac{2}{3}, \quad E(1 - \exp(-\alpha X))^2 = \frac{1}{3}.
\]

(R) **Characterization conditions in terms of moments of record values.** Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with cdf \( F \) and pdf \( f \). For a fixed \( k \geq 1 \) we define the sequence \( U_k(1), U_k(2), \ldots \) of \( k \)-(**upper**) record times of \( X_1, X_2, \ldots \) as follows:

\[
U_k(1) = 1,
\]

\[
U_k(n) = \min\{j > U_k(n - 1) : X_{j:j+k-1} > X_{U_k(n-1):U_k(n-1)+k-1}\}, \quad n = 2, 3, \ldots
\]

Write

\[
Y^{(k)}_n := X_{U_k(n):U_k(n)+k-1}, \quad n \geq 1.
\]
The sequence \( \{Y_n^{(k)}(k), n \geq 1\} \) is called the sequence of \( k \)-(upper) record values of the above sequence. For convenience we also take \( Y_0^{(k)} = 0 \) and note that \( Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k) \) (cf. [1]).

We see that for \( k = 1, 2, \ldots \), the sequences \( \{Y_n^{(k)}, n \geq 1\} \) of \( k \)th record values can be obtained from \( \{X_n, n \geq 1\} \) by inspecting successively the samples \( X_1, (X_1, X_2), (X_1, X_2, X_3), \) and so on. For \( k = 1 \), \( Y_1^{(1)} = X_1 \), and the following terms are obtained by looking at the maxima of the successive samples; \( Y_2^{(1)} \) is the first maximum that exceeds \( Y_1^{(1)} \), \( Y_3^{(1)} \) is the first maximum that exceeds \( Y_2^{(1)} \), and so on. For \( k = 2 \), \( Y_1^{(2)} = \min(X_1, X_2) \), and the following terms are obtained by looking at the next-to-largest values in the successive samples; \( Y_2^{(2)} \) is the first next-to-largest value that exceeds \( Y_1^{(2)} \), \( Y_3^{(2)} \) is the next-to-largest value that exceeds \( Y_2^{(2)} \), and so on. And generally, \( Y_1^{(k)} = \min(X_1, \ldots, X_k) = X_{1:k} \), and the following \( k \)th record values are obtained by looking at the \( k \)th largest values in successive samples, i.e., looking at the order statistics \( X_{2:k+1} \) from \( (X_1, \ldots, X_{k+1}) \), \( X_{3:k+2} \) from \( (X_1, \ldots, X_{k+2}) \), and so on.

We have the following characterizations.

**Theorem 3** (cf. [4]). Let \( \{X_n, n \geq 1\} \) be a sequence of i.i.d. random variables with cdf \( F \). Assume that \( G \) is a nondecreasing right-continuous function from \( \mathbb{R} \) to \( (-\infty, 1] \), and let \( n, k, l \) be given integers such that \( k \geq 1 \) and \( n \geq l \geq 1 \). Then \( F(x) = G(x) \) on \( I(F) \) iff

\[
(1.5) \quad k^{2l}(n - l)!E[-\log(1 - G(Y_{n-l+1}^{(k)}))]^{2l} - 2n!k^l E[-\log(1 - G(Y_{n+1}^{(k)}))]^l + (n + l)! = 0.
\]

**Theorem 4** (cf. [5], [4]). Under the assumptions of Theorem 3, \( F(x) = G(x) \) on \( I(F) \) iff

\[
E[-\log(1 - G(Y_{n+1}^{(k)}))]^l = \frac{(n + l)!}{n!k^l},
\]

\[
E[-\log(1 - G(Y_{n-l+1}^{(k)}))]^{2l} = \frac{(n + l)!}{(n - l)!k^{2l}}.
\]

Following the observation after Theorem 2 we see that Theorem 4 is a consequence of Theorem 3.

**Corollary 2.** \( X \sim F \) and \( F \) is continuous iff

\[
(1.6) \quad E[-\log(1 - F(Y_1^{(k)}))]^2 - \frac{2}{k} E[-\log(1 - F(Y_2^{(k)}))] + \frac{2}{k^2} = 0
\]

or

\[
(1.7) \quad E[-\log(1 - F(Y_2^{(k)}))] = \frac{2}{k}, \quad E[-\log(1 - F(Y_1^{(k)}))]^2 = \frac{2}{k^2}.
\]
In particular:

(a) \( X \sim U(0, 1) \) iff

\[
E[\log(1 - Y_1^{(k)})] - \frac{2}{k} E[\log(1 - Y_2^{(k)})] + \frac{2}{k^2} = 0
\]

or

\[
E[\log(1 - Y_2^{(k)})] = \frac{2}{k}, \quad E[\log(1 - Y_1^{(k)})] = \frac{2}{k^2},
\]

(b) \( X \sim \text{Exp}(\alpha) \) iff

\[
\alpha^2 E(Y_1^{(k)})^2 - \frac{2}{k^2} \alpha E(Y_2^{(k)}) + \frac{2}{k^2} = 0
\]

or

\[
EY_2^{(k)} = \frac{2}{\alpha k}, \quad E(Y_1^{(k)})^2 = \frac{2}{\alpha^2 k^2}.
\]

2. **Goodness-of-fit tests based on characterizations via moments of order statistics.** The cases when parameters of \( F \) are specified and unknown will be treated separately.

(A) **Parameters of \( F \) are specified.** First we construct goodness-of-fit tests based on the characterization in (1.1) (see also (1.3)) which we can write in the form

\[
E(F(X_2;2)) - \frac{1}{2} (E(F^2(X_1) + F^2(X_2))) = \frac{1}{3}
\]

where \( X_1 \) and \( X_2 \) are i.i.d. as \( X \).

Let \( (X_1, \ldots, X_{2n}) \) be a sample from \( F \), where \( F \) is continuous and strictly increasing. Define

\[
Y_j = F^2(X_{2j-1}) + F^2(X_{2j}), \\
Z_j = F(\max(X_{2j-1}, X_{2j})), \quad j = 1, \ldots, n.
\]

Then \( Y_1, \ldots, Y_n \) are i.i.d. and \( Z_1, \ldots, Z_n \) are i.i.d. Writing \( Y := Y_1 = F^2(X_1) + F^2(X_2), Z := Z_1 = F(\max(X_1, X_2)) \) we state the following result.

