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CONVERGENCE DOMAINS UNDER ZABREJKO–ZINČENKO CONDITIONS USING RECURRENT FUNCTIONS

Abstract. We provide a semilocal convergence analysis for Newton-type methods using our idea of recurrent functions in a Banach space setting. We use Zabrejko–Zinčenko conditions. In particular, we show that the convergence domains given before can be extended under the same computational cost. Numerical examples are also provided to show that we can solve equations in cases not covered before.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$(1.1) \quad F(x) + G(x) = 0,$$

where F is a Fréchet-differentiable operator defined on an open convex subset \mathcal{D} of a Banach space \mathcal{X} with values in a Banach space \mathcal{Y} , and $G : \mathcal{D} \rightarrow \mathcal{Y}$ is a continuous operator.

A large number of problems in applied mathematics and also in engineering are solved by finding solutions of equations in the form (1.1) [4].

We shall use the Newton-type method (NTM)

$$(1.2) \quad \begin{aligned} y_{n+1} &= y_n - A(y_n)^{-1}P(y_n) \quad (n \geq 0) \quad (y_0 \in \mathcal{D}), \\ P(x) &= F(x) + G(x) \quad (x \in \mathcal{D}), \end{aligned}$$

where $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, to generate a sequence approximating x^* .

2010 *Mathematics Subject Classification*: 65G99, 65H10, 65B05, 65N30, 47H17, 49M15.

Key words and phrases: Newton-type methods, recurrent functions, Banach space, Zabrejko–Zinčenko-type conditions, Kantorovich hypothesis, majorizing sequences, Green’s kernel.

(a) If

$$(1.3) \quad A(x) = F'(x) \quad (x \in \mathcal{D}),$$

we obtain the Zabrejko–Nguen iteration [26]:

$$(1.4) \quad y_{n+1} = y_n - F'(y_n)^{-1}P(y_n) \quad (n \geq 0), \quad (y_0 \in \mathcal{D}).$$

(b) If

$$(1.5) \quad A(x) = F'(x) + [x, y; G] \quad (x \in \mathcal{D}),$$

where $[x, y; F]$ is a divided difference of order one for the operator G , then we obtain an iteration faster than (1.4), first considered by Cătinaş [9]:

$$(1.6) \quad y_{n+1} = y_n - (F'(y_n) + [y_n, y_{n-1}; G])^{-1}P(y_n) \quad (n \geq 0) \quad (y_{-1}, y_0 \in \mathcal{D}).$$

(c) If

$$(1.7) \quad A(x) = F'(x), \quad G(x) = 0 \quad (x \in \mathcal{D}),$$

then (NTM) reduces to Newton’s method:

$$(1.8) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0), \quad (x_0 \in \mathcal{D}).$$

Several other choices are possible [4]–[7].

A local as well as a semilocal convergence analysis for all these methods has been provided by many authors under Lipschitz-type conditions [1]–[3], [11]–[27].

A survey of such results can be found in [4], and the references there. We also refer the reader to the elegant related works by Proinov [22], [23], and Ezquerro–Hernández [15] whose results are also improved here in at least the Newton’s method case (see Section 3).

Let $x^0 \in \mathcal{D}$ and $R > 0$ be such that

$$(1.9) \quad U(x^0, R) = \{x \in \mathcal{X} : \|x - x^0\| < R\} \subseteq \mathcal{D}.$$

Chen and Yamamoto [11] provided a semilocal convergence for (NTM) for $y^0 \in U(x^0, R)$ under the condition

