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A UNIFYING CONVERGENCE ANALYSIS OF NEWTON'S METHOD FOR TWICE FRÉCHET-DIFFERENTIABLE OPERATORS

Abstract. We provide a local as well as a semilocal convergence analysis for Newton's method using unifying hypotheses on twice Fréchet-differentiable operators in a Banach space setting. Our approach extends the applicability of Newton's method. Numerical examples are also provided.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet-differentiable operator defined on a non-empty convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ with values in a Banach space $\mathcal{Y}$.

Many problems from computational sciences, physics and other disciplines can be brought into a form similar to equation (1.1) using mathematical modeling [9, 11, 13, 36–38, 47, 48, 50]. Solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The study of convergence of iterative procedures is usually of two types: semilocal and local convergence analysis. The semilocal convergence analysis is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local convergence analysis is, based on the information around a solution, to find estimates of the radii of convergence balls. Note that in computational science, the practice of numerical analysis for finding such solutions is essentially connected to Newton's method. The basic idea of Newton's method is linearization. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a differentiable

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function, and we would like to solve equation (1.1). Starting from an initial guess, we can have the linear approximation of $F(x)$ in the neighborhood of x_0: $F(x_0+q) \approx F(x_0) + F'(x_0)q$, and solve the resulting linear equation $F(x_0) + F'(x_0)q = 0$, leading to the recurrent Newton method (NM)

$$(1.2) \qquad x_{n+1} = x_n - [F'(x_{n-1})]^{-1}F(x_n) \quad \text{for } n = 0, 1, \ldots,\ x_0 \in \mathcal{D}.$$

Method (1.2) is undoubtedly the most popular one for generating a sequence $\{x_n\}$ quadratically (under certain hypotheses [11, 36, 37, 48]) converging to $x^\star$. Here, $[F'(x)]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, the space of bounded linear operators from $\mathcal{Y}$ into $\mathcal{X}$. There is an extensive literature on local as well as semilocal convergence for (NM) under Lipschitz-type conditions. A survey of such results can be found in [11] (see also [2, 3, 5–13, 17, 19, 20, 23, 25, 27, 29, 30, 33, 34, 36–39, 41–48, 50–52] and the references therein).

In this study we provide sufficient convergence conditions under more general conditions than before. This way we expand the applicability of (NM) and also provide a tighter error analysis.

Semilocal case. We assume that there exist a non-decreasing continuous functions $\omega_0, \omega : [0,\infty) \to [0,\infty)$ (or $\omega_1 : [0,\infty) \to [0,\infty)$) with $\omega_1(0) = 0$ and $x_0 \in \mathcal{D}$ such that

$$(1.3) \qquad \|[F'(x_0)]^{-1}F''(x_0 + \theta(x - x_0))\| \le \omega_0(\theta\|x - x_0\|)$$

and either

$$(1.4) \qquad \|[F'(x_0)]^{-1}F''(x + \theta(y - x))\| \le \omega(\|x - x_0\| + \theta\|y - x_0\|)$$

or

$$(1.5) \qquad \|[F'(x_0)]^{-1}(F'(x) - F'(x_0)\| \le \omega_1(\theta\|x - x_0\|)$$

for all $x, y \in \mathcal{D}$, $\theta \in [0,1]$. Our results are obtained using (1.3) and (1.4) (see Theorem 3.1) or (1.4) and (1.5) (see Theorem 3.3).

Note that condition (1.4) always implies (1.3). That is, (1.3) is not an additional (to (1.4)) hypothesis. Moreover, in general

$$(1.6) \qquad \omega_0(t) \le \omega(t)$$

and ω/ω_0 can be arbitrarily large [5–9, 11–13]. Similarly, we may have

$$(1.7) \qquad \omega_1(t) \le t\int_0^1 \omega(\theta t)\, dt,$$

or even

$$(1.8) \qquad \omega_1(t) \le t\int_0^1 \omega_0(\theta t)\, dt.$$

Estimate (1.4) is used to obtain upper bounds on $\|[F'(x_n)]^{-1}F(x_0)\|$ [3, 17, 19, 20, 23, 25, 27, 29, 30, 33, 34, 36–39, 41–48, 50–52]. However, conditions (1.3) and (1.5) are used, since they are more precise or cheaper, respectively, than condition (1.4). This modification generates tighter majorizing sequences, which in turn lead to weaker sufficient more precise (1.3) or the cheaper (1.5) are really needed. This modification generates tighter majorizing sequences which in turn lead to weaker sufficient convergence conditions under less computational cost since the computation of ω requires that of ω_0 (or since (1.5) is cheaper than (1.4)).

Note that L. V. Kantorovich first provided sufficient convergence conditions in a Banach space setting using only (1.4) in the special case when ω is a constant [36], that is, when

$$\|[F'(x_0)]^{-1}F''(x)\| \le L, \quad L > 0. \tag{1.9}$$

However, the number of equations that can be solved using (1.9) and (NM) is limited, since it is not easy to see that F'' is bounded on $\mathcal{D}$. Moreover, it is not easy either to find a domain containing $x^\star$ where F'' is bounded.

Our approach provides more information about the operator F, not just that F'' is bounded.

Local case. We assume there exist non-decreasing continuous functions $\nu_0, \nu : [0,\infty) \to [0,\infty)$ (or $\nu_1 : [0,\infty) \to [0,\infty)$) with $\nu_1(0) = 0$, and $x^\star \in \mathcal{D}$ such that $[F'(x^\star)]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$, $F(x^\star) = 0$,

$$\|[F'(x^\star)]^{-1}F''(x^\star + \theta(x - x^\star))\| \le \nu_0(\theta\|x - x^\star\|), \tag{1.10}$$

and either

$$\|[F'(x^\star)]^{-1}F''(x + \theta(x^\star - x))\| \le \nu((1+\theta)\|x^\star - x\|) \tag{1.11}$$

or

$$\|[F'(x^\star)]^{-1}(F'(x) - F'(x^\star))\| \le \nu_1(\|x - x^\star\|) \tag{1.12}$$

for all $x \in \mathcal{D}$, $\theta \in [0,1]$ with benefits similar to the semilocal case (see Section 4). The results immediately extend to the case when the continuity of ω, ν is dropped and is replaced by $\lim_{t\to\infty} \omega(t) = 0$ [3, 7, 10, 11, 48].

The paper is organized as follows. Majorizing sequences for (NM) are studied in Section 2. The semilocal convergence of (NM) is given in Section 3. The local convergence of (NM) is provided in Section 4, whereas the applications can be found in the concluding Section 5.

2. Majorizing sequences for (NM). We need a result on majorizing sequences for (NM) using the functions ω and ω_0.

Lemma 2.1. *Let $\eta \geq 0$ and let non-decreasing functions $\omega_0, \omega : [0,\infty) \to [0,\infty)$ be given. Define a scalar sequence $\{t_n\}$ by*

$$\text{(2.1)} \quad \begin{cases} t_0 = 0, \quad t_1 = \eta, \\ t_{n+2} = t_{n+1} + \dfrac{\int_0^1 \omega(t_n + \theta(t_{n+1} - t_n))(1-\theta)(t_{n+1}-t_n)^2 \, d\theta}{1 - \int_0^1 \omega_0(\theta t_{n+1}) t_{n+1} \, d\theta}, \end{cases}$$

define sequences of functions $\{f_n\}$, $\{g_n\}$ on $(0,1)$ by

$$\text{(2.2)} \quad f_n(t) = t^n \eta \int_0^1 \omega\big((1 + t + \cdots + t^{n-1})\eta + \theta t^n \eta\big)(1-\theta) \, d\theta + t(1 + t + \cdots + t^n)\eta \int_0^1 \omega_0(\theta(1 + t + \cdots + t^n)\eta) \, d\theta - t,$$

$$\begin{aligned} \text{(2.3)} \quad g_n(t) &= f_{n+1}(t) - f_n(t) \\ &= t^{n+1}\eta \int_0^1 \omega\big((1 + t + \cdots + t^n)\eta + \theta t^{n+1}\eta\big)(1-\theta) \, d\theta \\ &\quad - t^n \eta \int_0^1 \omega\big((1 + t + \cdots + t^{n-1})\eta + \theta t^n \eta\big)(1-\theta) \, d\theta \\ &\quad + t(1 + t + \cdots + t^{n+1})\eta \int_0^1 \omega_0(\theta(1 + t + \cdots + t^{n+1})\eta) \, d\theta \\ &\quad - t(1 + t + \cdots + t^n)\eta \int_0^1 \omega_0(\theta(1 + t + \cdots + t^n)\eta) \, d\theta \end{aligned}$$

and define a function f_∞ on $(0,1)$ by

$$\text{(2.4)} \quad f_\infty(t) = t\left[\frac{\eta}{1-t} \int_0^1 \omega_0\left(\frac{\theta\eta}{1-t}\right) d\theta - 1\right].$$

