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A SIMPLE DERIVATION OF THE EIGENVALUES OF A TRIDIAGONAL MATRIX ARISING IN BIOGEOGRAPHY

Abstract. In investigating a certain optimization problem in biogeography, Simon [IEEE Trans. Evolutionary Comput. 12 (2008), 702–713] encountered a certain specially structured tridiagonal matrix and made a conjecture regarding its eigenvalues. A few years later, the validity of the conjecture was established by Igelnik and Simon [Appl. Math. Comput. 218 (2011), 195–201]. In this paper, we give another proof of this conjecture that is much shorter, almost computation-free, and does not resort to the eigenvectors of the matrix.

1. Introduction. In [2], D. Simon conjectured that the eigenvalues of the $(n+1)\times(n+1)$ tridiagonal matrix

$$A = \begin{bmatrix} -1 & \frac{1}{n} & 0 & \cdots & 0 \\ \frac{n}{n} & -1 & \frac{2}{n} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 & \frac{n}{n} \\ 0 & \cdots & 0 & \frac{1}{n} & -1 \end{bmatrix} \tag{1}$$

are given by

$$\lambda = -\frac{2k}{n}, \qquad k = 0, 1, \ldots, n.$$

In [1, Theorem 1], B. Igelnik and D. Simon proved the validity of the conjecture. The first part of their proof consists in finding a general form of the eigenvectors of A for nonzero eigenvalues and for the zero eigenvalue.

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This is done by transforming the system $AV(x) = xV(x)$ into an equivalent system. This new system is obtained by adding the first equation in the original system to the second equation, giving a new second equation, then adding this second equation to the third equation in the original system, thus obtaining a new third equation, and so on. The last equation in the new system has the form $x\sum_{i=0}^{n} v_i(x) = 0$. This implies that either $x = 0$ or $\sum_{i=0}^{n} v_i(x) = 0$. In both cases, a general form of $V(x)$ is derived.

The second part of the proof of [1, Theorem 1] uses induction to show that the eigenvectors obtained in the first part must be the eigenvectors of the conjectured eigenvalues. The basic induction step starts with $n = 6$, although the conjecture is set for $n \geq 4$. It is mentioned that when $n < 6$, not enough structure is provided to complete the inductive proof.

In contrast, our proof, which we give in Theorem 2 of Section 2, is quite short and does not involve any heavy computations. It is also direct, in the sense that it does not refer in any way to the eigenvectors.

2. The main result. In this section, we give in Theorem 2 a proof of the conjecture mentioned in the introduction.

We start by defining a matrix $T(c, n+1)$, where n is a nonnegative integer, and where c can stand for any polynomial, and in particular any number, or indeed any element in any commutative ring with 1. The matrix $T(c, n+1)$ is defined to be the $(n+1) \times (n+1)$ tridiagonal matrix whose main diagonal is the $(n+1)$ vector $[c, \ldots, c]$, whose lower diagonal is the n-vector $[1, \ldots, n]$, and whose upper diagonal is the n-vector $[n, n-1, \ldots, 1]$. Formally, the entries $t_{ij}, 1 \leq i, j \leq n+1$, of $T(c, n+1)$ are defined by

$$t_{ij} = \begin{cases} c & \text{if } i = j, \\ n+1-i & \text{if } j = i+1, \\ i-1 & \text{if } j = i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

For example, if λ is an indeterminate, then

$$T(2-\lambda, 5) = \begin{bmatrix} 2-\lambda & 4 & 0 & 0 & 0 \\ 1 & 2-\lambda & 3 & 0 & 0 \\ 0 & 2 & 2-\lambda & 2 & 0 \\ 0 & 0 & 3 & 2-\lambda & 1 \\ 0 & 0 & 0 & 4 & 2-\lambda \end{bmatrix}.$$

The following lemma will be used in the proof of Theorem 1, which in turn will be used in proving the main theorem, i.e., Theorem 2.