(C) $A(x^0)$ exists, and for any $x, y \in \bar{U}(x^0, R)$:

$$\begin{aligned} \|A(x^0)^{-1}(A(x) - A(x^0))\| &\leq \bar{\nu}_0(\|x - x^0\|) + a, \\ \|A(x^0)^{-1}(F'(x + t(x - y)) - A(x))\| \\ &\leq \bar{\nu}(\|x - x^0\| + t\|y - x\|) - \bar{\nu}_0(\|x - x^0\|) + b, \quad t \in [0, 1], \\ \|A(x^0)^{-1}(G(x) - G(y))\| &\leq \omega(r)\|x - y\|, \end{aligned}$$

where $\bar{\nu}(r+t) - \bar{\nu}_0(r)$, $t \geq 0$, and $\omega(r)$ are non-decreasing, non-negative functions with

$$\omega(0) = \bar{\nu}_0(0) = \bar{\nu}(0) = 0,$$

$\bar{\nu}_0(r)$ is differentiable, $\bar{\nu}_0'(r) > 0$ at every point of $[0, R]$, and the constants a, b satisfy $a, b \geq 0$, and $a + b < 1$.

Set

$$\begin{aligned} \|A(x^0)^{-1}P(x^0)\| &\leq \eta, \\ \phi(r) &= \eta - r + \int_0^1 \bar{\nu}(t) dt, \quad \psi(r) = e \int_0^r \omega(t) dt, \\ \chi(r) &= \phi(r) + \psi(r) + (a + b)r. \end{aligned}$$

Further, assume

$$\chi(R) \leq 0,$$

and define a scalar sequence $\{s_n\}$ by

$$s_0 \in [0, R], \quad s_{n+1} = s_n + \frac{u(s_n)}{p(s_n)} \quad (n \geq 0),$$

where

$$u(r) = \chi(r) - \chi^*, \quad p(r) = 1 - \bar{\nu}_0(r) - a,$$

χ^* is the minimal value of $\chi(r)$ in $[0, R]$, and s^* denotes the minimal point. Moreover, t^* denotes the unique zero of χ in $(0, s^*]$.

Under these assumptions, there exists a unique solution $x^* \in \bar{U}(x^0, t^*)$ such that

$$\|y_{n+1} - y_n\| \leq s_{n+1} - s_n, \quad \|x^* - y_n\| \leq s^* - s_n.$$

We shall use the more general set of conditions $(\mathcal{H}) \equiv (\mathcal{H}_1) - (\mathcal{H}_4)$, where

$$(\mathcal{H}_1) \quad A(x^0) \text{ exists, and for any } x, y \in \bar{U}(x^0, R) \text{ (} 0 < r \leq R \text{),}$$

$$\|A(x^0)^{-1}(A(x) - A(x^0))\| \leq \nu_0(\|x - x^0\|) + a,$$

$$(\mathcal{H}_2) \quad \|A(x^0)^{-1}(F'(x + t(x - y)) - A(x))\| \leq \nu(t\|y - x\|) + \omega_0(\|x - x^0\|) + b, \quad t \in [0, 1],$$

$$(\mathcal{H}_3) \quad \|A(x^0)^{-1}(G(x) - G(y))\| \leq \omega(r)\|x - y\|,$$

$$(\mathcal{H}_4) \quad \bar{U}(x^0, R) \subseteq \mathcal{D},$$

where ν_0, ν , and ω_0 are non-decreasing, non-negative functions on $[0, R]$ with

$$\nu_0(0) = \nu(0) = \omega_0(0) = \omega(0) = 0,$$

and a and b are non-negative constants. Some more hypotheses are given in Lemma 2.2.

A semilocal convergence analysis is provided in Section 2 of this study under the (\mathcal{H}) conditions, whereas in Section 3 we compare the two sets of hypotheses. Numerical examples are also provided in Section 3 to show how we can solve equations in cases not covered before [1]–[27].