Assume that either (I) *or* (II) *below holds:*

(I) *there exists $\alpha \in (0,1)$ such that*

$$\text{(2.5)} \quad 0 \leq \frac{\int_0^1 \omega(\theta\eta)(1-\theta)\eta \, d\theta}{1 - \int_0^1 \omega_0(\theta\eta)\eta \, d\theta} \leq \alpha,$$

$$\text{(2.6)} \quad \frac{\eta}{1-\alpha} \int_0^1 \omega_0\left(\frac{\theta\eta}{1-\alpha}\right) d\theta \leq 1,$$

$$\text{(2.7)} \quad g_n(\alpha) \geq 0 \quad \textit{for all } n,$$

(II) *there exists* $\alpha \in (0,1)$ *such that*

$$f_1(\alpha) \le 0, \tag{2.8}$$

$$0 \le \frac{\int_0^1 \omega(\theta\eta)(1-\theta)\eta\, d\theta}{1-\int_0^1 \omega_0(\theta\eta)\eta\, d\theta} \le \alpha,$$

$$g_n(\alpha) \le 0 \quad \textit{for all } n. \tag{2.9}$$

Then the sequence $\{t_n\}$ *is well defined, non-decreasing, bounded from above by*

$$t^{\star\star} = \frac{\eta}{1-\alpha} \tag{2.10}$$

and converges to its unique least upper bound $t^\star$ *satisfying*

$$\eta \le t^\star \le t^{\star\star}. \tag{2.11}$$

Moreover,

$$0 \le t_{n+1} - t_n \le \alpha^n \eta, \tag{2.12}$$

$$0 \le t^\star - t_n \le \frac{\alpha^n \eta}{1-\alpha}. \tag{2.13}$$

Proof. (I) We shall show by induction that

$$0 \le \frac{\int_0^1 \omega(t_n + \theta(t_{n+1}-t_n))(1-\theta)(t_{n+1}-t_n)\, d\theta}{1-\int_0^1 \omega_0(\theta t_{n+1})t_{n+1}\, d\theta} \le \alpha. \tag{2.14}$$

Estimate (2.14) holds for $n=0$ by the initial conditions and (2.5). It then follows from (2.1) that

$$0 \le t_2 - t_1 \le \alpha(t_1 - t_0) = \alpha\eta.$$

Assume that (2.14) holds for all $n \le k$. Then by the induction hypothesis,

$$0 \le t_{k+1} - t_k \le \alpha^k \eta \quad \text{and} \quad t_{k+1} \le \frac{1-\alpha^{k+1}}{1-\alpha}\eta \le t^{\star\star}.$$

Estimate (2.14) can be rewritten as

$$\begin{aligned} \int_0^1 \omega\Big(\frac{1-\alpha^k}{1-\alpha}\eta + \alpha^k\theta\eta\Big)(1-\theta)\alpha^k\eta\, d\theta \\ + \alpha\int_0^1 \omega_0\Big(\theta\frac{1-\alpha^{k+1}}{1-\alpha}\eta\Big)\frac{1-\alpha^{k+1}}{1-\alpha}\eta\, d\theta - \alpha \le 0. \end{aligned} \tag{2.15}$$

Estimate (2.15) motivates defining recurrent functions f_k given by (2.2) and showing instead of (2.15) that

$$f_k(\alpha) \le 0. \tag{2.16}$$

We need a relationship between two consecutive functions f_k. By (2.2) and (2.3) we have

$$f_{k+1}(\alpha) = f_k(\alpha) + g_k(\alpha). \tag{2.17}$$

Moreover, by hypothesis (2.7),

$$f_k(\alpha) \le f_{k+1}(\alpha). \tag{2.18}$$

Furthermore, define a function f_∞ on $[0, 1)$ by

$$f_\infty(\alpha) = \lim_{k\to\infty} f_k(\alpha). \tag{2.19}$$

Then by (2.3) we obtain

$$f_\infty(\alpha) = \alpha\left[\frac{\eta}{1-\alpha}\int_0^1 \omega_0\left(\frac{\theta\eta}{1-\alpha}\right) d\theta - 1\right]. \tag{2.20}$$

In view of (2.16)–(2.20) instead of (2.16) we can show that

$$f_\infty(\alpha) \le 0, \tag{2.21}$$

which is true by (2.6).

The induction for (2.14) is thus completed. It follows that the sequence $\{t_n\}$ is non-decreasing and bounded above by $t^{\star\star}$, and as such it converges to $t^\star$. Estimate (2.13) follows from (2.12) (which is implied by (2.14) and (2.1)). That completes the proof of part (I).

(II) In this case, by (2.9), (2.16) and (2.17) we can show instead of (2.16) that $f_1(\alpha) \le 0$, which is true by (2.8). The rest of the proof follows as in part (I). ∎

Remark 2.2. Define a scalar sequence $\{\bar{t}_n\}$ by

$$\begin{cases} \bar{t}_0 = 0, \quad \bar{t}_1 = \eta, \\ \bar{t}_2 = \bar{t}_1 + \dfrac{\int_0^1 \omega_0(\bar{t}_0 + \theta(\bar{t}_1 - \bar{t}_0))(1-\theta)(\bar{t}_1 - \bar{t}_0)^2\, d\theta}{1 - \int_0^1 \omega_0(\theta\bar{t}_1)\bar{t}_1\, d\theta}, \\ \bar{t}_{n+2} = \bar{t}_{n+1} + \dfrac{\int_0^1 \omega(\bar{t}_n + \theta(\bar{t}_{n+1} - \bar{t}_n))(1-\theta)(\bar{t}_{n+1} - \bar{t}_n)^2\, d\theta}{1 - \int_0^1 \omega_0(\theta\bar{t}_{n+1})\bar{t}_{n+1}\, d\theta}. \end{cases} \tag{2.22}$$

The sequence $\{\bar{t}_n\}$ is finer than $\{t_n\}$ and clearly converges under the hypotheses of Lemma 2.1. Moreover, a simple induction argument shows that

$$\bar{t}_n \le t_n, \quad n \ge 2, \tag{2.23}$$

$$\bar{t}_{n+1} - \bar{t}_n \le t_{n+1} - t_n, \quad n \ge 2, \tag{2.24}$$

$$\bar{t}^\star \le t^\star, \tag{2.25}$$

$$\bar{t}^\star = \lim_{n\to\infty} \bar{t}_n. \tag{2.26}$$

Furthermore, if $\omega_0 < \omega$ then strict inequality holds in (2.23) and (2.24).

Later we shall show that $\{t_n\}$, $\{\bar{t}_n\}$ are majorizing sequences for $\{x_n\}$. However, before doing that let us show that these sequences are finer than other majorizing sequences already in the literature.

Under (1.4), the majorizing iteration $\{u_n\}$ given by

$$\tag{2.27} \begin{cases} u_0 = 0, \quad u_1 = \eta, \\ u_{n+2} = u_{n+1} + \dfrac{\int_0^1 \omega(u_n + \theta(u_{n+1} - u_n))(1-\theta)(u_{n+1} - u_n)^2 \, dt}{1 - \int_0^1 \omega(\theta u_{n+1}) u_{n+1} \, d\theta} \\ \qquad = u_{n+1} - f(u_n)/f'(u_n) \end{cases}$$

was essentially used in [23] (see also [11]) where

$$\tag{2.28} f(t) = \int_0^t \int_0^\theta \omega(\xi) \, d\xi \, dt - t + \eta.$$

If there exists $\beta > 0$ such that

$$\tag{2.29} f(\beta) \leq 0$$

then $\{u_n\}$ is non-decreasing and converges to some

$$\tag{2.30} u^\star \leq \beta.$$

Note that hypothesis (2.29) is different from the corresponding ones of Lemma 2.1. However, in view of (1.6) we have, for $n \geq 2$,

$$\tag{2.31} t_n \leq u_n,$$

$$\tag{2.32} t_{n+1} - t_n \leq u_{n+1} - u_n,$$

$$\tag{2.33} t^\star \leq u^\star.$$

Moreover, in case strict inequality holds in (1.6), then (2.31) and (2.32) also hold as strict inequalities.

