APPLICATIONES MATHEMATICAE
28,2 (2001), pp. 211–224

P. Sztuba and A. Weron (Wrocław)

PRICING FORWARD-START OPTIONS IN THE HJM FRAMEWORK; EVIDENCE FROM THE POLISH MARKET

Abstract. We show how to use the Gaussian HJM model to price modified forward-start options. Using data from the Polish market we calibrate the model and price this exotic option on the term structure. The specific problems of Central Eastern European emerging markets do not permit the use of the popular lognormal models of forward LIBOR or swap rates. We show how to overcome this difficulty.

1. The Gaussian HJM model. Let $[0,\tau]$, for $\tau > 0$, be a trading interval and $(\Omega, \{\mathcal{F}_t : t \in [0,\tau]\}, \boldsymbol{P})$ be a probability space, where $\mathcal{F}_t$ is the $\boldsymbol{P}$-augmentation of the natural filtration generated by an n-dimensional Brownian motion $W(t) = \{W_1(t), \ldots, W_n(t)\}$. Assume that there are zero-coupon bonds in the market with all maturities and face value 1. For $t \le T$ let $P(t,T)$ denote the t time price of such a T maturity bond. Of course $P(T,T) \equiv 1$ must hold for all T. We require that $\partial \log P(t,T)/\partial T$ exists. We would like to consider a rate at time t of lending 1 unit at time T, which shall be returned after period δ. The loan costs $P(t,T) - P(t,T+\delta)$. Therefore the rate R of return of such a loan at time t can be evaluated from the equation

$$-1 \cdot P(t,T) + (1+\delta R)P(t,T+\delta) = 0,$$

hence

$$R = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)}.$$

To define the basic process in the Heath–Jarrow–Morton (HJM) model [3], we take the limit of the above expression as $\delta \to 0$. Thus the instantaneous

2000 *Mathematics Subject Classification*: Primary 90A09; Secondary 60G44, 60H30.
Key words and phrases: exotic option, hedging, term structure of interest rates.

forward rate $f(t,T)$ for $t<T$ is defined as

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}.$$

Solving this differential equation we obtain

$$P(t,T) = \exp\Big(-\int_t^T f(t,s)\,ds\Big).$$

When $t=T$ the forward rate $f(t,t)$ is called the *spot rate* and is denoted by $r(t)$. Similarly one can interpret $r(t)$ as the rate of loan at time t which is returned an instant later. The dynamics of the forward rate $f(t,T)$ is assumed to be driven by the standard Brownian motion as given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)^* dW(t),$$

where * denotes transposition.

Under certain assumptions about the processes $\alpha(t,T)$ and $\sigma(t,T)$ (see [3]), the process $P(t,T)$ can be expressed as a solution of the SDE

$$dP(t,T) = P(t,T)([r(t)+b(t,T)]dt + a(t,T)^* dW(t)),$$

where $a(t,T) = -\int_t^T \sigma(t,v)\,dv$ and $b(t,T) = -\int_t^T \alpha(t,v)\,dv + \frac{1}{2}|a(t,T)|^2$.

Let the price process of a savings account be given by $B(t) = \exp(\int_0^t r(s)\,ds)$. If one invests 1 unit in the cash market, after time t one will get the amount $B(t)$. The conditions on the processes $\alpha(t,T)$ and $\sigma(t,T)$ which assure existence and uniqueness of a measure Q (making the discounted bond price process a martingale) are known [3].

Thus if $\widetilde{W}(t)$ denotes Brownian motion with respect to Q we get

$$df(t,T) = \sigma(t,T)^* \int_t^T \sigma(t,y)\,dy\,dt + \sigma(t,T)^* d\widetilde{W}(t)$$

and

$$dP(t,T) = P(t,T)r(t)dt - P(t,T)\int_t^T \sigma(t,v)^*\,d\widetilde{W}(t).$$

In this case each financial instrument with pay-off function X, which is an $\mathcal{F}_T$-measurable random variable, can be priced at time t by using the formula

$$\mathrm{E}^Q\bigg(\frac{B(t)}{B(T)}X \,\bigg|\, \mathcal{F}_t\bigg). \tag{1}$$

In this paper we will assume that the volatility $\sigma(t,T)$ of the forward rate is a deterministic function. In the literature such a case is referred to as the *Gaussian HJM model* (see [6]).