**Lemma 1.** Under the above assumptions, the density function of \( (Y, Z) \) is given by

\[
f(y, z) = \begin{cases} 
1/\sqrt{y - z^2}, & 0 \leq z \leq 1, \ z^2 < y \leq 2z^2, \\
0, & \text{otherwise},
\end{cases}
\]

and

\[
EY = \frac{2}{3}, \ Var(Y) = \frac{8}{45}, \ EZ = \frac{2}{3}, \ Var(Z) = \frac{1}{15}, \ Cov(Y, Z) = \frac{4}{45}.
\]

Now we define

\[
R_j = Z_j - \frac{1}{2} Y_j, \quad j = 1, \ldots, n.
\]
We see that
\[ ER_j = EZ_j - \frac{1}{3} EY_j = \frac{1}{3}, \]
\[ \text{Var } R_j = \text{Var } Z_j + \frac{1}{4} \text{Var } Y_j - \text{Cov}(Z_j, Y_j) = \frac{1}{90}, \quad j = 1, \ldots, n. \]
Write
\[ \overline{R}_n = \frac{1}{n} \sum_{j=1}^{n} R_j; \]
then by the CLT
\[ 3\sqrt{10n} \left( \overline{R}_n - \frac{1}{3} \right) \overset{D}{\to} V \sim N(0, 1), \]
and hence
\[ D_n^{(1)} := 45 \cdot 2n \left( \overline{R}_n - \frac{1}{3} \right)^2 \overset{D}{\to} \chi^2(1), \]
and so \( D_n^{(1)} \) provides a simple asymptotic test of the hypothesis \( X \sim F \).

Setting \( X_j^* = \max(X_{2j-1}, X_{2j}) \), \( j = 1, \ldots, n \), we note that \( D_n^{(1)} \) in (2.1) has the form
\[ D_n^{(1)} = 45 \cdot 2n \left( \frac{1}{n} \sum_{j=1}^{n} F(X_j^*) - \frac{1}{2n} \sum_{j=1}^{2n} F^2(X_j) - \frac{1}{3} \right)^2. \]

Next we construct goodness-of-fit tests based on the characterization in (1.2) (see also (1.4)), which we write in the form
\[ EF(\max(X_1, X_2)) = \frac{2}{3}, \quad EF^2(X_1) = \frac{1}{3}. \]
Define
\[ W_j = \begin{pmatrix} Y_j \\ Z_j \end{pmatrix}, \quad j = 1, \ldots, n, \]
\[ \mu = EW_1 = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \]
\[ \Sigma := \text{Var}(W_1) = E(W_1 - EW_1)(W_1 - EW_1)' = \begin{pmatrix} 8/45 & 4/45 \\ 4/45 & 1/18 \end{pmatrix}, \]
and write \( \overline{W}_n = n^{-1} \sum_{j=1}^{n} W_j \). The CLT says that
\[ \sqrt{n} \left( \overline{W}_n - \mu \right) \overset{D}{\to} V \sim N(0, \Sigma), \]
whence
\[ D_n^{(2)} := n(\overline{W}_n - \mu)' \Sigma^{-1} (\overline{W}_n - \mu) \overset{D}{\to} V' \Sigma^{-1} V \sim \chi^2(2). \]
But \( D_n^{(2)} \) is a reasonable measure of the “size” of \( (\overline{W}_n - \mu) \) and so by (2.3) provides a test of the hypothesis that \( X \) has the distribution function \( F \). And since
\[ \Sigma^{-1} = 45 \begin{pmatrix} 5/8 & -1 \\ -1 & 2 \end{pmatrix}, \]
it follows that in extended form

\[(2.4) \quad D_n^{(2)} = 45n\left(\frac{5}{8}(Y_n - \frac{2}{3})^2 + 2(Z_n - \frac{2}{3})^2 - 2(Y_n - \frac{2}{3})(Z_n - \frac{2}{3})\right).\]

In terms of $X_j^*$, $D_n^{(2)}$ in (2.4) has the form

\[(2.5) \quad D_n^{(2)} = 45 \cdot 2n\left[\frac{1}{4}(F^2(X_{2n}) - \frac{1}{3})^2 + (F^2(X_{2n}) - F(X_n^*) + \frac{1}{3})^2\right].\]

By (2.5) and (2.2) we have

**Lemma 2.**

\[D_n^{(2)} = \frac{45}{4} \cdot 2n(F^2(X_{2n}) - \frac{1}{3})^2 + D_n^{(1)}.\]

Special cases:

(a) If $X \sim U(\alpha, \beta)$ then

\[D_n^{(1)} = 45 \cdot 2n\left(\frac{1}{(\beta - \alpha)^2}X_{2n}^2 - \frac{\beta + \alpha}{(\beta - \alpha)^2}X_{2n} - \frac{1}{\beta - \alpha}X_n^* + \frac{\alpha\beta}{\beta - \alpha} + \frac{1}{3}\right)^2,\]

\[D_n^{(2)} = 45 \cdot 2n\left(\frac{X_{2n}^2}{(\beta - \alpha)^2} - 2\alpha \frac{X_{2n} - X_n^*}{(\beta - \alpha)^2} + \frac{\alpha^2}{(\beta - \alpha)^2} - \frac{1}{3}\right)^2 + D_n^{(1)}.\]

**Remark.** If $X \sim U(0, \beta)$ then

\[D_n^{(1)} = 45 \cdot 2n\left(X_{2n}^2/\beta^2 - X_{2n}/\beta - X_n^*/\beta + \frac{1}{3}\right)^2,\]

\[D_n^{(2)} = 45 \cdot 2n\left(X_{2n}^2/\beta^2 - \frac{1}{3}\right)^2 + D_n^{(1)}.\]