Let us now provide another majorizing sequence for (NM) using the functions ω and ω_1. The proof is omitted since it can be obtained from Lemma 2.1 by exchanging the roles of ω_0 and ω_1.

LEMMA 2.3. *Let $\eta \geq 0$ and let $\omega_1, \omega : [0, \infty) \to [0, \infty)$ be non-decreasing functions with $\omega_1(0) = 0$. Define a scalar sequence $\{r_n\}$ by*

$$\tag{2.34} \begin{cases} r_0 = 0, \quad r_1 = \eta, \\ r_{n+2} = r_{n+1} + \dfrac{\int_0^1 \omega(r_n + \theta(r_{n+1} - r_n))(1-\theta)(r_{n+1} - r_n)^2 \, dt}{1 - \omega_1(r_{n+1})}, \end{cases}$$

define sequences of functions $\{f_n^1\}$, $\{g_n^1\}$ *on* $(0,1)$ *by*

$$f_n^1(t) = t^n\eta\int_0^1 \omega\big((1+t+\cdots+t^{n-1})\eta + \theta t^n\eta\big)(1-\theta)\,d\theta + t\omega_1((1+t+\cdots+t^n)\eta) - t, \tag{2.35}$$

$$\begin{aligned} g_n^1(t) &= f_{n+1}^1(t) - f_n^1(t) \\ &= t^{n+1}\eta\int_0^1 \omega\big((1+t+\cdots+t^n)\eta + \theta t^{n+1}\eta\big)(1-\theta)\,d\theta \\ &\quad - t^n\theta\int_0^1 \omega\big((1+t+\cdots+t^{n-1})\eta + \theta t^n\eta\big)(1-\theta)\,d\theta \\ &\quad + t\big(\omega_1((1+t+\cdots+t^{n+1})\eta) - \omega_1((1+t+\cdots+t^n)\eta)\big), \end{aligned} \tag{2.36}$$

and define a function f_∞^1 *on* $(0,1)$ *by*

$$f_\infty^1(t) = t\left(\omega_1\left(\frac{\eta}{1-t}\right) - 1\right). \tag{2.37}$$

Assume that either (I) *or* (II) *below holds:*

(I) *there exists* $\gamma \in (0,1)$ *such that*

$$0 \le \frac{\int_0^1 \omega(\theta\eta)(1-\theta)\eta\,d\theta}{1-\omega_1(\eta)} \le \gamma, \tag{2.38}$$

$$\omega_1\left(\frac{\eta}{1-\gamma}\right) \le 1, \tag{2.39}$$

$$g_n^1(\gamma) \ge 0 \quad \textit{for all } n, \tag{2.40}$$

(II) *there exists* $\gamma \in (0,1)$ *such that*

$$f_1^1(\gamma) \le 0, \tag{2.41}$$

$$0 \le \frac{\int_0^1 \omega(\theta\eta)(1-\theta)\eta\,d\theta}{1-\omega_1(\eta)} \le \gamma,$$

$$g_n^1(\gamma) \le 0 \quad \textit{for all } n. \tag{2.42}$$

Then the sequence $\{r_n\}$ *is well defined, non-decreasing, bounded from above by*

$$r^{\star\star} = \frac{\eta}{1-\gamma} \tag{2.43}$$

and converges to its unique least upper bound $r^\star$ *satisfying*

$$\eta \le r^\star \le r^{\star\star}. \tag{2.44}$$

Moreover,

$$0 \le r_{n+1} - r_n \le \gamma^n \eta, \tag{2.45}$$

$$0 \le r^\star - r_n \le \frac{\gamma^n \eta}{1-\gamma}. \tag{2.46}$$

REMARK 2.4. Define a scalar sequence $\{\overline{r}_n\}$ by

$$\begin{cases} \overline{r}_0 = 0, \quad \overline{r}_1 = \eta, \\ \overline{r}_2 = \overline{r}_1 + \dfrac{\int_0^1 \omega_1(\overline{r}_1 + \theta(\overline{r}_1 - \overline{r}_0))(1-\theta)(\overline{r}_1 - \overline{r}_0)^2\, dt}{1 - \omega_1(\overline{r}_1)}, \\ \overline{r}_{n+2} = \overline{r}_{n+1} + \dfrac{\int_0^1 \omega(\overline{r}_n + \theta(\overline{r}_{n+1} - \overline{r}_n))(1-\theta)(\overline{r}_{n+1} - \overline{r}_n)^2\, dt}{1 - \omega_1(\overline{r}_{n+1})}. \end{cases} \tag{2.47}$$

It follows from (1.7), (2.1) and (2.47) that $\{\overline{r}_n\}$ is a finer sequence than $\{\overline{r}_n\}$ (see also Remark 2.2).

In the next section we provide sufficient convergence conditions for (NM).

3. Semilocal convergence for (NM). Next, the semilocal convergence of (NM) is shown using functions ω and ω_0.

THEOREM 3.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be twice Fréchet-differentiable. Assume there exist $x_0 \in \mathcal{D}$, $\eta \ge 0$, and non-decreasing continuous functions $\omega_0, \omega : [0,\infty) \to [0,\infty)$ such that*

$$[F'(x_0)]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \tag{3.1}$$

$$\begin{aligned} &\|[F'(x_0)]^{-1} F(x_0)\| \le \eta, \\ &\|[F'(x_0)]^{-1} F''(x_0 + \theta(x - x_0))\| \le \omega_0(\theta \|x - x_0\|), \\ &\|[F'(x_0)]^{-1} F''(x + \theta(y - x))\| \le \omega(\|x - x_0\| + \theta\|y - x\|) \end{aligned} \tag{3.2}$$

for all $x, y \in \mathcal{D}$, $\theta \in [0,1]$. Moreover, assume that the hypotheses of Lemma 2.1 *hold and*

$$\overline{U}(x_0, t^\star) = \{x \in \mathcal{X} : \|x - x_0\| \le t^\star\} \subseteq \mathcal{D}. \tag{3.3}$$

Then the sequence $\{x_n\}$ generated by (NM) *is well defined, remains in $\overline{U}(x_0, t^\star)$ for all $n \ge 0$ and converges to a solution $x^\star \in \overline{U}(x_0, t^\star)$ of the equation $F(x) = 0$.*

Moreover,

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \tag{3.4}$$

$$\|x_n - x^\star\| \le t^\star - t_n. \tag{3.5}$$

Furthermore, if there exists $R \geq t^\star$ *such that*

$$\overline{U}(x_0, R) \subseteq \mathcal{D}, \tag{3.6}$$

$$\int_0^1\int_0^1 \omega_0(\theta(\lambda R + (1-\lambda)t^\star))(\lambda R + (1-\lambda)t^\star)\, d\lambda\, d\theta < 1, \tag{3.7}$$

then $x^\star$ *is a unique solution of the equation* $F(x) = 0$ *in* $\overline{U}(x_0, R)$.

Proof. By induction we shall show that

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \tag{3.8}$$

$$\overline{U}(x_{k+1}, t^\star - t_{k+1}) \subseteq \overline{U}(x_k, t^\star - t_k). \tag{3.9}$$

For every $z \in \overline{U}(x_1, t^\star - t_1)$,

$$\|z - x_0\| \leq \|z - x_1\| + \|x_1 - x_0\| \leq t^\star - t_1 + t_1 - t_0 = t^\star - t_0$$

implies $z \in \overline{U}(x_0, t^\star - t_0)$.