2. Semilocal convergence analysis of (NTM). We need to define some parameters, functions and sequences.

DEFINITION 2.1. Let $y_0 \in U(x^0, r)$. Define parameters r_0, r_1 , iteration $\{r_n\}$, functions f_n, h_n, p_n on $[0, 1)$, and q on $I_q = \{(t, s, \lambda, \beta_0, \beta, \gamma_0) : 0 \leq t, s < 1, 0 \leq \lambda \leq r_1 - r_0, r_1 - r_0 \leq \beta_0, \beta, \gamma_0 \leq \frac{r_1 - r_0}{1 - s}\}$ by

$$(2.1) \quad \begin{aligned} r_0 &\geq \|y_0 - x^0\|, \quad r_1 > r_0 + \|A(y_0)^{-1}(F(y_0) + G(y_0))\|, \\ r_{n+1} &= r_n + \frac{\int_0^1 \nu(t(r_n - r_{n-1})) dt + \omega_0(r_n) + \omega(r_n) + b}{1 - a - \nu_0(r_n)} (r_n - r_{n-1}), \end{aligned}$$

$$(2.2) \quad f_n(s) = \int_0^1 \nu(ts^{n-1}(r_1 - r_0)) dt + \omega_1((1 + s + \dots + s^{n-1})(r_1 - r_0)) + c,$$

$$(2.3) \quad \begin{aligned} h_n(s) &= \int_0^1 (\nu(ts^n(r_1 - r_0)) - \nu(ts^{n-1}(r_1 - r_0))) dt \\ &\quad + \omega_1((1 + s + \dots + s^n)(r_1 - r_0)) \\ &\quad - \omega_1((1 + s + \dots + s^{n-1})(r_1 - r_0)), \end{aligned}$$

where

$$(2.4) \quad c = b - \alpha(1 - a) \quad \text{for some } \alpha \in (0, 1),$$

$$(2.5) \quad \omega_1(s) = \omega_0(s) + \omega(s) + \alpha\nu_0(s),$$

$$(2.6) \quad \begin{aligned} p_n(s) &= \int_0^1 (\nu(ts^{n+1}(r_1 - r_0)) \\ &\quad + \nu(ts^{n-1}(r_1 - r_0)) - 2\nu(ts^n(r_1 - r_0))) dt \\ &\quad + \omega_1((1 + s + \dots + s^{n+1})(r_1 - r_0)) \\ &\quad + \omega_1((1 + s + \dots + s^{n-1})(r_1 - r_0)) \\ &\quad - 2\omega_1((1 + s + \dots + s^n)(r_1 - r_0)), \end{aligned}$$

$$(2.7) \quad \begin{aligned} q(t, s, \lambda, \beta_0, \beta, \gamma_0) &= \int_0^1 (\nu(t\lambda s^2) + \nu(t\lambda) - 2\nu(t\lambda s)) dt \\ &\quad + \omega_1(\beta_0 + \beta + \gamma_0) + \omega_1(\beta) - 2\omega_1(\beta_0 + \beta). \end{aligned}$$

Define a function f_∞ on $[0, 1)$ by

$$(2.8) \quad f_\infty(s) = \lim_{n \rightarrow \infty} f_n(s).$$

It then follows from (2.2) and (2.8) that

$$(2.9) \quad f_\infty(s) = \omega_1\left(\frac{r_1 - r_0}{1 - s}\right) + c.$$

It can also be easily seen from (2.2), (2.3), (2.6), and (2.7) that

$$(2.10) \quad f_{n+1}(s) = f_n(s) + h_n(s),$$

$$(2.11) \quad h_{n+1}(s) = h_n(s) + p_n(s),$$

$$(2.12) \quad q(t, s, s^{n-1}(r_1 - r_0), s^n(r_1 - r_0), \\ (1 + s + \dots + s^{n-1})(r_1 - r_0), s^{n+1}(r_1 - r_0)) = p_n(s).$$

We need the following result on majorizing sequences for (NTM).

LEMMA 2.2. *Let constants a, b , parameters r_0, r_1 , and functions $\nu_0, \nu, \omega_0, \omega$ be as in the Introduction, and let parameters α, c and functions $\omega_1, f_n, h_n, p_n, q$ be as in Definition 2.1. Assume there exists $\alpha \in (0, 1)$ such that*

$$(2.13) \quad \nu_0(r_1) + a < 1,$$

$$(2.14) \quad \frac{\int_0^1 \nu(t(r_1 - r_0)) dt + \omega_0(r_1) + \omega(r_1) + b}{1 - a - \nu_0(r_1)} \leq \alpha,$$