(b) If $X \sim \text{Pow}(\alpha)$ (power distribution), i.e. $F(x) = 1 - (1 - x/\alpha)^\alpha$, $0 \leq x \leq \alpha$, $0 < \alpha \leq 1$, then

\[D_n^{(1)} = 45 \cdot 2n\left(\frac{1}{2n} \sum_{j=1}^{2n} (1 - (1 - X_j/\alpha)^\alpha)^2 + \frac{1}{n} \sum_{j=1}^{n} (1 - X_j^*/\alpha)^\alpha - \frac{2}{3}\right)^2,\]

\[D_n^{(2)} = 45 \cdot 2n\left(\frac{1}{2n} \sum_{j=1}^{2n} (1 - (1 - X_j/\alpha)^\alpha)^2 - \frac{1}{3}\right)^2 + D_n^{(1)}.\]

(c) If $X \sim \text{Exp}(\alpha)$ then

\[D_n^{(1)} = 45 \cdot 2n\left(\frac{1}{2n} \sum_{j=1}^{2n} (1 - e^{-\alpha X_j})^2 + \frac{1}{n} \sum_{j=1}^{n} e^{-\alpha X_j} - \frac{2}{3}\right)^2,\]

\[D_n^{(2)} = 45 \cdot 2n\left(\frac{1}{2n} \sum_{j=1}^{2n} (1 - e^{-\alpha X_j})^2 - \frac{1}{3}\right)^2 + D_n^{(1)}.\]
(d) If $X \sim W(\beta, \alpha)$ (Weibull distribution), i.e. $F(x) = 1 - \exp(-\alpha x^\beta)$, $x > 0$, $\alpha > 0$, $\beta > 0$, then

$$D_n^{(1)} = 45 \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - \exp(-\alpha X_j^\beta))^2 + \frac{1}{n} \sum_{j=1}^{n} \exp(-\alpha X_j^\beta) - \frac{2}{3} \right)^2,$$

$$D_n^{(2)} = \frac{45}{4} \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - \exp(-\alpha X_j^\beta))^2 - \frac{1}{3} \right)^2 + D_n^{(1)}.$$

(e) If $X \sim \text{Par}_S(\alpha, \sigma)$ (single-parameter Pareto distribution), i.e. $F(x) = 1 - (\sigma/x)^\alpha$, $x > \sigma$, $\alpha > 0$, $\sigma > 0$, then

$$D_n^{(1)} = 45 \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - \left( \frac{\sigma}{X_j} \right)^\alpha)^2 + \frac{1}{n} \sum_{j=1}^{n} \left( \frac{\sigma}{X_j} \right)^\alpha - \frac{2}{3} \right)^2,$$

$$D_n^{(2)} = \frac{45}{4} \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} \left( 1 - \left( \frac{\sigma}{X_j} \right)^\alpha \right)^2 - \frac{1}{3} \right)^2 + D_n^{(1)}.$$

(f) If $X \sim \text{Par}_T(\alpha, \theta)$ (two-parameter Pareto distribution), i.e. $F(x) = 1 - \left( \frac{\theta}{x+\theta} \right)^\alpha$, $x > 0$, $\alpha > 0$, $\theta > 0$, then

$$D_n^{(1)} = 45 \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - \left( \frac{\theta}{X_j + \theta} \right)^\alpha)^2 + \frac{1}{n} \sum_{j=1}^{n} \left( \frac{\theta}{X_j + \theta} \right)^\alpha - \frac{2}{3} \right)^2,$$

$$D_n^{(2)} = \frac{45}{4} \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} \left( 1 - \left( \frac{\theta}{X_j + \theta} \right)^\alpha \right)^2 - \frac{1}{3} \right)^2 + D_n^{(1)}.$$

(g) If $X \sim \text{Log}(\alpha, \beta)$ (logistic distribution), i.e.

$$F(x) = [1 + \exp(-(x - \alpha)/\beta)]^{-1}, \quad -\infty < x < \infty, \quad \alpha \in \mathbb{R}, \quad \beta > 0,$$

then

$$D_n^{(1)} = 45 \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 + \exp(-(X_j - \alpha)/\beta))^{-2} - \frac{1}{n} \sum_{j=1}^{n} (1 + \exp(-(X_j^* - \alpha)/\beta))^{-1} + \frac{1}{3} \right)^2,$$

$$D_n^{(2)} = \frac{45}{4} \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 + \exp(-(X_j - \alpha)/\beta))^{-2} - \frac{1}{3} \right)^2 + D_n^{(1)}.$$

**Unknown parameters.** We discuss asymptotic tests obtained from $D_n^{(1)}$ and $D_n^{(2)}$ in (A) when parameters are replaced by estimators.
Proposition 1. Goodness-of-fit tests for $F(x) = x/\beta$, $x \in (0, \beta)$, $\beta > 0$, are given by

$$
\hat{D}_n^{(1)} := D_n^{(1)}(\hat{\beta}_n) = 45 \cdot 2n \left( \frac{X_{2n}^2}{\beta_n^2} - \frac{X_{2n}}{\beta_n} + \frac{1}{3} \right)^2 \xrightarrow{D} \chi^2(1),
$$

$$
\hat{D}_n^{(2)} := D_n^{(2)}(\hat{\beta}_n) = \frac{45}{4} \cdot 2n \left( \frac{X_{2n}^2}{\beta_n^2} - \frac{1}{3} \right)^2 + D_n^{(1)}(\hat{\beta}_n) \xrightarrow{D} \chi^2(2),
$$

where $\hat{\beta}_n = \max(X_1, \ldots, X_{2n})$.