By (2.1) and (3.2) we have

$$\|x_1 - x_0\| = \|[F'(x_0)]^{-1}F(x_0)\| \leq \eta = t_1 - t_0.$$

Hence, estimates (3.8) and (3.9) hold for $k = 0$. Assume they hold for $n \leq k$. Then

$$\|x_{k+1} - x_0\| \leq \sum_{i=1}^{k+1} \|x_i - x_{i-1}\| \leq \sum_{i=1}^{k+1} (t_i - t_{i-1}) = t_{k+1} - t_0 = t_{k+1} \leq t^{\star\star}$$

and

$$\|x_k + \theta(x_{k+1} - x_k) - x_0\| \leq t_k + \theta(t_{k+1} - t_k) \leq t^{\star\star} \quad \text{for all } \theta \in [0, 1].$$

Using (3.3) and the induction hypotheses, we get

$$\begin{aligned}
&\|[F'(x_0)]^{-1}(F'(x_0) - F'(x_{k+1}))\| \\
&\quad = \Big\| \int_0^1 F''(x_0 + \theta(x_{k+1} - x_0))(x_{k+1} - x_0)\, d\theta \Big\| \\
&\quad \leq \int_0^1 \omega_0(\theta\|x_{k+1} - x_0\|)\, d\theta\, \|x_{k+1} - x_0\| \leq \int_0^1 \omega_0(\theta t_{k+1}) t_{k+1}\, d\theta < 1.
\end{aligned} \tag{3.10}$$

It follows from (3.10) and the Banach lemma on invertible operators [7, 36] that $[F'(x_{k+1})]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and

$$\|[F'(x_{k+1})]^{-1}F'(x_0)\| \leq \frac{1}{1 - \int_0^1 \omega_0(\theta t_{k+1}) t_{k+1}\, d\theta}. \tag{3.11}$$

Using (1.2) we obtain the approximation

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - F'(x_k)(x_{k+1} - x_k) \tag{3.12}$$
$$= \int_0^1 F''(x_k + \theta(x_{k+1} - x_k))(1-\theta)(x_{k+1} - x_k)^2\, d\theta.$$

By (3.3), (3.12), (2.1) and the induction hypotheses we in turn get

$$\begin{aligned} &\|[F'(x_0)]^{-1}F(x_1)\| \\ &= \Big\|[F'(x_0)]^{-1}\int_0^1 F''(x_0 + \theta(x_1 - x_0))(1-\theta)(x_1 - x_0)^2\, d\theta\Big\| \\ &\le \int_0^1 \omega_0(\theta\|x_1 - x_0\|)(1-\theta)\|x_1 - x_0\|^2\, d\theta \\ &\le \int_0^1 \omega_0(\theta t_1)(1-\theta)t_1^2\, d\theta \le \int_0^1 \omega(\theta t_1)(1-\theta)t_1^2\, d\theta, \end{aligned} \tag{3.13}$$

and for $k \ge 1$,

$$\begin{aligned} &\|[F'(x_0)]^{-1}F(x_{k+1})\| \\ &= \Big\|\int_0^1 [F'(x_0)]^{-1}F''(x_k + \theta(x_{k+1} - x_k))(1-\theta)(x_{k+1} - x_k)^2\, d\theta\Big\| \\ &\le \int_0^1 \omega\big(\|x_k - x_0\| + \theta\|x_{k+1} - x_k\|\big)(1-\theta)\|x_{k+1} - x_k\|^2\, d\theta \\ &\le \int_0^1 \omega(t_k + \theta(t_{k+1} - t_k))(1-\theta)(t_{k+1} - t_k)^2\, d\theta. \end{aligned} \tag{3.14}$$

Then, in view of (2.1), (3.11)–(3.14), we get

$$\begin{aligned} \|x_{k+2} - x_{k+1}\| &\le \|[F'(x_{k+1})]^{-1}F'(x_0)\|\, \|[F'(x_0)]^{-1}F(x_{k+1})\| \\ &\le \frac{\int_0^1 \omega(t_k + \theta(t_{k+1} - t_k))(1-\theta)(t_{k+1} - t_k)^2\, d\theta}{1 - \int_0^1 \omega_0(\theta t_{k+1})t_{k+1}\, d\theta} = t_{k+2} - t_{k+1}, \end{aligned} \tag{3.15}$$

with completes the induction for (3.8). Moreover, for every $z \in \overline{U}(x_{k+2}, t^\star - t_{k+2})$, we obtain

$$\|z - x_{k+1}\| \le \|z - x_{k+2}\| + \|x_{k+2} - x_{k+1}\| \le t^\star - t_{k+2} + t_{k+2} - t_{k+1},$$

so $z \in \overline{U}(x_{k+1}, t^\star - t_{k+1})$, which completes the induction for (3.9). Lemma 2.1 implies that $\{t_n\}$ is a Cauchy sequence. It follows from (3.8) and (3.9) that $\{x_n\}$ is a Cauchy sequence too in the Banach space $\mathcal{X}$, and as such it converges to some $x^\star \in \overline{U}(x_0, t^\star)$. By letting $k \to \infty$ in (3.14) we get

$F(x^\star) = 0$. Estimate (3.5) follows from (3.4) by using standard majorization techniques [7, 11, 36–38].

Finally, to show the uniqueness part, let $y^\star \in \overline{U}(x_0, R)$ be a solution of the equation $F(x) = 0$. Set $x_\lambda = x^\star + \lambda(y^\star - x^\star)$ for $\lambda \in [0,1]$. Define an operator M by

$$M = \int_0^1 F'(x_\lambda)\, dx_\lambda. \tag{3.16}$$

Then, using (3.3), (3.7) and (3.16) we get

$$\begin{aligned} &\|[F'(x_0)]^{-1}(F'(x_0) - M)\| \\ &\quad = \Big\| \int_0^1\!\!\int_0^1 [F'(x_0)]^{-1} F''(x_0 + \theta(x_\lambda - x_0))(x_\lambda - x_0)\, dx_\lambda\, d\theta \Big\| \\ &\quad \le \int_0^1\!\!\int_0^1 \omega_0(\theta\|x_\lambda - x_0\|)\|x_\lambda - x_0\|\, dx_\lambda\, d\theta \\ &\quad \le \int_0^1\!\!\int_0^1 \omega_0\big(\theta(\lambda R + (1-\lambda)t^\star)\big)(\lambda R + (1-\lambda)t^\star)\, d\lambda\, d\theta < 1. \end{aligned} \tag{3.17}$$

Then it follows from (3.17) and the Banach lemma on invertible operators that $M^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Moreover, using the identity

$$F(y^\star) - F(x^\star) = M(y^\star - x^\star), \tag{3.18}$$

we obtain $x^\star = y^\star$. ∎

Remark 3.2. (a) It follows from the proof of Theorem 3.1 (see (3.13)) that $\{\bar{t}_n\}$ given by (2.22) is also a majorizing sequence for $\{x_n\}$.

(b) The point $t^{\star\star}$ given in closed form by (2.10) can replace $t^\star$ in Theorem 3.1.

We shall show a semilocal convergence result for (NM) using the functions ω_1 and ω. The proof is obtained from Theorem 3.1 by exchanging the roles of ω_0 and ω_1.

Theorem 3.3. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be twice Fréchet-differentiable. Assume that there exist $x_0 \in \mathcal{D}$, $\eta \ge 0$, and non-decreasing functions $\omega_1, \omega : [0,\infty) \to [0,\infty)$ with $\omega_1(0) = 0$ such that*

$$\begin{aligned} &[F'(x_0)]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \\ &\|[F'(x_0)]^{-1} F(x_0)\| \le \eta, \\ &\|[F'(x_0)]^{-1}(F'(x) - F'(x_0))\| \le \omega_1(\|x - x_0\|), \\ &\|[F'(x_0)]^{-1} F''(x + \theta(y - x))\| \le \omega(\|x - x_0\| + \theta\|y - x\|) \end{aligned}$$

for all $x, y \in \mathcal{D}$ and $\theta \in [0,1]$. Moreover, assume that the hypotheses of Lemma 2.3 *hold and $\overline{U}(x_0, r^\star) \subseteq \mathcal{D}$. Then the sequence $\{x_n\}$ generated by* (NM) *is well defined, remains in $\overline{U}(x_0, r^\star)$ for all $n \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, r^\star)$ of the equation $F(x) = 0$.*

Moreover,

$$\|x_{n+1} - x_n\| \leq r_{n+1} - r_n, \qquad \|x_n - x^\star\| \leq r^\star - r_n.$$

Furthermore, if there exists $R_1 \geq r^\star$ such that

$$\int_0^1 \omega_1((1-\theta)r^\star + \theta R_1)\, d\theta \leq 1$$

then $x^\star$ is a unique solution of the equation $F(x) = 0$ in $U(x_0, R_1)$.

Proof. As noted above, the proof follows that of Theorem 3.1 apart from the uniqueness part.