2. Forward-start call options. In this section we will focuss on the so-called *forward-start options*, i.e. options that are paid for at present but received by holders at a prespecified future date. In the classic Black–Scholes model the forward-start call option has terminal pay-off $(S_T - S_{T_0})^+$, where $T_0 < T$ and its price can be easily obtained since it suffices to consider the option's value at the delivery date T_0. That is,

$$\mathbf{C}_{T_0}^{\mathrm{FS}} = S_{T_0}(\Phi(\widetilde{d}_+) - e^{-r(T-T_0)}\Phi(\widetilde{d}_-))$$

where $\Phi(\cdot)$ is the normal cumulative distribution function and

$$\widetilde{d}_\pm = \frac{(r \pm \sigma^2/2)(T-T_0)}{\sigma\sqrt{T-T_0}}$$

(see [6], p. 207). However, in the case of a fixed-income market the situation is more complex.

Let us consider a call option with maturity u on a zero-coupon bond $P(\cdot,T)$, $u < T$, where the strike price K is $e^R P(s,T)$, $s < u$. If the price of the bond increases over the interval (s,u) at a rate higher than R, than the buyer gets $P(u,T) - e^R P(s,T)$. If not, the pay-off is 0. To price this security we will need the following lemmas, which are crucial to the application of the forward measure technique.

LEMMA 2.1 ([7]). *Define a new forward measure by letting, for all* $A \in \mathcal{F}_T$,

$$Q_T(A) = \int_A (P(0,T)B(T))^{-1}\,dQ.$$

Then for any random variable X which is $\mathcal{F}_T$-measurable and such that $\mathrm{E}|X|^p < \infty$ *for a certain $p > 1$ we have*

$$\mathrm{E}^Q\left(\frac{B(t)}{B(T)}X \,\middle|\, \mathcal{F}_t\right) = P(t,T)\mathrm{E}^{Q_T}(X \mid \mathcal{F}_t).$$

LEMMA 2.2. *For $t \le u \le T$ and every $\mathcal{F}_u$-measurable pay-off function X the following relation holds*:

$$\mathrm{E}^Q\left(\frac{B(t)}{B(u)}P(u,T)X \,\middle|\, \mathcal{F}_t\right) = \mathrm{E}^Q\left(\frac{B(t)}{B(T)}X \,\middle|\, \mathcal{F}_t\right).$$

Proof. Because $B(t)$, $B(u)$ and X are $\mathcal{F}_u$-measurable we have

$$\begin{aligned}\mathrm{E}^Q\left(\frac{B(t)}{B(T)}X \,\middle|\, \mathcal{F}_t\right) &= \mathrm{E}^Q\left(\mathrm{E}^Q\left(\frac{B(t)}{B(u)}\frac{B(u)}{B(T)}X \,\middle|\, \mathcal{F}_u\right) \,\middle|\, \mathcal{F}_t\right)\\ &= \mathrm{E}^Q\left(\frac{B(t)}{B(u)}X\mathrm{E}^Q\left(\frac{B(u)}{B(T)} \,\middle|\, \mathcal{F}_u\right) \,\middle|\, \mathcal{F}_t\right)\\ &= \mathrm{E}^Q\left(\frac{B(t)}{B(u)}XP(u,T) \,\middle|\, \mathcal{F}_t\right). \ \blacksquare\end{aligned}$$

LEMMA 2.3. *For any $0 \le t \le u \le T$ the inverted bond price process has the following representation*:

$$P(u,T)^{-1} = \frac{P(t,u)}{P(t,T)} \exp\Big(\int_t^u \int_u^T \sigma(x,s)^*\,ds\,dW(x,T) - \frac{1}{2}\int_t^u \Big|\int_u^T \sigma(x,s)\,ds\Big|^2 dx\Big),$$

where $W(t,T)$ denotes the Brownian motion with respect to the forward measure Q_T.

Proof. The bond price process satisfies the following SDE:

$$dP(t,T) = P(t,T)\Big(r(t)dt - \int_t^T \sigma(t,s)^*\,ds\,dW(t)\Big),$$

therefore by Itô's lemma,

$$d\frac{P(t,u)}{P(t,T)} = \frac{P(t,u)}{P(t,T)} \int_u^T \sigma(t,s)^*\,ds\Big(\int_t^T \sigma(t,s)\,ds\,dt + dW(t)\Big),$$
$$= \frac{P(t,u)}{P(t,T)} \int_u^T \sigma(t,s)^*\,ds\,dW(t,T).$$

Hence

$$(2)\quad \frac{P(t,u)}{P(t,T)} = \frac{P(0,u)}{P(0,T)} \exp\Big(\int_0^t \int_u^T \sigma(x,s)^*\,ds\,dW(x,T) - \frac{1}{2}\int_0^t \Big|\int_u^T \sigma(x,s)\,ds\Big|^2 dx\Big)$$

for any $t \le u$. This yields the result because $P(u,T)^{-1} = P(u,u)/P(u,T)$. ∎