Proposition 2. Goodness-of-fit tests for $F(x) = \frac{x-\alpha}{\beta-\alpha}$, $x \in (\alpha, \beta)$, $\alpha < \beta$, are given by

$$
\hat{D}_n^{(1)} := D_n^{(1)}(\hat{\alpha}_n, \hat{\beta}_n) = 45 \cdot 2n \left( \frac{X_{2n}^2}{(\hat{\beta}_n - \hat{\alpha}_n)^2} - (\hat{\beta}_n + \hat{\alpha}_n) \frac{X_{2n}}{(\hat{\beta}_n - \hat{\alpha}_n)^2} \right.
$$

$$
\left. - \frac{\hat{\alpha}_n}{(\hat{\beta}_n - \hat{\alpha}_n)^2} + \frac{1}{3} \right)^2 \xrightarrow{D} \chi^2(1),
$$

$$
\hat{D}_n^{(2)} := D_n^{(2)}(\hat{\alpha}_n, \hat{\beta}_n) = \frac{45}{4} \cdot 2n \left( \frac{X_{2n}^2}{(\hat{\beta}_n - \hat{\alpha}_n)^2} - 2 \frac{\hat{\alpha}_n X_{2n}}{(\hat{\beta}_n - \hat{\alpha}_n)} \right.
$$

$$
\left. + \frac{\hat{\alpha}_n^2}{(\hat{\beta}_n - \hat{\alpha}_n)^2} - \frac{1}{3} \right)^2 + D_n^{(1)}(\hat{\alpha}_n, \hat{\beta}_n) \xrightarrow{D} \chi^2(2),
$$

where $\hat{\beta}_n = \max(X_1, \ldots, X_{2n})$ and $\hat{\alpha}_n = \min(X_1, \ldots, X_{2n})$.

The proofs of Propositions 1 and 2 are given in [6] and [7]. For the following propositions concerning exponential and normal distributions we use a general theorem based on results in [8] and [9].

Theorem 5 ([8]). Let $\hat{T}_n = T_n(X_1, \ldots, X_n; \hat{\lambda}_n)$, where $\hat{\lambda}_n = \hat{\lambda}_n(X_1, \ldots, \ldots, X_n)$ is an estimator of a parameter $\lambda$ of the distribution of $X$, and let $T_n = T_n(X_1, \ldots, X_n; \lambda)$ (here $T_n$, $\lambda$ and $\hat{\lambda}_n$ may be vectors). Suppose that:

(i) For each $\lambda$, $\sqrt{n} \left( \frac{T_n}{\hat{\lambda}_n - \lambda} \right) \xrightarrow{D} T \sim N(0, V)$,

where

$$
V = \begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}
$$

and $V_{22}$ is nonsingular.

(ii) There is a matrix $B$, possibly depending continuously on $\lambda$, such that $\sqrt{n} \hat{T}_n = \sqrt{n} T_n + B \sqrt{n} (\hat{\lambda}_n - \lambda) + o_p(1)$. 
(iii) $\hat{\lambda}_n$ is asymptotically efficient (cf. [8]).

Then

$$\sqrt{n} \hat{T}_n \xrightarrow{D} T^* \sim N(0, V_{11} - BV_{22}B').$$

Note that (ii) is satisfied when $T_n$ is differentiable in $\lambda$, and then

$$B = \lim_{n \to \infty} E \left[ \frac{\partial}{\partial \lambda} T_n \right].$$

The following result is a consequence of Theorem 5.

**Theorem 6.** Let $(X_1, \ldots, X_{2n})$ be a sample with an absolutely continuous distribution function $F(x; \lambda)$ differentiable with respect to the $m \times 1$ vector $\lambda$. Set

$$\mathbf{W}_n := \begin{pmatrix} \mathbf{Y}_n \\ \mathbf{Z}_n \end{pmatrix} := \begin{pmatrix} \mathbf{Y}_n(\lambda) \\ \mathbf{Z}_n(\lambda) \end{pmatrix} = \mathbf{W}_n(\lambda),$$

where

$$\mathbf{Y}_n = \frac{1}{n} \sum_{j=1}^{2n} F^2(X_j; \lambda), \quad \mathbf{Z}_n = \frac{1}{n} \sum_{j=1}^{n} F(X_j^*; \lambda),$$

and $X_j^* = \max(X_{2j-1}, X_{2j})$, $j = 1, \ldots, n$. Write

$$\widehat{\mathbf{W}}_n = \mathbf{W}_n(\mathbf{\hat{\lambda}}_{2n}) = \begin{pmatrix} \mathbf{\hat{Y}}_n \\ \mathbf{\hat{Z}}_n \end{pmatrix},$$

where

$$\mathbf{\hat{Y}}_n := \mathbf{Y}_n(\mathbf{\hat{\lambda}}_{2n}), \quad \mathbf{\hat{Z}}_n := \mathbf{Z}_n(\mathbf{\hat{\lambda}}_{2n}).$$

and $\mathbf{\hat{\lambda}}_{2n}$ is the MLE of $\lambda$. Suppose that $F$ is such that the MLE $\mathbf{\hat{\lambda}}_{2n}$ is “regular” in the sense that

$$\sqrt{2n} (\mathbf{\hat{\lambda}}_{2n} - \lambda) \xrightarrow{D} \gamma \sim N(0, I^{-1}),$$

where $I = I(\lambda)$ is the information matrix for $\lambda$ based on a single observation. Then

$$\sqrt{n} (\mathbf{W}_n(\mathbf{\hat{\lambda}}_{2n}) - \mu) \xrightarrow{D} W \sim N(0, \Sigma_1),$$

$$\hat{D}_n^{(1)} := 45 \cdot 2n (\mathbf{\hat{F}}(X_{n}^*) - \mathbf{\hat{F}}^2(X_{2n}) - \frac{1}{3})^2 \to \chi^2(1),$$

$$\hat{D}_n^{(2)} := \frac{45}{4 - \frac{1}{2}} 2n (\mathbf{\hat{F}}^2(X_{2n}) - \frac{1}{3})^2 + \hat{D}_n^{(1)} \to \chi^2(2),$$

where $\Sigma_1 = \Sigma - B(2I)^{-1}B$, $\mu$ and $\Sigma$ are taken from (2.3), $\mathbf{\hat{F}}(x) := F(x, \mathbf{\hat{\lambda}})$, and

$$B = 2 \left( \frac{2}{1} \right)^d,$$
where
\[ d := E\left( F(X; \lambda) \frac{dF(X; \lambda)}{d\lambda} \right) \] is \( m \times 1 \),
and
\[ b := b(\lambda) = 180d' I^{-1}d. \]