To show the uniqueness, let $y^\star \in U(x_0, R_1)$ be such that $F(y^\star) = 0$. Define an operator M by

$$M = \int_0^1 F'(x^\star + \theta(y^\star - x^\star))\, d\theta.$$

Then using (1.5) we obtain

$$\begin{aligned}
\|[F'(x_0)]^{-1}(M - F'(x_0))\| &\leq \int_0^1 \omega_1\big(\|x^\star + \theta(y^\star - x^\star) - x_0\|\big)\, d\theta \\
&\leq \int_0^1 \omega_1\big((1-\theta)\|x^\star - x_0\| + \theta\|y^\star - x_0\|\big)\, d\theta \\
&< \int_0^1 \omega_1((1-\theta)r^\star + \theta R_1)\, d\theta \leq 1.
\end{aligned}$$

It follows that $M^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. Using the identity

$$F(y^\star) - F(x^\star) = M(y^\star - x^\star),$$

we deduce $x^\star = y^\star$. ■

Remark 3.4. (a) The sequence $\{\overline{r}_n\}$ can replace $\{r_n\}$ in Theorem 3.3 (see also Theorem 3.1 and (3.13)).

(b) The point $r^{\star\star}$ given in closed from by (2.43) can replace $r^\star$ in Theorem 3.3.

4. Local convergence of (NM). We provide a local convergence result for (NM) using the functions ν_0 and ν.

THEOREM 4.1. *Let $F : \mathcal{D} \subseteq \mathcal{X} \to \mathcal{Y}$ be twice Fréchet-differentiable. Assume there exist $x^\star \in \mathcal{D}$, $\delta > 0$ and non-decreasing continuous functions $\nu_0, \nu : [0,\infty) \to [0,\infty)$ such that*

$$\begin{aligned}
&[F'(x^\star)]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X}), \quad F(x^\star) = 0, \\
&\|[F'(x^\star)]^{-1} F''(x^\star + \theta(x - x^\star))\| \le \nu_0(\theta \|x - x^\star\|), \\
&\|[F'(x^\star)]^{-1} F''(x + \theta(x^\star - x))\| \le \nu(\|x - x^\star\| + \theta\|x^\star - x\|)
\end{aligned} \tag{4.1}$$

for all $x \in \mathcal{D}$ and $\theta \in [0,1]$. Moreover, assume that

$$\int_0^1 \big(\nu((1+\theta)\delta)(1-\theta) + \nu_0(\theta\delta)\big)\delta \, d\theta < 1, \tag{4.2}$$

$$\overline{U}(x^\star, \delta) \subseteq \mathcal{D}. \tag{4.3}$$

Then the sequence $\{x_n\}$ generated by (NM) *is well defined, remains in $\overline{U}(x^\star, \delta)$ for all $n \ge 0$ and converges to $x^\star$ provided that $x_0 \in \overline{U}(x^\star, \delta)$ so that*

$$\|x_1 - x^\star\| \le \frac{\int_0^1 \nu_0((1+\theta)\|x_0 - x^\star\|)(1-\theta)\|x_0 - x^\star\|^2 \, d\theta}{1 - \int_0^1 \nu_0(\theta\|x_0 - x^\star\|)\|x_0 - x^\star\| \, d\theta}, \tag{4.4}$$

and for $n \ge 1$,

$$\|x_{n+1} - x^\star\| \le \frac{\int_0^1 \nu((1+\theta)\|x_n - x^\star\|)(1-\theta)\|x_n - x^\star\|^2 \, d\theta}{1 - \int_0^1 \nu_0(\theta\|x_n - x^\star\|)\|x_n - x^\star\| \, d\theta}. \tag{4.5}$$

Proof. By hypothesis $x_0 \in \overline{U}(x^\star, \delta)$. Assume $x_k \in \overline{U}(x^\star, \delta)$. Then, using (1.10) and (1.11), respectively, we obtain the estimates

$$\begin{aligned}
&\|[F'(x^\star)]^{-1}(F'(x^\star) - F'(x_k))\| \\
&\qquad = \left\| \int_0^1 [F'(x^\star)]^{-1} F''(x^\star + \theta(x_k - x^\star))(x_k - x^\star) \, d\theta \right\| \\
&\qquad \le \int_0^1 \nu_0(\theta\|x_k - x^\star\|)\|x_k - x^\star\| \, d\theta \\
&\qquad \le \int_0^1 \nu_0(\theta\delta)\delta \, d\theta < 1 \quad \text{(by (4.2))}
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}(4.7)\qquad &\big\|[F'(x^\star)]^{-1}\big(F(x^\star)-F(x_k)-F'(x_k)(x^\star-x_k)\big)\big\|\\ &=\left\|F'(x^\star)\int_0^1 F''(x_k+\theta(x^\star-x_k))(1-\theta)(x^\star-x_k)^2\,d\theta\right\|\\ &\le\int_0^1\nu((1+\theta)\|x_k-x^\star\|)(1-\theta)\|x_k-x^\star\|^2\,d\theta.\end{aligned}$$

It follows from (4.6) that $[F'(x_k)]^{-1}\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ and

$$\|[F'(x_k)]^{-1}F'(x^\star)\|\le\frac{1}{1-\int_0^1\nu_0(\theta\|x_k-x^\star\|)\|x_k-x^\star\|\,d\theta}. \tag{4.8}$$

Using the approximation

$$\begin{aligned}(4.9)\qquad x^\star-x_{k+1}&=([F'(x_k)]^{-1}F'(x^\star))[F'(x^\star)]^{-1}\\ &\quad\times\int_0^1 F''\big(x_k+\theta(x^\star-x_k)\big)(1-\theta)(x^\star-x_k)^2\,d\theta\end{aligned}$$

together with (4.7) and (4.8) we arrive at (4.4) and (4.5).

It then follows from (4.4), (4.5) and (4.2) that

$$\|x_{k+1}-x^\star\|<\|x_k-x^\star\|, \tag{4.10}$$

which implies $\lim_{k\to\infty}x_k=x^\star$ and $x_{k+1}\in\overline{U}(x^\star,\delta)$. ■

Finally, we provide a local convergence analysis for (NM) using the functions ν_1 and ν. The proof can be obtained from Theorem 4.1 by exchanging the roles of ν_0 and ν_1.

THEOREM 4.2. *Let $F:\mathcal{D}\subseteq\mathcal{X}\to\mathcal{Y}$ be twice Fréchet-differentiable. Assume there exist $x^\star\in\mathcal{D}$, $\delta_1>0$ and non-decreasing continuous functions $\nu_1,\nu:[0,\infty)\to[0,\infty)$ with $v_1(0)=0$ such that*

$$[F'(x^\star)]^{-1}\in\mathcal{L}(\mathcal{Y},\mathcal{X}),\quad F(x^\star)=0,$$
$$\|[F'(x^\star)]^{-1}(F'(x)-F'(x^\star))\|\le\nu_1(\|x-x^\star\|),$$
$$\|[F'(x^\star)]^{-1}F''(x+\theta(x^\star-x))\|\le\nu(\|x-x^\star\|+\theta\|x^\star-x\|)$$

for all $x\in\mathcal{D}$ and $\theta\in[0,1]$. Assume also that

$$\int_0^1\nu((1+\theta)\delta_1)(1-\theta)\delta_1\,d\theta+\nu_1(\delta_1)<1, \tag{4.11}$$

$$\overline{U}(x^\star,\delta_1)\subseteq\mathcal{D}. \tag{4.12}$$

Then the sequence $\{x_n\}$ generated by (NM) *is well defined, remains in $\overline{U}(x^\star,\delta_1)$ for all $n\ge0$ and converges to $x^\star$ provided that $x_0\in\overline{U}(x^\star,\delta_1)$.*

Moreover,

$$\|x_1 - x^\star\| \le \frac{\int_0^1 \nu_1((1+\theta)\|x_0 - x^\star\|)(1-\theta)\|x_0 - x^\star\|^2\, d\theta}{1 - \nu_1(\|x_0 - x^\star\|)}, \tag{4.13}$$

and for $n \ge 1$,

$$\|x_{n+1} - x^\star\| \le \frac{\int_0^1 \nu((1+\theta)\|x_n - x^\star\|)(1-\theta)\|x_n - x^\star\|^2\, d\theta}{1 - \nu_1(\|x_n - x^\star\|)}. \tag{4.14}$$