$$(2.15) \quad c < 0,$$

$$(2.16) \quad q(t, s, \lambda, \beta_0, \beta, \gamma_0) \geq 0 \quad \text{on } I_q,$$

$$(2.17) \quad h_1(\alpha) \geq 0,$$

$$(2.18) \quad f_\infty(\alpha) \leq 0.$$

Then the scalar sequence $\{r_n\}$ ($n \geq 0$) given by (2.1) is non-decreasing, bounded from above by

$$(2.19) \quad r^{**} = \frac{r_1 - r_0}{1 - \alpha},$$

and converges to its least upper bound r^* satisfying $r^* \in [0, r^{**}]$.

Moreover the following estimates hold for all $n \geq 0$:

$$(2.20) \quad 0 \leq r_{n+1} - r_n \leq \alpha(r_n - r_{n-1}) \leq \alpha^n(r_1 - r_0),$$

$$(2.21) \quad r^* - r_n \leq \frac{r_1 - r_0}{1 - \alpha} \alpha^n.$$

Proof. Estimate (2.20) is true if

$$(2.22) \quad 0 \leq \frac{\int_0^1 \nu(t(r_n - r_{n-1})) dt + \omega_0(r_n) + \omega(r_n)}{1 - a - \nu_0(r_n)} \leq \alpha$$

for all $n \geq 1$.

In view of (2.1), (2.13), (2.14), estimate (2.22) holds true for $n = 1$. Also (2.22) implies $0 \leq r_2 - r_1 \leq \alpha(r_1 - r_0)$.

Let us assume that (2.20) and (2.22) hold for all $k \leq n$. Then

$$(2.23) \quad r_n \leq \frac{1 - \alpha^n}{1 - \alpha} (r_1 - r_0).$$

By the induction hypothesis, and (2.23), estimate (2.22) is true if

$$(2.24) \quad \int_0^1 \nu(t\alpha^{n-1}(r_1 - r_0)) dt + \omega_1 \left(\frac{1 - \alpha^n}{1 - \alpha} (r_1 - r_0) \right) + c \leq 0,$$

where c and ω_1 are given by (2.4) and (2.5), respectively. Estimate (2.24) (for $s = \alpha$) motivates us to introduce the function f_n given by (2.2), and show, instead of (2.24),

$$(2.25) \quad f_n(\alpha) \leq 0 \quad (n \geq 1).$$

By (2.10)–(2.12) (for $s = \alpha$) and (2.17) we have

$$(2.26) \quad f_{n+1}(\alpha) \geq f_n(\alpha) \quad (n \geq 1).$$

In view of (2.8), (2.9), and (2.26), estimate (2.25) holds if (2.18) is true, since

$$(2.27) \quad f_n(\alpha) \leq f_\infty(\alpha) \quad (n \geq 1).$$

The induction is completed. It follows that the iteration $\{r_n\}$ is non-decreasing, bounded above by r^{**} (given by (2.19)), and so converges to r^* . Finally, estimate (2.21) follows from (2.20) by using standard majorization techniques [4], [17], [18].

That completes the proof of Lemma 2.2. ■

The hypotheses (\mathcal{H}) and those of Lemma 2.2 will be called (\mathcal{A}).

We can now show the main semilocal convergence result for (NTM).

THEOREM 2.3. *Assume hypotheses (\mathcal{A}) hold. Then the sequence $\{y_n\}$ ($n \geq 0$) generated by (NTM) is well defined, remains in $\bar{U}(x^0, r^*)$ for all $n \geq 0$, and converges to a solution $x^* \in \bar{U}(x^0, r^*)$ of the equation $F(x) + G(x) = 0$.*

Moreover, the following estimates hold for all $n \geq 0$:

$$(2.28) \quad \|y_{n+1} - y_n\| \leq r_{n+1} - r_n,$$

$$(2.29) \quad \|y_n - x^*\| \leq r^* - r_n,$$

where the sequence $\{r_n\}$ ($n \geq 0$) and r^* are given in Lemma 2.2.