LEMMA 2.4. *Assume that a random variable Z has normal $N(-\operatorname{Var}\zeta/2, \operatorname{Var}\zeta)$ distribution. Then for every positive constant K,*

$$\mathrm{E}((1-K\exp(Z))^+) = \Phi\Big(\frac{-\log K + \frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\Big) - K\Phi\Big(\frac{-\log K - \frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\Big).$$

Proof. Set $ke^{-rT} = K^{-1}$, $S_0 = 1$, $\sigma^2 T = \operatorname{Var}\zeta$. Then

$$\mathrm{E}((1-K\exp(Z))^+) = \frac{e^{rT}}{k}\mathrm{E}(e^{-rT}(k - S_0 e^{Z+rT})^+)$$

and the expectation is equal to the value of a European put option on a stock for

$$S_0 e^{Z+rT} = S_0 e^{\sigma B_T(-\sigma^2/2+r)T} = S_T.$$

Thus the Black–Scholes formula yields

$$\begin{aligned}\mathrm{E}((1-K\exp(Z))^+) &= \frac{e^{rT}}{k}\biggl(ke^{-rT}\Phi\biggl(\frac{\log k/S_0-(r-\sigma^2/2)T}{\sigma\sqrt{T}}\biggr)\\ &\qquad - S_0\Phi\biggl(\frac{\log k/S_0-(r+\sigma^2/2)T}{\sigma\sqrt{T}}\biggr)\biggr)\\ &= K\biggl(K^{-1}\Phi\biggl(\frac{\log K^{-1}+\frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\biggr)\\ &\qquad -\Phi\biggl(\frac{\log K^{-1}-\frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\biggr)\biggr). \quad \blacksquare\end{aligned}$$

Now, we will price the forward-start options.

THEOREM 2.1. *The price of a forward-start call option with maturity* $u < T$ *and strike price* $e^R P(s,T)$, *for* $s < u$, *on a zero-coupon bond with price process* $P(\cdot,T)$ *is given by the following formula*:

$$\mathbf{C}_t^{\mathrm{FS}} = P(t,T)\biggl[\Phi(d_+) - \frac{P(t,u)}{P(t,s)}e^{R+A}\Phi(d_-)\biggr], \tag{3}$$

where

$$d_\pm = \frac{\log\dfrac{P(t,s)}{P(t,u)} - R - A \pm B/2}{\sqrt{B}},$$

$$A = \frac{1}{2}\int_t^s\biggl(\biggl|\int_s^T \sigma(x,y)\,dy\biggr|^2 + \biggl|\int_s^u \sigma(x,y)\,dy\biggr|^2 - \biggl|\int_u^T \sigma(x,y)\,dy\biggr|^2\biggr)dx,$$

$$B = \int_s^u\biggl|\int_u^T \sigma(y,x)\,dx\biggr|^2 dy + \int_t^s\biggl|\int_s^u \sigma(y,x)\,dx\biggr|^2 dy.$$

Proof. Let $t < s < u < T$. We start by applying formula (1):

$$\begin{aligned}\mathbf{C}_t^{\mathrm{FS}} &\equiv \mathrm{E}^Q\biggl(\frac{B(t)}{B(u)}(P(u,T)-e^R P(s,T))^+ \biggm| \mathcal{F}_t\biggr)\\ &= \mathrm{E}^Q\biggl(\frac{B(t)}{B(T)}\biggl(1-e^R\frac{P(s,T)}{P(u,T)}\biggr)^+ \biggm| \mathcal{F}_t\biggr)\end{aligned}$$

because of Lemma 2.2.

Using Lemmas 2.3 and 2.1 we can write

$$\mathbf{C}_t^{\mathrm{FS}} = P(t,T)\mathrm{E}^{Q_T}\bigg(\bigg[1 - \frac{P(t,u)}{P(t,s)}\exp\bigg(R + \int_t^u\int_u^T \sigma(y,x)\,dx\,dW(y,T) - \int_t^s\int_s^T \sigma(y,x)\,dx\,dW(y,T) - \frac{1}{2}\int_t^u\bigg|\int_u^T \sigma(y,x)\,dx\bigg|^2 dy + \frac{1}{2}\int_t^s\bigg|\int_s^T \sigma(y,x)\,dx\bigg|^2 dy\bigg)\bigg]^+ \bigg|\, \mathcal{F}_t\bigg).$$