5. Special cases and numerical examples

Semilocal case. Let $\omega(t) = L > 0$ and $\omega_1(t) = L_0 t$. Then the sequences $\{u_n\}$ (used in [36–38, 45, 48]), $\{r_n\}$, $\{\overline{r}_n\}$ reduce to

$$\begin{cases} u_0 = 0, \quad u_1 = \eta, \\ u_{n+1} = u_n + \dfrac{L(u_n - u_{n-1})^2}{2(1 - L u_n)} \quad \text{for } n = 1, 2, \ldots, \end{cases}$$

$$\begin{cases} r_0 = 0, \quad r_1 = \eta, \\ r_{n+1} = r_n + \dfrac{L(r_n - r_{n-1})^2}{2(1 - L_0 r_n)} \quad \text{for } n = 1, 2, \ldots \end{cases}$$

and

$$\begin{cases} \overline{r}_0 = 0, \quad \overline{r}_1 = \eta, \quad \overline{r}_2 = \overline{r}_1 + \dfrac{L_0(\overline{r}_1 - \overline{r}_0)^2}{2(1 - L_0 \overline{r}_1)}, \\ \overline{r}_{n+1} = \overline{r}_n + \dfrac{L(\overline{r}_n - \overline{r}_{n-1})^2}{2(1 - L_0 \overline{r}_n)} \quad \text{for } n = 2, 3, \ldots. \end{cases}$$

The sequence $\{u_n\}$ converges provided that the famous Newton–Kantorovich hypothesis

$$h_k = 2L\eta \le 1$$

is satisfied [36].

Let us now look at Lemma 2.3, case (I). It can easily be seen that

$$f_n^1(t) = \big(Lt^{n-1} + 2L_0(1 + t + \cdots + t^n)\big)\eta - 2,$$
$$g_n^1(t) = \tfrac{1}{2} g(t) t^{n-1} \eta, \quad g(t) = 2L_0 t^2 + Lt - L,$$
$$\omega_1(t) = \frac{L_0 \eta}{1 - \gamma}, \quad f_\infty^1(t) = t\left(\frac{L_0\eta}{1-t} - 1\right), \quad \gamma = \frac{2L}{L + \sqrt{L^2 + 8L_0 L}}.$$

Then hypotheses (2.38)–(2.40) are satisfied if

$$h_1 = L_1 \eta \le 1,$$

where

$$L_1 = \tfrac{1}{4}\big(L + 4L_0 + \sqrt{L^2 + 8L_0 L}\big).$$

For the sequence $\{\overline{r}_n\}$ we have (see also [11, 13])

$$\begin{aligned}
f_n^1(t) &= \frac{L}{2}(\overline{r}_2 - \overline{r}_1)t^n + tL_0(1 + t + \cdots + t^n)(\overline{r}_2 - \overline{r}_1) - (1 - L_0\overline{r}_1)t,\\
g_n^1(t) &= \tfrac{1}{2}g(t)t^n(\overline{r}_2 - \overline{r}_1),\\
f_\infty^1(t) &= \left(\frac{L_0(\overline{r}_2 - \overline{r}_1)}{1-\alpha} + L_0\overline{r}_1 - 1\right)t.
\end{aligned}$$

Then hypotheses (2.38)–(2.40) are satisfied if

$$h_2 = L_2\eta \le 1,$$

where

$$L_2 = \tfrac{1}{4}\big(4L_0 + \sqrt{L_0L + 8L_0^2} + \sqrt{L_0L}\big).$$

Note that

$$h_k \le 1 \;\Rightarrow\; h_1 \le 1 \;\Rightarrow\; h_2 \le 1$$

but not necessarily vice versa unless $L_0 = L$.

We also have

$$\frac{h_2}{h_k} \to \frac{1}{4}, \quad \frac{h_2}{h_k} \to 0, \quad \frac{h_2}{h_1} \to 0 \quad \text{as } \frac{L_0}{L} \to 0.$$

Hence, the applicability of (NM) is extended under the same computational cost as in the Kantorovich theorem, since in practice computing L requires the same computation power as L_0.

Local case. Let $\nu(t) = \ell$ and $\nu_1(t) = \ell_1$. Then the hypotheses of Theorem 4.2 are satisfied provided that

$$\delta_1 = \frac{2}{2\ell_1 + \ell}.$$

The hypotheses in the literature using only ν are satisfied provided that the radius δ_2 found by Traub [37, 50] is given by

$$\delta_2 = \frac{2}{3\ell}.$$

Note that if $\ell_1 \le \ell$, then

$$\delta_2 \le \delta_1,$$

and if $M_1 < M$, then

$$\delta_2 < \delta_1.$$

Moreover, the error bounds on the distances are also tighter in this case (see (4.13), (4.14) and the numerical examples).

5.1. Applications: semilocal case

EXAMPLE 5.1. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $x_0 = 1$ and $\mathcal{D} = U(1, 1-a)$ for $a \in (0, 1)$. Define a scalar function F on $\mathcal{D}$ by

$$F(x) = x^3 - a. \tag{5.1}$$

Then, using the Newton–Kantorovich method (1.2), we get $\eta = \frac{1}{3}(1-a)$, $L = 2(2-a)$ and $L_0 = 3-a$. The convergence criterion for the Fréchet-differentiable operator F' is given by h_k, but is not satisfied if $a = 0.49$ because $h_k = 2L\eta = 1.0268 > 1$. Hence there is no guarantee that Newton's method starting at $x_0 = 1$ converges to $x^* = \sqrt[3]{0.49}$. Instead, we see in Table 1 that the sequences r_n and $\overline{r}_n$ (for (2.34) and (2.47) respectively) converge, and we find in Table 2 that $|\overline{r}_{n+1} - \overline{r}_n| < |r_{n+1} - r_n|$ and $\overline{r}^* = 0.10379135\ldots < r^* = 0.17312593\ldots$. However, for $a = 0.60$, the convergence criterion for F' given by h_k is satisfied ($h_k = 0.746667 < 1$) for the method (1.2). Thus, the sequence $\{u_n\}$ in (1.3) converges, and so do the sequences for (2.34) and (2.47) (see Table 3). We have $|\overline{r}_{n+1} - \overline{r}_n| < |r_{n+1} - r_n| < |u_{n+1} - u_n|$, and with the sequence $\{\overline{r}_n\}$ we obtain the best a priori error bounds (see Table 4). In addition, $\overline{r}^* = 0.10379135\ldots < r^* = 0.17312593\ldots < u^* = 0.17738489\ldots$.

Table 1. The sequences $\overline{r}_n$ and r_n

Iteration	$\overline{r}_n$	r_n
1	$0.10378514\ldots$	$0.16993464\ldots$
2	$0.10379135\ldots$	$0.17310190\ldots$

Table 2. A priori error bounds for $\overline{r}_n$ and r_n

Iteration	$\lvert\overline{r}_{n+1} - \overline{r}_n\rvert$	$\lvert r_{n+1} - r_n\rvert$
1	$6.2052\ldots \times 10^{-6}$	$0.00316726\ldots$
2	$7.1790\ldots \times 10^{-11}$	$0.00002402\ldots$

Table 3. The sequences $\overline{r}_n$, r_n and u_n

Iteration	$\overline{r}_n$	r_n	u_n
1	$0.10378514\ldots$	$0.16993464\ldots$	$0.17304964\ldots$
2	$0.10379135\ldots$	$0.17310190\ldots$	$0.17733384\ldots$

Table 4. A priori error bounds for $\overline{r}_n$, r_n and u_n

Iteration	$\lvert\overline{r}_{n+1} - \overline{r}_n\rvert$	$\lvert r_{n+1} - r_n\rvert$	$\lvert u_{n+1} - u_n\rvert$
1	$6.2052\ldots \times 10^{-6}$	$0.00316726\ldots$	$0.00428420\ldots$
2	$7.1790\ldots \times 10^{-11}$	$0.00002402\ldots$	$0.00005103\ldots$

EXAMPLE 5.2. Let $X = Y = \mathcal{C}([0,1])$ be equipped with the max-norm. Consider the following nonlinear boundary value problem [11]:

$$\begin{cases} u'' = -u^3 - \gamma u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 \mathcal{Q}(s,t)(u^3(t) + \gamma u^2(t))\, dt, \tag{5.2}$$

where $\mathcal{Q}$ is the Green function

$$\mathcal{Q}(s,t) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0\le s\le 1} \int_0^1 |\mathcal{Q}(s,t)|\, dt = \frac{1}{8}.$$

Thus, problem (5.2) is in the form (1.1), where $F : D \to Y$ is defined by

$$[F(x)](s) = x(s) - s - \int_0^1 \mathcal{Q}(s,t)(x^3(t) + \gamma x^2(t))\, dt.$$

Set $u_0(s) = s$ and $D = U(u_0, R_0)$. It is easy to verify that $U(u_0, R_0) \subset U(0, R_0 + 1)$ since $\|u_0\| = 1$. If $2\gamma < 5$, the operator F' satisfies $L\eta \le 1$ and $L_0\eta \le 1$ with

$$\eta = \frac{1+\gamma}{5-2\gamma}, \quad L = \frac{\gamma + 6R_0 + 3}{4}, \quad L_0 = \frac{2\gamma + 3R_0 + 6}{8}.$$

Note that it is easy to see that $L_0 < L$ for all γ and $R_0 > 0$.