Furthermore, if there exists

$$(2.30) \quad R_0 \in [r^*, R]$$

such that

$$(2.31) \quad \int_0^1 \nu(t(R_0 + r_0)) dt + \omega_0(r^*) + \omega(R_0) + \nu_0(r^*) + a + b \leq 1,$$

then the solution x^* of equation (1.1) is unique in $U(x^0, R_0)$.

Proof. We shall show by induction that

$$(2.32) \quad \|y_n - y_{n-1}\| \leq r_n - r_{n-1},$$

$$(2.33) \quad \|y_n - x^0\| \leq r_n.$$

Estimates (2.32) and (2.33) hold for $n = 1$, by (1.4) and (2.1). Assume they hold for all $n \leq k$. Using (\mathcal{H}_1) and (2.13), we get

$$(2.34) \quad \|A(x_0)^{-1}[A(y_1) - A(x^0)]\| \leq \nu_0(\|y_1 - x^0\|) + a \leq \nu_0(r_1) + a < 1.$$

It follows from (2.34) and the Banach lemma on invertible operators [4], [18] that $A(y_1)^{-1}$ exists, and

$$(2.35) \quad \|A(y_1)^{-1}A(x^0)\| \leq (1 - a - \nu_0(r_1))^{-1}.$$

We also showed in Lemma 2.2 that

$$(2.36) \quad \nu_0(r_k) + a < 1.$$

It then follows as in (2.34) with r_k, y_k replacing r_1, y_1 , respectively, that $A(y_k)^{-1}$ exists, and

$$(2.37) \quad \|A(y_k)^{-1}A(x^0)\| \leq (1 - a - \nu_0(r_k))^{-1}.$$

Using (1.2), (\mathcal{H}_2) , (\mathcal{H}_3) , (2.1), (2.32), (2.33), and (2.37), we obtain in turn

$$(2.38) \quad \begin{aligned} \|y_{k+1} - y_k\| &= \|A(y_k)^{-1}(F(y_k) + G(y_k))\| \\ &\leq \|A(y_k)^{-1}A(x^0)\| \|A(x^0)^{-1}(F(y_k) \\ &\quad + G(y_k) - A(y_{k-1})(y_k - y_{k-1}) - F(y_{k-1}) - G(y_{k-1}))\| \\ &\leq (1 - a - \nu_0(r_k))^{-1} \left(\int_0^1 \|A(x^0)^{-1}[F'(y_{k-1} + \theta(y_k - y_{k-1})) \right. \\ &\quad \left. - A(y_{k-1})]\| \|y_k - y_{k-1}\| d\theta \right. \\ &\quad \left. + \|A(x^0)^{-1}(G(y_k) - G(y_{k-1}))\| \right) \\ &\leq (1 - a - \nu_0(r_k))^{-1} \left(\int_0^1 \nu(t\|y_k - y_{k-1}\|) dt \right. \\ &\quad \left. + \omega_0(\|y_k - x^0\|) + \omega(\|y_k - x^0\|) + b \right) \|y_k - y_{k-1}\| \\ &\leq (1 - a - \nu_0(r_k))^{-1} \left(\int_0^1 \nu(t(r_k - r_{k-1})) dt + \omega_0(r_k) \right. \\ &\quad \left. + \omega(r_k) + b \right) (r_k - r_{k-1}) = r_{k+1} - r_k. \end{aligned}$$

Moreover, we have

$$(2.39) \quad \begin{aligned} \|y_{k+1} - x^0\| &\leq \|y_{k+1} - y_k\| + \|y_k - x^0\| \\ &\leq (r_{k+1} - r_k) + r_k = r_{k+1} \leq r^*. \end{aligned}$$

The induction for (2.32) and (2.33) is completed.

In view of Lemma 2.2, (2.32), and (2.33), the sequence $\{y_n\}$ ($n \geq 0$) is Cauchy in the Banach space \mathcal{X} , and so it converges to some $x^* \in \overline{U}(x^0, r^*)$ (since $\overline{U}(x^0, r^*)$ is a closed set).