**Proof.** The statement (2.7) follows directly from (2.3) and (2.6). Now note that
\[
E \frac{\partial Y_n}{\partial \lambda_j} = 2E \frac{\partial F^2(X; \lambda)}{\partial \lambda_j} = 4E \left( F \frac{\partial F}{\partial \lambda_j} \right)
= 4 \int F(x; \lambda) \frac{\partial F}{\partial \lambda_j} f(x; \lambda) \, dx, \quad j = 1, \ldots, m,
\]
and correspondingly
\[
E \left( \frac{\partial Z_n}{\partial \lambda_j} \right) = E \left( \frac{\partial F(\max(X_{2i-1}, X_{2i}); \lambda)}{\partial \lambda_j} \right) = \frac{1}{2} E \left( \frac{\partial Y_n}{\partial \lambda_j} \right), \quad j = 1, \ldots, m,
\]
since the pdf of \( X_i^* = \max(X_{2i-1}, X_{2i}) \) is \( 2F(x^*; \lambda)f(x^*; \lambda), i = 1, \ldots, n \).

It follows that
\[
B = \lim_{n \to \infty} E \left( \frac{\partial W_n}{\partial \lambda} \right) = \left( \frac{4}{2} \right) \left( E \left( F \frac{\partial F}{\partial \lambda_1} \right) \ldots E \left( F \frac{\partial F}{\partial \lambda_m} \right) \right) = 2 \binom{2}{1} d',
\]
and hence that
\[
B(2I)^{-1}B' = 2d' I^{-1}d \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} = \frac{b}{90} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.
\]
Thus we have
\[
\Sigma_1 = \Sigma - B(2I)^{-1}B' = \frac{1}{90} \begin{pmatrix} 16 & 8 \\ 8 & 5 \end{pmatrix} - \frac{b}{90} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}
= \frac{1}{90} \begin{pmatrix} 4(4-b) & 2(4-b) \\ 2(4-b) & 5-b \end{pmatrix},
\]
and
\[
\Sigma_1^{-1} = 90 \begin{pmatrix} 5 - b/(4(4-b)) & -1/2 \\ -1/2 & 1 \end{pmatrix}.
\]
Therefore
\[
\hat{D}_n^{(2)} = n(\hat{W} - \mu)' \Sigma_1^{-1}(\hat{W} - \mu) \xrightarrow{D} \chi^2(2),
\]
which (in extended form) proves (2.9).

Finally, writing \( a = (-1/2) \) we see that
\[
\sqrt{n} \left( \hat{Y}_n - \frac{1}{2} \hat{Y}_n - \frac{1}{3} \right) = a'(\sqrt{n} (\hat{W} - \mu)) \xrightarrow{D} a'W \sim N(0, a' \Sigma_1 a),
\]
and \( a' \Sigma_1 a = 1/90 \), which shows (2.8) and completes the proof of Theorem 6.
Proposition 3. Goodness-of-fit tests for \( X \sim \text{Exp}(\alpha) \) are given by

\[
\begin{align*}
\hat{D}_n^{(1)} &:= D_n^{(1)}(\hat{\alpha}_{2n}) \\
&= 45 \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - e^{-\hat{\alpha}_{2n}X_j})^2 + \frac{1}{n} \sum_{j=1}^{n} e^{-\hat{\alpha}_{2n}X_j^*} - \frac{2}{3} \right)^2 \\
&\xrightarrow{D} \chi^2(1), \\
\hat{D}_n^{(2)} &:= D_n^{(2)}(\hat{\alpha}_{2n}) \\
&= \frac{45 \cdot 36}{19} \cdot 2n \left( \frac{1}{2n} \sum_{j=1}^{2n} (1 - e^{-\hat{\alpha}_{2n}X_j})^2 - \frac{1}{3} \right)^2 + \hat{D}_n^{(1)} \xrightarrow{D} \chi^2(2),
\end{align*}
\]

where \( \hat{\alpha}_{2n} = 1/\overline{X}_{2n} \).

Proof. The first statement of Proposition 3 follows from (c) after Lemma 2 and Theorem 6. To prove the second statement it is enough to see that for \( X \sim \text{Exp}(\alpha) \) we have \( I(\alpha) = 1/\alpha^2 \),

\[
d = \alpha \int_0^\infty (xe^{-2\alpha x} - xe^{-3\alpha x}) \, dx = 5/(36\alpha),
\]

and \( b = 125/36 \), which by (2.9) gives the test statistic \( \hat{D}_n^{(2)} \).

Proposition 4. Goodness-of-fit tests for \( X \sim N(\mu, \sigma^2) \) with

\[
\begin{align*}
F(x) &= \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^x e^{-(t-\mu)^2/(2\sigma^2)} \, dt, \\
f(x) &= \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/(2\sigma^2)}, \quad -\infty < x < \infty,
\end{align*}
\]

are given by

\[
\begin{align*}
\hat{D}_n^{(1)} &:= D_n^{(1)}(\hat{\mu}_{2n}, \hat{\sigma}_{2n}^2) \\
&= 45 \cdot 2n \left( (X_{2n} - \hat{\mu}_{2n})/\hat{\sigma}_{2n} \right) \left( \Phi^2 ((X_{2n} - \hat{\mu}_{2n})/\hat{\sigma}_{2n}) - \Phi((X_{2n}^* - \hat{\mu}_{2n})/\hat{\sigma}_{2n}) + \frac{1}{3} \right)^2 \\
&\xrightarrow{D} \chi^2(1), \\
\hat{D}_n^{(2)} &:= D_n^{(2)}(\hat{\mu}_{2n}, \hat{\sigma}_{2n}^2) \\
&= \frac{45 \cdot 8\pi^2}{32\pi^2 - 15(6\pi + 1)} \cdot 2n \left( \Phi^2 ((X_{2n} - \hat{\mu}_{2n})/\hat{\sigma}_{2n}) - \frac{1}{3} \right)^2 + \hat{D}_n^{(1)},
\end{align*}
\]

where

\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} \, dt,
\]
\[
\Phi^2((X_{2n} - \hat{\mu}_{2n})/\hat{\sigma}_{2n}) = \frac{1}{2n} \sum_{j=1}^{2n} \Phi^2((X_j - \hat{\mu}_{2n})/\hat{\sigma}_{2n}),
\]
\[
\Phi((X_{2n}^* - \hat{\mu}_{2n})/\hat{\sigma}_{2n}) = \frac{1}{n} \sum_{j=1}^{n} \Phi((X_j^* - \hat{\mu}_n)/\hat{\sigma}_n),
\]
and
\[
\hat{\mu}_{2n} = X_{2n}, \quad \hat{\sigma}_{2n}^2 = \frac{1}{2n} \sum_{j=1}^{2n} (X_j - X_{2n})^2.
\]