For $\gamma = 0.1$ and $R_0 = 0.5$ we obtain the corresponding sequences u_n, r_n and $\overline{r}_n$ defined earlier and compare them. In Tables 5 and 6 we can see that $\overline{r}_{n+2} < r_{n+2} < u_{n+2}$, $|r_{n+2} - \overline{r}_{n+1}| < |r_{n+2} - r_{n+1}| < |u_{n+2} - u_{n+1}|$. Moreover, $\overline{r}^* = 0.19773026\ldots < r^* = 0.28330864\ldots < u^* = 0.29595236\ldots$.

Table 5. The sequences $\overline{r}_n$, r_n and u_n

Iteration	$\overline{r}_{n+2}$	r_{n+2}	u_{n+2}
1	0.19772934...	0.28054349...	0.29072424...
2	0.19773026...	0.28330067...	0.29591492...

Table 6. A priori error bounds for $\overline{r}_n$, r_n and u_n

Iteration	$\lvert\overline{r}_{n+2} - \overline{r}_{n+1}\rvert$	$\lvert r_{n+2} - r_{n+1}\rvert$	$\lvert u_{n+2} - u_{n+1}\rvert$
1	$9.2117\ldots \times 10^{-7}$	0.00275718...	0.00519067...
2	$7.9911\ldots \times 10^{-13}$	$7.9697\ldots \times 10^{-6}$	0.00003743...

EXAMPLE 5.3. In the Newton–Kantorovich theorem appearing in [36], the most demanding condition on the operator F is $\|[F'(x_0)]^{-1}F''(x)\| \leq L$ for $x, x_0 \in \mathcal{D}$. There are operators that do not satisfy this condition, for example the following nonlinear integral equation of the Hammerstein type:

$$x(s) = u(s) + \lambda \int_a^b G(s,t)H(x(t))\,dt, \quad s \in [a,b],\ \lambda \in \mathbb{R}, \tag{5.3}$$

where $-\infty < a < b < \infty$, the function $u(s)$ is continuous on $[a,b]$, the kernel $G(s,t)$ is the Green function and $H(\xi)$ is a polynomial function.

Hammerstein equations of the second kind [40] are a particular case of (5.3). They have strong physical background and arise from the electromagnetic fluid dynamics [49]. These equations appeared in the 1930s as general models for the study of semilinear boundary value problems, where the kernel $G(s,t)$ typically arises as the Green function of a differential operator [31]. So, this type of equations can be reformulated as a two-point boundary value problem with a certain nonlinear boundary condition [15]. Also multi-dimensional analogues of these equations appear as reformulations of elliptic partial differential equations with nonlinear boundary conditions [14]. The Hammerstein equations appear very often in several applications to real world problems [16]. For example, some problems in vehicular traffic theory, biology and queuing theory lead to integral equations of this type [22]. These equations are also applied in the theory of radiative transfer and the theory of neutron transport as well in the kinetic theory of gases (see [35], among others). They also play a significant role in several applications [21], for example in the dynamic models of chemical reactors [18], which are governed by control equations [32].

As Hammerstein equations of the form (5.3) cannot be solved exactly, we can use iterative methods to solve them, as we can see in [1, 4]. In this paper, we apply Newton's method and use its theoretical properties to draw conclusions about the convergence of the method. Solving equation (5.3) is equivalent to solving the equation $F(x) = 0$, where $F : \mathcal{C}([a,b]) \to \mathcal{C}([a,b])$ and

$$[F(x)](s) = x(s) - u(s) - \lambda \int_a^b G(s,t)H(x(t))\,dt.$$

For this operator, we have

$$[F'(x)y](s) = y(s) - \lambda \int_a^b G(s,t)H'(x(t))y(t)\,dt,$$

$$[F''(x)(yz)](s) = -\lambda \int_a^b G(s,t)H''(x(t))z(t)y(t)\,dt.$$

Observe that Kantorovich's condition $\|[F'(x_0)]^{-1}F''(x)\| \leq L$ is not satisfied in general because $\|F''(x)\|$ is not bounded in $\mathcal{C}([a,b])$ unless $H(\xi)$ is a polynomial of degree less than or equal to two. In other case, it is not easy to find a domain $\Omega \subseteq \mathcal{C}([a,b])$ where $\|F''(x)\|$ is bounded and contains a solution of the equation that one tries to find.

To solve the last problem, a common option is to first locate a solution $x^*(s)$ of (5.3) in a domain $\Omega \subseteq \mathcal{C}([a,b])$ and look for a bound for $\|F''(x)\|$ in Ω (see [28]). So, taking into account the max-norm, the solution $x^*(s)$ of (5.3) must satisfy

$$\|x^*(s)\| - B - |\lambda| N \|H(x^*(s))\| \leq 0, \tag{5.4}$$

where $B = \|u(s)\|$ and $N = \max_{[a,b]} \int_a^b |G(s,t)|\, dt$. From (5.4), we try to find a region $\Omega \subseteq \mathcal{C}([a,b])$ containing $x^*(s)$. We find a solution $x^*(s)$ such that $\|x^*(s)\| \in [0,\rho]$ where ρ is a positive real root of the scalar equation $\xi - B - |\lambda| N \|H(\xi)\| = 0$.

Now, we illustrate the study presented in the last section with a particular integral equation of the form (5.3). Our study improves those of Kantorovich under his conditions and the results appearing in [24]. We use the max-norm.

Consider

$$x(s) = \frac{1}{2} + \int_0^1 G(s,t)x(t)^3\, dt, \tag{5.5}$$

where $x \in \mathcal{C}([0,1])$, $t \in [0,1]$ and the kernel G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases} \tag{5.6}$$

Solving (5.5) is equivalent to solving $F(x) = 0$, where we have $F : \Omega \subseteq \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$,

$$[F(x)](s) = x(s) - \frac{1}{2} - \int_0^1 G(s,t)x(t)^3\, dt, \quad s \in [0,1],$$

and Ω is a suitable non-empty open convex domain. Moreover,

$$[F'(x)y](s) = y(s) - 3\int_0^1 G(s,t)x(t)^2 y(t)\, dt,$$

$$[F''(x)(yz)](s) = -6\int_0^1 G(s,t)x(t)z(t)y(t)\, dt,$$

$$[F'''(x)(yzw)](s) = -6\int_0^1 G(s,t)w(t)z(t)y(t)\, dt.$$

Observe that $\|F''(x)\|$ is not bounded in a general domain Ω. Taking into account that a solution $x^*(s)$ of (5.5) in $\mathcal{C}([0,1])$ must satisfy

$$\|x^*\| - \frac{1}{2} - \frac{1}{8}\|x^*\|^3 \leq 0,$$

it follows that $\|x^*\| \leq \sigma = 0.5173\ldots$, where σ is the smallest real positive root of $t^3/8 - t + 1/2 = 0$. We see in this example that we improve the a priori error bounds obtained by Kantorovich and the results appearing in [24].

We take for example $\sigma = 2$ and choose, as is usually done [26], the starting point $x_0(s) = u(s) = 1/2$. Since $\|I - F'(x_0)\| \leq 3/32 < 1$, the operator $[F'(x_0)]^{-1}$ is well defined and $\|[F'(x_0)]^{-1}F''(x)\|$ is bounded. Hence, Kantorovich's theory [36] can be applied. Consequently, Newton's method can approximate a solution $x^\star \in U(0,\sigma)$.