Estimate (2.29) follows from (2.28) by using standard majorization techniques [4], [17], [18].

Using (2.38), we obtain

$$(2.40) \quad \|A(x^0)^{-1} (F(x_k) + G(x_k))\| \leq \left(\int_0^1 \nu(t(r_k - r_{k-1})) dt + \omega_0(r_k) + \omega(r_k) + b \right) (r_k - r_{k-1}).$$

By letting $k \rightarrow \infty$ in (2.40), we obtain $F(x^*) + G(x^*) = 0$. Finally to show that x^* is the unique solution of equation (1.1) in $\bar{U}(x^0, R_0)$, let $y^* \in \bar{U}(x^0, R_0)$, with $F(y^*) + G(y^*) = 0$.

Using the approximation

$$(2.41) \quad y^* - y_{k+1} = y^* - y_k + A(y_k)^{-1}(F(y_k) + G(y_k)) - A(y_k)^{-1}(F(y^*) + G(y^*)),$$

as in (2.38), we obtain in turn

$$(2.42) \quad \|y^* - y_{k+1}\| \leq (1 - a - \nu_0(r_k))^{-1} \left(\int_0^1 \|A(x^0)^{-1}[F'(y_k + \theta(y^* - y_k)) - A(y_k)]\| \|y^* - y_k\| d\theta + \|A(x^0)^{-1}(G(y^*) - G(y_k))\| \right) \leq (1 - a - \nu_0(r_k))^{-1} \left(\int_0^1 \nu(t\|y^* - y_k\|) dt + \omega_0(\|y_k - x^0\|) + \omega(\|y_k - x^0\|) + b \right) \|y^* - y_k\| < (1 - a - \nu_0(r^*))^{-1} \left(\int_0^1 \nu(t(R_0 + r_0)) dt + \omega_0(r^*) + \omega(R_0) + b \right) \|y_k - y^*\| \leq \|y_k - y^*\| \quad (\text{by (2.31)}).$$

It follows from (2.42) that $\lim_{k \rightarrow \infty} y_k = y^*$. But we showed $\lim_{k \rightarrow \infty} y_k = x^*$. Hence, $x^* = y^*$.

That completes the proof of Theorem 2.3. ■

3. Special cases and applications

APPLICATION 3.1 (Newton’s method). Let $A(x) = F'(x)$ and $G(x) = 0$ ($x \in \mathcal{D}$). Then, in the case of the (C) conditions, we have

$$y_0 = x^0 = x_0, \quad s_0 = 0, \quad s_1 = \eta, \quad a = b = 0, \\ \bar{\nu}_0(r) = L_0r, \quad \bar{\nu}(r) = Lr, \quad \omega(r) = 0,$$

$$\begin{aligned}
 \chi(r) = \phi(r) &= \frac{L}{2}r^2 - r + \eta, & \chi^* &= \frac{2L\eta - 1}{2L}, \\
 s^* &= \frac{1}{L}, & t^* &= \frac{1 - \sqrt{1 - 2L\eta}}{L}, & R &\in [t^*, s^*], \\
 u(r) &= \frac{1}{2L}(Lr - 1)^2, & p(r) &= 1 - L_0r, \\
 (3.1) \qquad & & h_{CY} &= L\eta \leq 1/2,
 \end{aligned}$$

and

$$\begin{aligned}
 (3.2) \qquad s_0 &= 0, & s_1 &= \eta, \\
 s_{n+1} &= s_n + \frac{(Ls_n - 1)^2}{2L(1 - L_0s_n)} = s_n + \frac{L(s_n - s_{n-1})^2}{2(1 - L_0s_n)} \quad (n \geq 1),
 \end{aligned}$$