Let a random variable ζ be given by the formula

$$\zeta = \int_s^u\int_u^T \sigma(y,x)\,dx\,dW(y,T) - \int_t^s\int_s^u \sigma(y,x)\,dx\,dW(y,T).$$

Then ζ has normal distribution with respect to the measure Q_T with mean 0 and variance

$$\operatorname{Var}\zeta = \int_s^u\bigg|\int_u^T \sigma(y,x)\,dx\bigg|^2 dy + \int_t^s\bigg|\int_s^u \sigma(y,x)\,dx\bigg|^2 dy.$$

Finally we get

$$\mathbf{C}_t^{\mathrm{FS}} = P(t,T)\mathrm{E}^{Q_T}\big(\big(1 - K\exp(\zeta - \tfrac{1}{2}\operatorname{Var}\zeta)\big)^+ \,\big|\, \mathcal{F}_t\big)$$

for

$$K = \frac{P(t,u)}{P(t,s)}\exp\bigg(R + \frac{1}{2}\int_t^s\bigg(\bigg|\int_s^T \sigma(x,y)\,dy\bigg|^2 + \bigg|\int_s^u \sigma(x,y)\,dy\bigg|^2 - \bigg|\int_u^T \sigma(x,y)\,dy\bigg|^2\bigg)dx\bigg)$$

and because K is $\mathcal{F}_t$-measurable and ζ is independent of $\mathcal{F}_t$ we can apply Lemma 2.4 to obtain the final result:

$$\mathbf{C}_t^{\mathrm{FS}} = P(t,T)\bigg(\Phi\bigg(\frac{-\log K + \frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\bigg) - K\Phi\bigg(\frac{-\log K - \frac{1}{2}\operatorname{Var}\zeta}{\sqrt{\operatorname{Var}\zeta}}\bigg)\bigg). \blacksquare$$

REMARKS. It is easy to observe that:

- $\lim_{R\to\infty}\mathbf{C}_t^{\mathrm{FS}} = 0$.
- $\lim_{s\to u}\mathbf{C}_t^{\mathrm{FS}} = 0$ for $R \ge 0$.
- For $s = t$ the price $\mathbf{C}_t^{\mathrm{FS}}$ is equal to the price of a European call option on a bond struck (at time t) at the current price of the bond with maturity T, i.e. $K = e^R P(t,T)$.

3. Calibration of the model. In this section the one-factor model with constant volatility function $\sigma(t,T) \equiv \sigma$ will be considered.

The standard procedure when dealing with an interest rate in the HJM model is as follows:

- at time $t_0 = 0$ use market data to calibrate (fit) the model to the observed bond prices,
- use the calibrated model to compute prices of various interest rate derivatives,
- at time $t_1 = t + \theta$, repeat the above procedure in order to recalibrate the model, etc.

Let us fix times $0 = t_0 < t_1 < \ldots < t_n$. We can get the following t_i-prices of $(t_i + \delta)$-maturity bills from the market: $P(t_0, t_0 + \delta), P(t_1, t_1 + \delta), \ldots, P(t_n, t_n + \delta)$ for some fixed δ. For example, in the case studied in the next section, $\delta = 364/365$, which corresponds to 52-week bills traded on the Polish money market. We work with a model with continuously compounded rates. The yield curve determined by the market at time $t_0 = 0$ is denoted by $Y(0,\cdot)$. Thus for each $T > 0$ we have

$$P(0,T) = \exp(-T \cdot Y(0,T)).$$

According to [3] in a one-factor HJM model with a constant volatility σ, the price of a zero-coupon bond is given by the expression

$$P(t,T) = \frac{P(0,T)}{P(0,t)} \exp[-(\sigma^2/2)Tt(T-t) - \sigma(T-t)\widetilde{W}(t)].$$

The formula for the volatility σ is provided by the following theorem:

THEOREM 3.1. *For a one-factor HJM model the maximum likelihood estimator $\widehat{\sigma}$ of the constant volatility σ is given by the formula*

$$\widehat{\sigma} = \sqrt{\frac{-n + \sqrt{n^2 + 4\sum_{i=0}^{n-1} a(i)^2 \sum_{i=0}^{n-1} x(i)^2}}{2\sum_{i=0}^{n-1} a(i)^2}}, \tag{4}$$

where $a(i) = \sqrt{t_{i+1} - t_i}\,(\delta + t_i + t_{i+1})/2$ *and*

$$x(i) = \frac{1}{\delta\sqrt{t_{i+1} - t_i}} \ln \frac{P(t_{i+1}, t_{i+1} + \delta)}{P(t_i, t_i + \delta)}$$
$$+ \frac{(t_{i+1} + \delta)Y(0, t_{i+1} + \delta) - t_{i+1}Y(0, t_{i+1}) - (t_i + \delta)Y(0, t_i + \delta) + t_iY(0, t_i)}{\delta\sqrt{t_{i+1} - t_i}}$$

for times $0 = t_0 < t_1 < \ldots < t_n$, $i = 0, \ldots, n-1$, $\delta > 0$.