On the other hand, using $[F'(x_0)]^{-1} = 32/29$ and conditions (1.3)–(1.5) we obtain

$$\omega(t) = \omega_0(t) = \frac{36}{29}, \qquad \omega_1(t) = \frac{12}{29}(1+t)t,$$

and we can construct the sequences t_n and r_n corresponding to (2.1) and (2.34). These sequences converge since the conditions of Theorems 3.1 and 3.3 hold as we can see in Table 7. Moreover, $r_n < t_n$ and $r^\star = 0.0174272591\ldots < t^\star = 0.0174299467\ldots$.

Table 7. The sequences r_n and t_n

Iteration	r_n	t_n
1	$0.01724137\ldots$	$0.01724137\ldots$
2	$0.01742723\ldots$	$0.01742992\ldots$
3	$0.01742725\ldots$	$0.01742994\ldots$

Next, we apply Newton's method for approximating a solution with the features mentioned above. For this, we use a discretization process. Thus, we approximate the integral of (5.5) by the following Gauss–Legendre quadrature formula with eight nodes:

$$\int_0^1 \phi(t)\,dt \simeq \sum_{j=1}^{8} \omega_j \phi(t_j),$$

where the nodes t_j and the weights ω_j are known. Moreover, we denote $x(t_i)$ by x_i, $i = 1, \ldots, 8$, so that equation (5.5) is now transformed into the following system of nonlinear equations:

$$x_i = \frac{1}{2} + \sum_{j=1}^{8} a_{ij}\, x_j^3,$$

where

$$a_{ij} = \begin{cases} \omega_j t_j(1-t_i) & \text{if } j \le i, \\ \omega_j t_i(1-t_j) & \text{if } j > i. \end{cases}$$

Then we write the above system in the following matrix form:

$$F(\overline{x}) = \overline{x} - \overline{v} - A\overline{w} = 0, \tag{5.7}$$

where $\overline{x} = (x_1, \dots, x_8)^T$, $\overline{v} = (1/2, \dots, 1/2)^T$, $A = (a_{ij})_{i,j=1}^8$ and $\overline{w} = (x_1^3, \dots, x_8^3)^T$. We have

$$F'(\overline{x}) = I - 3A \operatorname{diag}\{x_1^2, \dots, x_8^2\}.$$

Since $x_0(s) = 1/2$ has been chosen as starting point for the theoretical study, a reasonable choice of initial approximation for Newton's method seems to be the vector $\overline{x}_0 = (1/2, \dots, 1/2)^T$. After four iterations we obtain the numerical approximation to the solution $\overline{x}^* = (x_1^*, \dots, x_8^*)^T$ which is given in Table 8. Observe that $\|\overline{x}^*\| = 0.5168\ldots \le \sigma = 0.5173\ldots$.

Table 8. Numerical solution of (5.5)

i	x_i^*	i	x_i^*
1	$0.501329\ldots$	5	$0.516824\ldots$
2	$0.506285\ldots$	6	$0.512542\ldots$
3	$0.512542\ldots$	7	$0.506285\ldots$
4	$0.516824\ldots$	8	$0.501329\ldots$

Next, we see in Table 9 that

$$|r_{n+1} - r_n| < |t_{n+1} - t_n| < |z_{n+1} - z_n| < |q_{n+1} - q_n|$$

where the sequence q_n is for the Kantorovich conditions [36], z_n corresponds to the new majorizing sequence appearing in [24], and the sequences r_n and t_n are for the new sequences (2.34) and (2.1) of the last section respectively. The majorizing sequence r_n provides better a priori error estimates than the others.

Table 9. A priori error estimates for r_n, t_n, z_n and q_n

n	$\lvert r_{n+1} - r_n\rvert$	$\lvert t_{n+1} - t_n\rvert$	$\lvert z_{n+1} - z_n\rvert$	$\lvert q_{n+1} - q_n\rvert$
1	$2.1599\ldots \times 10^{-8}$	$2.2552\ldots \times 10^{-8}$	$0.00006266\ldots$	$0.00025323\ldots$
2	$2.9143\ldots \times 10^{-16}$	$3.2265\ldots \times 10^{-16}$	$8.4667\ldots \times 10^{-10}$	$5.4655\ldots \times 10^{-8}$

By interpolating the values of Table 8 and taking into account that the solutions of (5.5) satisfy $x(0) = x(1) = 1/2$, we obtain the solution denoted by $\widetilde{x}$ and drawn in Figure 1.

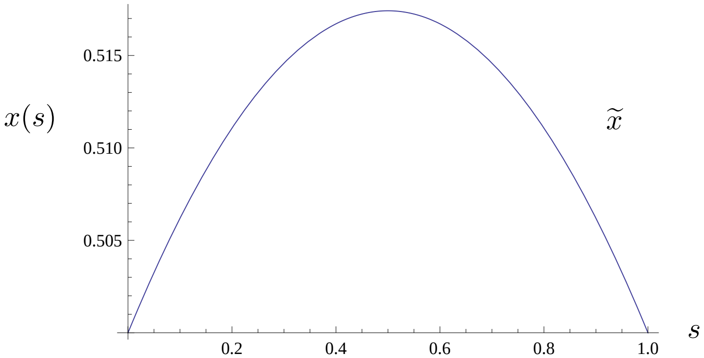

Fig. 1. Approximate solution of (5.5)

5.2. Applications: local case

EXAMPLE 5.4. Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}$, $\mathcal{D} = \overline{U}(0,1)$, $x^* = 0$ and define a function f on $\mathcal{D}$ by

$$f(x) = e^x - 1. \tag{5.8}$$

Through Theorems 4.1 and 4.2 we see that since $f'(x^*)^{-1} = 1$, we can define functions

$$\nu(t) = e = \nu_0(t) \quad \text{and} \quad \nu_1(t) = (e-1)t.$$

Using (4.2) and (4.11) we find that

$$\delta = 0.24525296\ldots < \delta_1 = 0.32494723\ldots,$$

where δ is the radius of convergence obtained in [50]. Thus, the new radius, δ_1, is bigger.

Furthermore, with the starting point $x_0 = 0.2$ we obtain better a priori error bounds by using (4.13) and (4.14) for $\beta_n^{\delta_1}$ than (4.4) and (4.5) for β_n^{δ}, as we can see in Table 10.

Table 10. A priori error bounds for $\beta_n^{\delta_1}$ and β_n^{δ}

n	$\beta_n^{\delta_1}$	β_n^{δ}
1	$0.01396245\ldots$	$0.11913311\ldots$
2	$0.00027147\ldots$	$0.02852845\ldots$
3	$1.0021\ldots \times 10^{-7}$	$0.00119916\ldots$

EXAMPLE 5.5. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}([0,1])$ and $\mathcal{D} = \overline{U}(0,1)$. Define a function F on $\mathcal{D}$ by

$$F(h)(x) = h(x) - 5\int_0^1 x\theta h(\theta)^3\, d\theta. \tag{5.9}$$

Then

$$F'(h[u])(x) = u(x) - 15\int_0^1 x\theta h(\theta)^2 u(\theta)\,d\theta \quad \text{for all } u \in \mathcal{D}.$$

Using (5.9) we see that the hypotheses of Theorems 4.1 and 4.2 hold for $x^\star(x) = 0$, where $x \in [0,1]$,

$$\nu(t) = \nu_0(t) = 15 \quad \text{and} \quad \nu_1(t) = 7.5t.$$

Moreover, we have $\delta' = 0.04444445\ldots < \delta_1' = 0.06666667\ldots$ from (4.2) and (4.11). For $x_0 = 0.02$ as starting point, in Table 11 we can see again that the a priori error bounds for this case are better when using (4.13) and (4.14) for $\zeta_n^{\delta_1'}$ than when using (4.4) and (4.5) for $\zeta_n^{\delta'}$.

Table 11. A priori error bounds for $\zeta_n^{\delta_1'}$ and $\zeta_n^{\delta'}$

n	$\zeta_n^{\delta_1'}$	$\zeta_n^{\delta'}$
1	$0.00004705\ldots$	$0.00428571\ldots$
2	$1.66149\ldots \times 10^{-8}$	$0.00014721\ldots$
3	$2.0704\ldots \times 10^{-15}$	$1.6291\ldots \times 10^{-7}$

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