**Proof.** Here
\[
I^{-1} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}
\]
and
\[
\frac{\partial F}{\sigma \mu} = -f, \quad \frac{\partial F}{\partial \sigma^2} = -\frac{1}{2\sigma^2} (x - \mu)f,
\]
so
\[
d_1 = -\int_{-\infty}^{\infty} F(x) f^2(x) \, dx, \quad d_2 = -\frac{1}{2\sigma^2} \int_{-\infty}^{\infty} (x - \mu) F(x) f^2(x) \, dx.
\]
To evaluate the integrals, write
\[
F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_1} e^{-y^2/2} \, dy = \frac{1}{2} + \psi(x_1),
\]
where
\[
x_1 = (x - \mu)/\sigma, \quad \psi(x_1) = \frac{1}{\sqrt{2\pi}} \int_{0}^{x_1} e^{-y^2/2} \, dy.
\]
Changing variables to \(x_1 = (x - \mu)/\sigma\) gives
\[
d_1 = -\frac{1}{2\pi \sigma} \int_{-\infty}^{\infty} \left( \frac{1}{2} + \psi(x_1) \right) e^{-x_1^2} \, dx_1 = -\frac{1}{4\pi \sigma} \int_{-\infty}^{\infty} e^{-x_1^2} \, dx_1 = -\frac{1}{4\sqrt{\pi} \sigma},
\]
where we have used the fact that \(\psi\) is an odd function. Similarly
\[
d_2 = -\frac{1}{4\pi \sigma^2} \int_{-\infty}^{\infty} \left( \frac{1}{2} + \psi(x_1) \right) x_1 e^{-x_1^2} \, dx_1 = -\frac{1}{4\pi \sigma^2} \int_{-\infty}^{\infty} \psi(x_1) x_1 e^{-x_1^2} \, dx_1
\]
\[
= -\frac{1}{8\pi \sigma^2} \int_{-\infty}^{\infty} \psi'(x_1) e^{-x_1^2} \, dx_1 = -\frac{1}{8\sqrt{3} \pi \sigma^2},
\]
where we have used integration by parts and the facts that \(x_1 e^{-x_1^2}\) is an odd function and
\[
\psi'(x_1) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2}.
\]
Hence
\[ b = 180(d_1, d_2) \left( \begin{array}{cc} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{array} \right) (d_1, d_2)' = \frac{15(6\pi + 1)}{8\pi^2}, \]
which by (2.9) leads us to the \( \hat{D}_n^{(2)} \) test. Then \( \hat{D}_n^{(1)} \) is obtained immediately from (2.8).

3. Goodness-of-fit tests based on characterizations via moments of record values. Suppose that \( X \) has df \( F \) and pdf \( f \). To simplify the notation we write
\[ g(x) = 1 - F(x) \quad \text{and} \quad h(x) = -\log(g(x)) \]
if \( F(x) < 1 \) and 0 otherwise.

Then Theorem 3 says (see (1.5)) that \( X \sim F \) iff
\[ k^{2l}(n - l)!Eh^{2l}(Y_{n-l+1}^{(k)}) - 2n!k^lEh^l(Y_{n+1}^{(k)}) + (n + l)! = 0. \]
Since the definition of \( Y_n^{(k)} \) requires an infinite sequence it is hard to see how a finite sample can be used to estimate \( EY_n^{(k)} \). So our procedure is as follows.

We consider the special case \( l = n \). Then \( X \sim F \) iff
\[ (3.1) \quad Eh^{2n}(X_{1:k}) - \frac{2n!}{k^n} Eh^n(Y_{n+1}^{(k)}) + \frac{(2n)!}{k^{2n}} = 0. \]
We know that the pdf of \( Y_n^{(k)} \) is
\[ f_{Y_n^{(k)}}(x) = \frac{k^n}{(n-1)!} h^{n-1}(x) g^{k-1}(x) f(x) \quad \text{(cf. [1])} \]
and that
\[ (3.2) \quad F_{Y_n^{(k+1)}}(x) = F_{Y_n^{(k)}}(x) - \frac{k^n}{n!} h^n(x) g^k(x) \]
\[ = 1 - g^k(x) \sum_{j=0}^{n} \frac{k^j}{j!} h^j(x) \quad \text{(cf. [2])}. \]
Hence
\[ Eh^n(Y_{n+1}^{(k)}) = Eh^n(Y_n^{(k)}) - \frac{k^n}{(n-1)!} Eh^{2n-1}(X) g^{k-1}(X) \]
\[ + \frac{k^{n+1}}{n!} Eh^{2n}(X) g^{k-1}(X). \]
Taking into account that
\[ Eg^{\alpha-1}(X)h^{\beta-1}(X) = \frac{\Gamma(\beta)}{\alpha^\beta} \quad \text{for} \ \alpha, \beta > 0 \]
as $X$ has df $F$, we get
\[ Eh^n(Y_{n+1}^{(k)}) = Eh^n(Y_n^{(k)}) + \frac{(2n)!}{2n! k^n}. \]

Hence by (3.1) we obtain
\[ Eh^{2n}(X_{1:k}) - \frac{2n!}{k^n} Eh^n(Y_n^{(k)}) = 0. \]

Letting $n = 1$ we have
\[ (3.3) \quad Eh^2(X_{1:k}) - \frac{2}{k} Eh(X_{1:k}) = 0. \]

Similarly using the second equality in (3.2) we get
\[ (3.4) \quad Eh^{2n}(X_{1:k}) - \frac{2n!}{k^n} Eh^n(X_{1:k}) - \frac{(2n)! - 2(n)!^2}{k^{2n}} = 0. \]