Moreover, in the case of the (A) conditions, we have:

$$\begin{aligned}
 (3.3) \qquad y_0 &= x^0 = x_0, & r_0 &= 0, & r_1 &= \eta, & a &= b = 0, \\
 \nu_0(r) &= L_0r, & \nu(r) &= Lr, & \omega(r) &= \omega_0(r) = 0, \\
 \alpha &= \frac{2L}{L + \sqrt{L^2 + 8L_0L}}, \\
 h_{AH} &= \bar{L}\eta \leq 1/2,
 \end{aligned}$$

where

$$\bar{L} = \frac{1}{8}(L + 4L_0 + \sqrt{L^2 + 8L_0L}),$$

and

$$\begin{aligned}
 (3.4) \qquad s_0 &= 0, & s_1 &= \eta, \\
 s_{n+1} &= s_n + \frac{L(s_n - s_{n-1})^2}{2(1 - L_0s_n)} \quad (n \geq 1).
 \end{aligned}$$

Note that

$$(3.5) \qquad L_0 \leq L$$

in general, and L/L_0 can be arbitrarily large [4]–[7].

Let us now compare the results. It follows from (3.1), (3.3), and (3.5) that

$$(3.6) \qquad h_{CY} \leq 1/2 \Rightarrow h_{AH} \leq 1/2,$$

but not necessarily vice versa unless $L_0 = L$.

Hence the convergence domains approach in [11] does not necessarily produce the weakest sufficient convergence conditions, even in the simplest possible case of a Newton-like method which is Newton’s method (1.8). The recurrent functions approach produces sufficient convergence (3.3) that can always replace (3.1).

Hence, the applicability of Newton’s method has been extended, under the same hypotheses and computational cost as in [11], [18]. Note that the results in [15], [22], [23] are also improved in at least the Newton’s method case, since their conditions also lead to (3.1) instead of (3.3).

In the Newton’s method case, although the majorizing sequences $\{s_n\}$ and $\{r_n\}$ coincide, the convergence domains approach fails to take advantage of the relationship between L_0 and L , since L_0 does not appear in (3.1). The same is happening in the general case, since the function χ does not depend on $\bar{\nu}_0$. However, our approach depends on ν_0 for the derivation of the sufficient convergence conditions. Under our method the ratio “ α ” of convergence for $\{s_n\}$ is known, but this is not true for the iteration $\{r_n\}$.

Next, we provide three examples where $L_0 < L$. Moreover, in the first example, (3.3) is satisfied but (3.1) is not.

EXAMPLE 3.2.

CASE 1: $A(x) = F'(x)$ and $G(x) = 0$ ($x \in \mathcal{D}$). Let $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$, equipped with the max-norm, $x^0 = (1, 1)^T$, and

$$\mathcal{D} = U_0 = \{x : |x - x^0| \leq 1 - \delta\}, \quad \delta \in [0, 1/2).$$

Define a function F on U_0 by

$$(3.7) \quad F(x) = (\xi_1^3 - \delta, \xi_2^3 - \delta), \quad x = (\xi_1, \xi_2)^T.$$

The Fréchet derivative of F is given by

$$F'(x) = \begin{bmatrix} 3\xi_1^2 & 0 \\ 0 & 3\xi_2^2 \end{bmatrix}.$$

Using the hypotheses of Theorem 2.3, we get

$$\eta = \frac{1}{3}(1 - \delta), \quad L = 3 - \delta, \quad K = 2(2 - \delta).$$

The Newton–Kantorovich condition (3.1) is violated, since

$$\frac{4}{3}(1 - \delta)(2 - \delta) > 1 \quad \text{for all } \delta \in [0, 1/2).$$

Hence, there is no guarantee that Newton’s method (1.2) converges to $x^* = (\sqrt[3]{\delta}, \sqrt[3]{\delta})^T$, starting at x^0 .

However, our condition (3.3) is true for all $\delta \in I = [.450339002, 1/2)$. Hence, the conclusions of our Theorem 2.3 can be applied to solve equation (3.7) for all $\delta \in I$.