Lemma 1. *For the matrix $T(c,n+1)$ defined above, and for all positive integers n, we have*

$$\det[T(c,n+1)] = (c+n)\det[T(c-1,n)].$$

Proof. Apply the sequence $\rho_1,\dots,\rho_{n+1}$ of column operations, defined by

$$\rho_i : C_i \mapsto C_i + C_{i+1} + \cdots + C_{n+1},$$

and follow this by the sequence $\tau_{n+1},\tau_n,\dots,\tau_2$ of row operations defined by

$$\tau_i : R_i \mapsto R_i - R_{i-1}.$$

Thus the operation ρ_i consists in adding to the i-th column all the columns that follow it, and the operation τ_i consists in subtracting from the i-th row the $(i-1)$-th row. Let the matrix resulting from applying these operations to $T(c,n+1)$ be denoted by B, and let the entries of B be denoted by $b_{i,j}$, $1 \le i,j \le n+1$. Then it is easy to see that

$$b_{i,j} = (t_{i,j} + t_{i,j+1} + \cdots + t_{i,n+1}) - (t_{i-1,j} + t_{i-1,j+1} + \cdots + t_{i-1,n+1})$$

for $1 < i,j \le n+1$. Using the definition of $t_{i,j}$ given in (2) above, we obtain

$$b_{i,j} = \begin{cases} c+n & \text{if } i=j=1,\\ c-1 & \text{if } i=j\ge 2,\\ n+1-i & \text{if } j=i+1 \text{ and } i\ge 1,\\ i-2 & \text{if } j=i-1 \text{ and } i>1,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, the matrix B has the form

$$B = \begin{bmatrix} c+n & n-1 & 0 & 0 & \cdots & 0\\ 0 & c-1 & n-2 & 0 & \ddots & 0\\ 0 & 1 & c-1 & n-3 & 0 & 0\\ 0 & 0 & 2 & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & c-1 & 1\\ 0 & \cdots & \cdots & 0 & n-1 & c-1 \end{bmatrix}.$$

Thus, $\det(B) = (c+n)\det(B_1)$, where B_1 is the matrix obtained from B by deleting the first row and the first column. However, B_1 is nothing but the matrix $T(c-1,n)$. Thus $\det(B) = (c+n)\det[T(c-1,n)]$, as claimed. ■

Theorem 1. *For the matrix $T(c,n+1)$ defined above, and for all non-negative integers n, we have*

$$\det[T(c,n+1)] = \prod_{k=0}^{n}[c-(-n+2k)].$$

Proof. The proof is by induction on n. Obviously the conclusion holds true for $n = 0$ and $n = 1$. Suppose that it is true for $n = r$ for some $r > 1$. Then

$$\begin{aligned}\det[T(c, r+2)] &= (c+r+1)\ \det[T(c-1, r+1)] \text{ by Lemma 1}\\ &= (c+r+1)\ \prod_{k=0}^{r}[(c-1)-(-r+2k)]\\ &\qquad\qquad \text{by the induction hypothesis}\\ &= (c+r+1)\ \prod_{k=0}^{r}[c-(-r+1+2k)]\\ &= \prod_{k=-1}^{r}\ [c-(-r+1+2k)]\\ &= \prod_{k=0}^{r+1}[c-(-r-1+2k)].\end{aligned}$$

Thus the conclusion holds for $n = r+1$, and the proof is complete. ∎

THEOREM 2. *Let A be the $(n+1)\times(n+1)$ matrix given in* (1)*. Then the eigenvalues of A are given by*

$$-\frac{2k}{n}, \qquad k = 0, 1, \dots, n. \tag{3}$$

Proof. The eigenvalues of A are the zeros of $\det(A-\lambda I)$, or equivalently, the zeros of $\det[n(A-\lambda I)]$. But $n(A-\lambda I)$ is nothing but the transpose of $T(n(-1-\lambda), n+1)$. Therefore

$$\begin{aligned}\det[n(A-\lambda I)] &= \det[T(n(-1-\lambda), n+1)]\\ &= \prod_{k=0}^{n}[n(-1-\lambda)-(-n+2k)] \quad \text{by Theorem 2}\\ &= \prod_{k=0}^{n}[-n\lambda-2k].\end{aligned}$$

Hence the eigenvalues are the solutions of the equations

$$-n\lambda - 2k = 0, \quad 0 \le k \le n,$$

which are nothing but the numbers given in (3). ∎

References

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[2] D. Simon, *Biogeography-based optimization*, IEEE Trans. Evolutionary Comput. 12 (2008), 702–713.

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