Proof. For each t_i, $i = 0, \ldots, n$,

$$P(t_i, t_i+\delta) = \frac{P(0, t_i+\delta)}{P(0, t_i)} \exp[-(\sigma^2/2)(t_i+\delta)t_i\delta - \sigma\delta\widetilde{W}(t_i)] \tag{5}$$

$$= \exp[-(t_i+\delta)Y(0, t_i+\delta) + t_iY(0, t_i) - (\sigma^2/2)(t_i+\delta)t_i\delta - \sigma\delta\widetilde{W}(t_i)].$$

Taking the logarithms of both sides we get

$$\begin{aligned}\ln P(t_i, t_i+\delta) = &-(t_i+\delta)Y(0, t_i+\delta) + t_iY(0, t_i)\\ &-(\sigma^2/2)(t_i+\delta)t_i\delta - \sigma\delta\widetilde{W}(t_i).\end{aligned}$$

Therefore

$$\begin{aligned}\ln\frac{P(t_{i+1}, t_{i+1}+\delta)}{P(t_i, t_i+\delta)} = &-(t_{i+1}+\delta)Y(0, t_{i+1}+\delta) + t_{i+1}Y(0, t_{i+1})\\ &-(\sigma^2/2)(t_{i+1}+\delta)t_{i+1}\delta - \sigma\delta\widetilde{W}(t_{i+1})\\ &+(t_i+\delta)Y(0, t_i+\delta) - t_iY(0, t_i)\\ &+(\sigma^2/2)(t_i+\delta)t_i\delta + \sigma\delta\widetilde{W}(t_i).\end{aligned}$$

After dividing by $\delta\sqrt{t_{i+1}-t_i}$ we obtain

$$\begin{aligned}&\frac{1}{\delta\sqrt{t_{i+1}-t_i}}\ln\frac{P(t_{i+1}, t_{i+1}+\delta)}{P(t_i, t_i+\delta)}\\ &+\frac{(t_{i+1}+\delta)Y(0, t_{i+1}+\delta) - t_{i+1}Y(0, t_{i+1}) - (t_i+\delta)Y(0, t_i+\delta) + t_iY(0, t_i)}{\delta\sqrt{t_{i+1}-t_i}}\\ &= -(\sigma^2/2)\sqrt{t_{i+1}-t_i}\,(\delta+t_i+t_{i+1}) - \frac{\sigma}{\sqrt{t_{i+1}-t_i}}(\widetilde{W}(t_{i+1}) - \widetilde{W}(t_i)).\end{aligned} \tag{6}$$

The right side of the above equation yields, for $i = 0, \ldots, n-1$, a sequence of n independent random variables. Denote these variables by X_i, $i = 0, \ldots, n-1$. Each X_i is normally distributed with mean $-(\sigma^2/2)\sqrt{t_{i+1}-t_i}$ $(\delta + t_i + t_{i+1})$ and variance σ^2 (see (6)). Simultaneously, for each $i = 0, \ldots, n-1$, we can obtain a realization of every X_i by computing the left-hand side of this equation using only the market data. Hence, X_i has the following density function:

$$f_i(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\biggl(-\frac{(x-\sigma^2a(i))^2}{2\sigma^2}\biggr),$$

where $a(i) = \sqrt{t_{i+1}-t_i}\,(\delta + t_i + t_{i+1})/2$ for $i = 0, \ldots, n-1$.

Now we can evaluate σ using the maximum likelihood method. Since the ML function is given by

$$L(x(0),\dots,x(n-1),\sigma)=\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n\exp\left(\frac{1}{2\sigma^2}\sum_{i=0}^{n-1}(x(i)-\sigma^2 a(i))^2\right)$$

the equality $\partial L/\partial\sigma=0$ is satisfied for $\widehat{\sigma}$ given by (4). ■

4. The empirical analysis of the Polish market. Statistics show that a steady recovery began in the Central Eastern European countries in 1994. According to a CASE forecast the economic growth in Poland should rise above 5% mark in 2000 and even more than 6% in 2001. A substantial growth is also expected in the other countries of the region.