To verify $H : X \sim F$ we use (3.3). Consider first the case $k = 1$. Then
\[ E(h^2(X_1) - 2h(X_1)) = 0. \]

The sample $(X_1, \ldots, X_n)$ provides an estimator of $EW_1$, where $W_1 = h^2(X_1) - 2h(X_1)$, of the form
\[ \overline{W_n} = \overline{h^2(X_n)} - 2\overline{h(X_n)}, \]

where
\[ \overline{h^2(X_n)} = \frac{1}{n} \sum_{j=1}^{n} h^2(X_j), \quad \overline{h(X_n)} = \frac{1}{n} \sum_{j=1}^{n} h(X_j). \]

It follows from the CLT that
\[ \sqrt{n} \overline{W_n} \Rightarrow N(0, \text{Var}(W_1)), \]

and hence that
\[ T_n^{(1)} := n\overline{W_n}^2 / \text{Var}(W_1) \Rightarrow \chi^2(1), \]

and so provides a simple asymptotic test of the hypothesis $X \sim F$ when the parameters of $F$ are known. Here
\[ \text{Var} W_1 = Eh^4(X_1) - 4Eh^3(X_1) + 4Eh^2(X_1) = 8 \]

since $h(X_1) \sim \text{Exp}(1)$ gives $Eh^m(X_1) = m!, \ m = 1, 2, \ldots$, and so
\[ T_n^{(1)} = \frac{n}{8} (\overline{h^2(X_n)} - 2\overline{h(X_n)})^2. \]

We have proved

**Proposition 5.** If $X_n \sim F$, $n \geq 1$, are independent then
\[ (3.5) \quad T_n^{(1)} = \frac{n}{8} (\overline{h^2(X_n)} - 2\overline{h(X_n)})^2 \Rightarrow \chi^2(1). \]
Now consider the case \( k = 2 \). Write \( U_1 := X_{1:2} = \min(X_1, X_2) \). Here from (3.3) we have to estimate \( EW'_1 \), where \( W'_1 = h^2(U_1) - h(U_1) \). The sample \( X_1, \ldots, X_{2n} \) provides the sample \( W'_1, \ldots, W'_n \), where \( W'_j = h^2(U_j) - h(U_j) \) and \( U_j = \min(X_{2j-1}, X_{2j}) \), \( j = 1, \ldots, n \). Then \( EW'_1 \) is estimated by
\[
W'_n = h^2(U_n) - h(U_n),
\]
and
\[
T^{(2)}_n := n(W'_n)^2 / \text{Var}(W'_1) \overset{D}{\to} \chi^2(1).
\]
Taking into account that \( h(U_1) \sim \text{Exp}(2) \) we see that \( \text{Var}(W'_1) = 1/2. \) Thus another simple asymptotic test is provided by

**Proposition 6.** If \( X_n \sim F, n \geq 1, \) are independent then
\[
(3.6) \quad T^{(2)}_n = 2n(h^2(U_n) - h(U_n))^2 \overset{D}{\to} \chi^2(1).
\]

The same argument leads to a similar test for the case \( k = 3, \ldots, n - 1 \) based on a sample of size \( kn \).

We now consider the case \( k = n \). Write \( U_n = \min(X_1, \ldots, X_n) \). Then by (3.3) we have to estimate \( E(h^2(U_n) - (2/n)h(U_n)) \). The obvious estimate is \( h^2(U_n) - (2/n)h(U_n) \) itself, and if the parameters of \( F \) are specified the test statistic is
\[
T^{(n)}_n := \left( h^2(U_n) - \frac{2}{n} h(U_n) \right)^2.
\]
As above, under \( H, h(U_n) \sim \text{Exp}(n) \), whence
\[
(3.7) \quad R_n := nh(U_n) \sim U \sim \text{Exp}(1), \quad n \geq 1.
\]
It follows that
\[
T^{(n)}_n = \frac{1}{n^4}(R_n^2 - 2R_n)^2
\]
and so an equivalent test statistic is \( T_n := (R_n^2 - 2R_n)^2 \sim T := (U^2 - 2U)^2, n \geq 1, \) which provides an exact test for \( H : X \sim F. \)

**Proposition 7 (cf. [7]).** The significance probability of the test using \( T_n \) is
\[
(3.8) \quad P_t := P[T_n > t]
\]
\[
= \begin{cases} 
  e^{-1-\sqrt{1+\sqrt{t}}} + e^{-1+\sqrt{1-\sqrt{t}}} - e^{-1-\sqrt{1-\sqrt{t}}} & \text{if } 0 < t \leq 1, \\
  e^{-1-\sqrt{1+\sqrt{t}}} & \text{if } t \geq 1.
\end{cases}
\]

**Proof.** The significance probability \( P[T_n > t] \) associated with an observed value \( t \) can be obtained by considering the graph of \( u^2(u - 2)^2 = t \) and using the fact that \( P[U < u] = 1 - e^{-u} \). One finds readily that (3.8) holds true.
In particular we consider the 5% test of \( H \), i.e. \( P_t = 0.05 \). But since
\[
P[T > 1] = e^{-(1+\sqrt{2})} > 0.05
\]
the 5% test rejects when \( R_n > u_0 \), where \( e^{-u_0} = 0.05 \), i.e. when \( u_0 = 3.00 \). Thus the exact 5% test rejects when \( nh(U_n) > 3 \).

Now we show that instead of \( T_n = [R_n^2 - 2R_n]^2 \) one can use more generally the statistics
\[
T_n^{[m]} := \{(R_n^m - m!)^2 - ((2m)! - (m!)^2)\}^2, \quad m \geq 1.
\]
We note that \( T_n = T_n^{[1]} \).