Uncertainty surrounding monetary growth plays an important role in reducing expectations for bond market risks. So as long as the US bond market remains stable, bond yields in Central Eastern Europe are also expected to decline. The expansion of liquidity in Poland and elsewhere appears to signal better growth ahead. However, the liquidity is likely to remain the main obstacle in implementation of advanced interest rate methodology from the hard-currency countries. Therefore in this section we demonstrate how one can deal with these difficulties for emerging markets, taking Poland as an example.

The HJM methodology of term structure modeling is based on the arbitrage-free dynamics of instantaneous continuously compounded forward rates. Hence, this methodology requires some degree of smoothness with respect to maturity of bond prices and their volatilities. Alternative approaches based on forward LIBOR and swap rates were developed recently (see [1], [4], [7] and [6]). However, on emerging markets we do not have yet the market practice of pricing caps, swaptions and other interest rate derivatives. Therefore we will develop a simple methodology based on the use of T-bills only.

On the Polish money market, every Monday T-bills are offered by the Polish National Bank (NBP). T-bills are zero-coupon securites with a face value of 100 PLN. They are offered at a discount and usually have three different maturites: 13, 26 and 52 weeks. For these T-bills average weighted rates of return are quoted in financial news services (an example is given in Fig. 1). On the secondary market the T-bills are traded before their maturities. A certain day is fixed to be time 0 at the beginning of the analysis. We assume that arbitrage is not possible between the T-bills and the cash market.

To obtain the yield curve we have chosen average weighted rates of return of 13, 26 and 52-week T-bills from the primary and the secondary market. In contrast to the well developed markets, the rates from the interbank cash market (WIBOR and WIBID) cannot be taken into account because as yet it is difficult to regard them as traded securities [8].

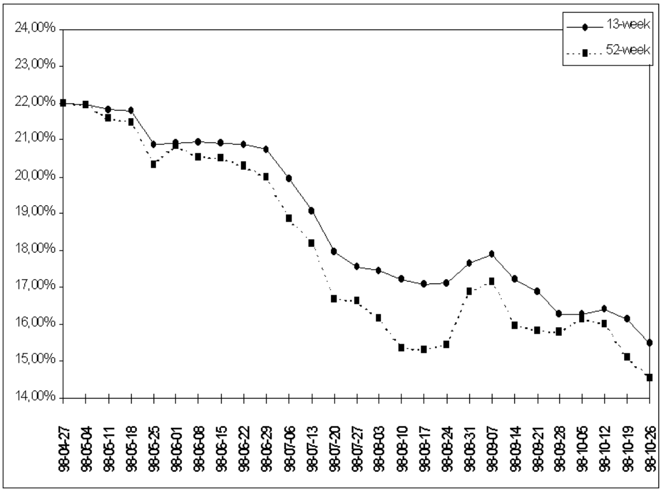

Fig. 1. Rates of Polish T-bills for two different maturities

Table 1. 52-week T-bills used for estimating volatility: average weighted prices and rates of return

i	Date	Model time t_i	Price $P(t_i, t_i + 364/365)$	Quoted yield r_{t_i}
0	98-06-01	0	82.49	20.85%
1	98-06-08	0.01918	82.76	20.53%
2	98-06-15	0.03836	82.8	20.50%
3	98-06-22	0.05753	82.92	20.31%
4	98-06-29	0.07671	83.12	20.00%
5	98-07-06	0.09590	83.96	18.89%
6	98-07-13	0.11509	84.46	18.20%
7	98-07-20	0.13427	85.53	16.69%
8	98-07-27	0.15342	85.55	16.65%
9	98-08-03	0.17260	85.92	16.18%
10	98-08-10	0.19178	86.45	15.36%
11	98-08-17	0.21096	86.45	15.33%
12	98-08-24	0.23014	86.10	15.46%
13	98-08-31	0.24932	85.14	16.89%
14	98-09-07	0.26849	85.14	17.15%

Table 1 presents results of consecutive auctions on Mondays. The date is the date of the offering, and the model time is a fraction of a 365-day year. The quoted yield is a rate computed for a 360-day year, that is,

$$e^{(364/365)\cdot Y(t,t+364/365)} = (1 + (364/360) \cdot r_t)$$

for every quoted rate r_t and time t. Following the algorithm of Section 3 with data taken from Table 1 we can use formula (4) to obtain the needed volatility $\widehat{\sigma}$. We get

$$\widehat{\sigma} = 0.04205.$$

Basing on the estimated volatility and the yield curve, Figure 2 shows the simulated process of the rate of return for a 13-week T-bill over the entire period from the day of emission to maturity (dotted line). The solid lines (quantile lines) determine the range which the rate of return is expected to be in with probability 0.9 during the next 13 weeks.