Writing (3.4) in the form
\[
Ek^{2m}h^{2m}(X_{1:k}) - 2m!Ek^m h^m(X_{1:k}) - ((2m)! - 2(m!)^2) = 0
\]
and letting \( k = n \) (sample size), we have
\[
E\{(nh(X_{1:n}))^m - m!\}^2 - ((2m)! - (m!)^2) = 0.
\]
Taking into account that \( R_n = nh(X_{1:n}) \sim \text{Exp}(1) \), \( n \geq 1 \), we see that
\[
T_n^{[m]} = \{(R_n^m - m!)^2 - a_m\}^2 \sim [(U^m - m!)^2 - a_m]^2
\]
where
\[
a_m = (2m)! - (m!)^2.
\]
It follows that the statistics \( T_n^{[m]} \) have for every \( n \geq 1 \) the distribution of \([(U^m - m!)^2 - a_m]^2\), and we reject \( H : X \sim F \) if \( T_n^{[m]} \) is large enough. Moreover, we can state the following result.

**Proposition 8.** The significance probability of the test using \( T_n^{[m]} \) is
\[
(3.9) \quad P_t^{[m]} := P[T_n^{[m]} > t]
\]
\[
= \begin{cases} 
1 - e^{-b_1^{(2)}(t)} + e^{-b_3^{(2)}(t)} & \text{if } 0 < t \leq t_m, \\
-e^{-b_1^{(1)}(t)} - e^{-b_3^{(1)}(t)} + e^{-b_3^{(3)}(t)} & \text{if } t_m < t \leq t_m', \\
e^{-b_3^{(3)}(t)} & \text{if } t > t_m',
\end{cases}
\]
where
\[
\begin{align*}
b_1^{(1)}(t) &= (m! - \sqrt{a_m - \sqrt{t}})^{1/m}, & b_1^{(2)}(t) &= (m! + \sqrt{a_m - \sqrt{t}})^{1/m}, \\
b_3^{(3)}(t) &= (m! + \sqrt{a_m + \sqrt{t}})^{1/m}, & t_m &= (a_m - (m!)^2)^2, & t_m' &= a_m^2.
\end{align*}
\]
The proof of (3.9) is similar to the proof of Proposition 7.
Corollary. $P_{t}^{[1]}$ is given by (3.8) and $P_{t}^{[2]}$ is given by the formula

$$P_{t}^{[2]} = P[T_{n}^{[2]} > t] = \begin{cases} 1 - e^{-\sqrt{2+\sqrt{20+\sqrt{t}}}} + e^{-\sqrt{2+\sqrt{20+\sqrt{t}}}} & \text{if } 0 < t \leq 256, \\
e^{-\sqrt{\frac{2}{\sqrt{20-\sqrt{t}}}}} - e^{-\sqrt{2+\sqrt{20-\sqrt{t}}}} + e^{-\sqrt{2+\sqrt{20+\sqrt{t}}}} & \text{if } 256 < t \leq 400, \\
e^{-\sqrt{2+\sqrt{20+\sqrt{t}}}} & \text{if } t > 400. \end{cases}$$

4. Tests for exponentiality. We consider corresponding tests for $X \sim \text{Exp}(\alpha)$ when $\alpha$ is not specified. Note that in this case $h(x) = -\log(1 - F(x)) = \alpha x$. Using $T_{n}^{(1)} := T_{n}^{(1)}(\alpha)$, $T_{n}^{(2)} := T_{n}^{(2)}(\alpha)$ in (3.5) and (3.6) respectively, we replace $\alpha$ by the estimator $\hat{\alpha}_{n}$. We have proved in [7] the following results.

Proposition 9. If $X_n \sim F$, $n \geq 1$, are independent then

$$\hat{T}_{n}^{(1)} := 2T_{n}^{(1)}(\hat{\alpha}_{n}) = \frac{n}{4n} \left(\frac{X_{n}^{2}}{X_{n}^{2}} - 2\right)^{2} \xrightarrow{D} \chi^{2}(1),$$

where $\hat{\alpha}_{n} = 1/X_{n}$.

Proposition 10. If $X_n \sim F$, $n \geq 1$, are independent then

$$\hat{T}_{n}^{(2)} := \frac{4}{3}T_{n}^{(2)}(\hat{\alpha}_{n}) = \frac{8}{3} \left(\frac{U_{n}^{2}}{n\hat{\alpha}_{n}} - \frac{1}{\hat{\alpha}_{n}} U_{n}^{2} \right)^{2} = \frac{8}{3} \left(\frac{U_{n}^{2}}{X_{2n}^{2}} - \frac{U_{n}^{2}}{X_{2n}^{2}} \right)^{2} \xrightarrow{D} \chi^{2}(1),$$

where $\hat{\alpha}_{n} = 1/X_{2n}$.

Proposition 11. Let $\hat{T}_{n} := T(\hat{\alpha}_{n}) = (\hat{U}_{n}^{2} - 2\hat{U}_{n})^{2}$ where $\hat{U}_{n} = n\hat{\alpha}_{n}U_{n} = nU_{n}/\overline{X}_{n}$, and let $\hat{P}_{t} := P[\hat{T}_{n} > t]$ stand for the associated significance probability. Then $\lim_{n \to \infty} \hat{P}_{t} = P_{t}$, where $P_{t}$ is given by Proposition 7.

Now by Proposition 8 we have the following generalization of Proposition 11.

Proposition 12. Let

$$\hat{T}_{n}^{[m]} := T_{n}^{[m]}(\hat{\alpha}_{n}) = \{(n\hat{\alpha}_{n}U_{n})^{m} - m!\}^{2} \xrightarrow{D} \chi^{2}(1),$$

and let $\hat{P}_{t}^{[m]} := P[\hat{T}_{n}^{[m]} > t]$ stand for the associated significance probability. Then

$$\lim_{n \to \infty} \hat{P}_{t}^{[m]} = P_{t}^{[m]} , \quad m \geq 1,$$

where $P_{t}^{[m]}$ is given by Proposition 8.

Proof. Since $\hat{\alpha}_{n} \xrightarrow{P} \alpha$, from (3.7) we get $n\hat{\alpha}_{n}U_{n} = (\hat{\alpha}_{n}/\alpha)R_{n} \xrightarrow{D} U$ and so

$$\hat{T}_{n}^{[m]} \xrightarrow{D} [(U^{m} - m!)^{2} - a_{m}]^{2},$$

which is distributed as $T_{n}^{[m]}$.
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