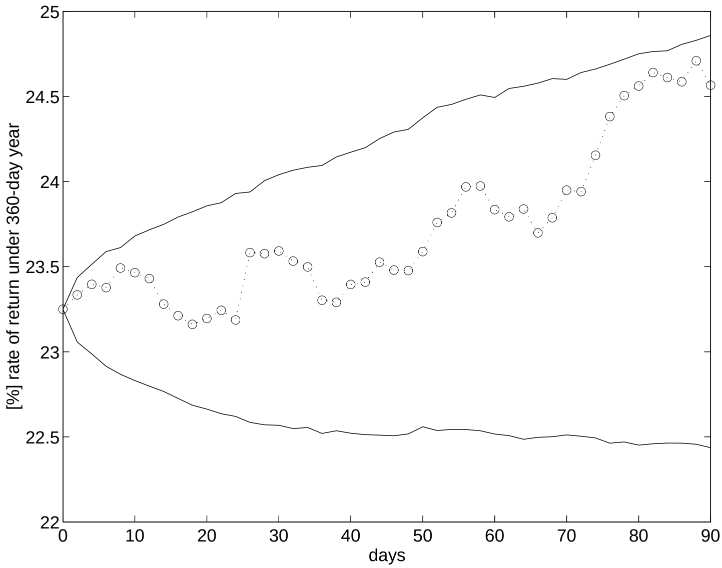

Fig. 2. The simulated process of the rate of return (dotted line) for a 13-week T-bill with maturity in 91 days and two quantile lines (solid lines)

Now we will show how to apply the presented results to hedging. Recall that in the HJM model, the price $\mathbf{C}_t$ at time t of a European call option on a zero-coupon bond struck at K with maturity u (see [3]) is given by

$$\mathbf{C}_t = P(t,T)\Phi(h) - KP(t,u)\Phi(h - \sigma(T-u)\sqrt{u-t}), \tag{7}$$

where

$$h = \frac{\log(P(t,T)/KP(t,u)) + \frac{1}{2}\sigma^2(T-u)^2(u-t)}{\sigma(T-u)\sqrt{u-t}}.$$

Assume that an investor intends to buy a 13-week T-bill on the next Monday with a fixed striking price K. He wants to protect his position from a possible fall in rates of return. Denote by u the date of the future auction of 13-week T-bills. Consider another investor with an investment horizon T, who possesses T-bills with maturity T in his portfolio, but needs to have cash at time s for a given period after which he again wants to invest in

T-bills (at time $u > s$). Therefore at time s he has to close his position, i.e. sell the T-bills, and at a later time u he has to buy them back. He wants to hedge his position against a possible fall of the rate of return over the period (s, u). He decides to protect himself from a more than $100 \cdot I\%$ increase of the T-bill price.

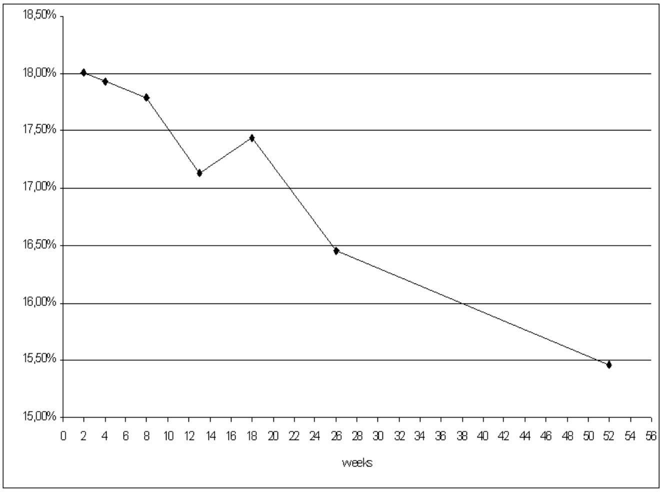

Fig. 3. Linearly interpolated yield curve constructed for the rates of return of Polish T-bills

To be more precise: let $t = 0$ correspond to the 24th August 1998. The current yield curve is given in Fig. 3. Quoted rates are linearly interpolated. Our goal is to price both options for any Monday auction. Such a date is denoted in the model by u. Then for 13-week T-bills we have $T = u+91/365$.

We choose $s = u - 91/365$ in order to match the price of 13-week T-bills with the price of 26-week bills in the forward-start option. The given $I\%$ is related to the rate R appearing in (3) through the formula $R = \log(1 + I)$. Now, having estimated the volatility σ from the data, we can evaluate the value of a European call option and a forward-start option directly from (3) and (7), respectively.

Figure 4 shows the prices for both derivatives with various expiry dates corresponding to expected Monday auctions (for different times u). In the case of a European call option, the longer the maturity, the higher the price. It should be pointed out that the price of a forward-start option does not behave in the same manner. The reason is that the entire uncertainty of evolution of the price of this derivative is "cut off" by the striking price $P(s, T)$. For this reason a buyer of a forward-start option pays only for the risk from the interval (s, u), in contrast to the case of a European call option. The price of a forward-start option considered as a function of the

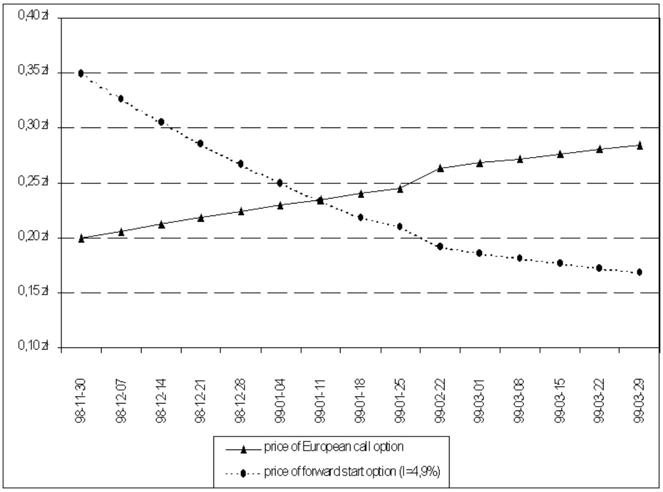

Fig. 4. European call options and forward-start options priced for 28.04.97

time to maturity may not only be increasing in time but decreasing as well. It depends on the shape of the initial yield curve. Finally, the pay-off function of forward-start options which have been priced above is given in Fig. 5.

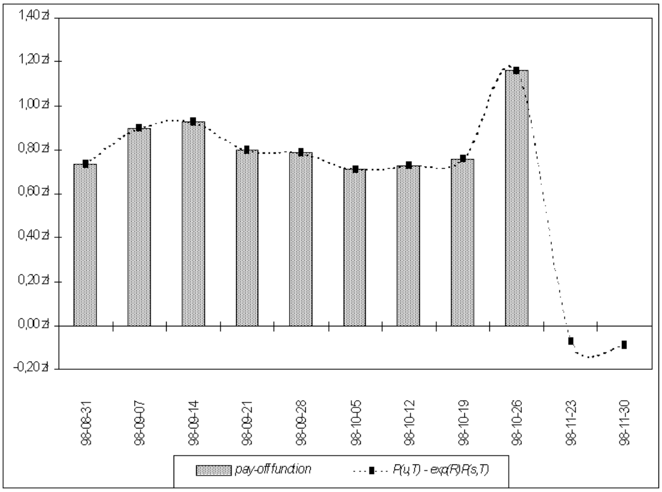

Fig. 5. Pay-off function of the forward start options

Acknowledgments. The authors would like to thank Marek Musiela for discussions on a preliminary version of this paper, and Marek Rutkowski for his comments improving the original proof of Theorem 2.1.

References

[1] A. Brace, D. Gątarek and M. Musiela, *The market model of interest rate dynamics*, Math. Finance 7 (1997), 127-154.
[2] A. Brace and M. Musiela, *A multifactor Gauss Markov implementation of Heath, Jarrow, and Morton*, ibid. 4 (1994), 259–283.
[3] D. Heath, R. Jarrow and A. Morton, *Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation*, Econometrica 60 (1992), 77–105.
[4] F. Jamshidian, *LIBOR and swap market models and measures*, Finance Stochast. 1 (1997), 293–330.
[5] M. Musiela and M. Rutkowski, *Continuous-time term structure models*: *Forward measure approach*, ibid., 261–291.
[6] —, —, *Martingale Methods in Financial Modeling,* Springer, Berlin, 1997.
[7] P. Sztuba and A. Weron, *Numerical methods in financial modelling. HJM model calibration*, Rynek Terminowy 5 (1999), 124–126 (in Polish).
[8] —, —, *Financial Engineering*; *Derivatives Pricing*, *Computer Simulations*, *Market Statistics*, WNT, Warszawa, 1998 (in Polish).

Hugo Steinhaus Center for Stochastic Methods
and
Institute of Mathematics
Wrocław University of Technology
50-370 Wrocław, Poland
E-mail: sztuba@im.pwr.wroc.pl
weron@im.pwr.wroc.pl

Received on 25.9.2000;
revised version on 30.11.2000 (1550)