CASE 2: $A(x) = F'(x)$ and $G(x) = (\epsilon|\xi_1 - 1|, \epsilon|\xi_2 - 1|)$ ($x \in \mathcal{D}$). Let us choose $\epsilon = .1$, $\delta = .49$, $y_0 = x^0$. Using the (\mathcal{C}) conditions, we have

$$r_0 = 0, \quad \bar{\nu}_0(r) = \bar{\nu}(r) = \ell r, \quad \omega(r) = |\epsilon|, \quad r \geq 0, \quad a = b = 0,$$

and

$$\chi(r) = \frac{\ell}{2}r^2 - (1 - |\epsilon|)r + \eta.$$

However, the discriminant of the function χ is negative, since

$$\frac{4}{3}(1 - \delta)(2 - \delta) > (1 - |\epsilon|)^2 \quad \text{for all } \delta \in [0, 1/2).$$

Hence, the (\mathcal{C}) conditions do not hold. That is, there is no guarantee that the iteration (1.2) converges to a solution x^* of equation (1.1).

We shall now check condition (\mathcal{A}) . We have

$$\begin{aligned} \nu_0(r) &= \ell_0 r, & \ell_0 &= 2.51, & \omega_0(r) &= 0, \\ \nu(r) &= \ell r, & \ell &= 3.02, & \omega(r) &= \ell_1 = |\epsilon| = .1. \end{aligned}$$

Choose $r_1 = .118$. Then (2.13)–(2.18) hold, since by Maple we have

$$.118 < \frac{1}{\ell_0} \simeq .398406374, \quad \frac{\int_0^1 \ell t r_1 dt + .1}{1 - \ell_0 r_1} \leq \alpha,$$

or

$$\alpha \geq .395243102,$$

so we choose

$$\alpha = .40353, \quad c = -\alpha < 0, \quad \omega_1(r) = .1 + \alpha \ell_0 r,$$

Then

$$\begin{aligned} f_\infty(\alpha) &= .1 + \frac{\alpha \ell_0 r_1}{1 - \alpha} - \alpha = -.1031552697 < 0, \\ h_1(r) &= \left(\left(\frac{\ell}{2} + \alpha \ell_0 \right) r + \alpha \ell_0 - \frac{\ell}{2} \right) r_1, \end{aligned}$$

so

$$h_1(\alpha) = .06146739381 \geq 0, \quad R = r^{**} = .1978305699,$$

and

$$\bar{U}(x^0, R) \subseteq U(x^0, 1 - \delta) = U(x^0, .51).$$

Hence, all hypotheses of Theorem 2.3 are satisfied. That is, the iteration (NTM) starting from x^0 converges to $x^* = (.788373516, .788373516)^T$.

EXAMPLE 3.3. Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ be the space of real-valued continuous functions defined on the interval $[0, 1]$ with the norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Let $\theta \in [0, 1]$ be a given parameter. Consider the “cubic” integral equation

$$(3.8) \quad u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s, t) u(t) dt + y(s) - \theta.$$

Here the kernel $q(s, t)$ is a continuous function of two variables defined on $[0, 1] \times [0, 1]$; λ is a real number called the “albedo” for scattering; $y(s)$ is a given continuous function defined on $[0, 1]$; and $x(s)$ is the unknown function sought in $\mathcal{C}[0, 1]$. Equations of the form (3.8) arise in the kinetic theory of gasses [4], [10]. For simplicity, we choose $u_0(s) = y(s) = 1$, and $q(s, t) = s/(s + t)$ for all $s \in [0, 1]$ and $t \in [0, 1]$ with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1 - \theta)$, and define the operator F on \mathcal{D} by

$$(3.9) \quad F(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s, t) x(t) dt + y(s) - \theta$$

for all $s \in [0, 1]$, then every zero of F satisfies equation (3.8). We have the estimate

$$\max_{0 \leq s \leq 1} \left| \int_0^1 \frac{s}{s + t} dt \right| = \ln 2.$$

Therefore, if we set $\xi = \|F'(u_0)^{-1}\|$, then it follows from the hypotheses of Theorem 2.3 that

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta),$$

$$L = 2\xi(|\lambda| \ln 2 + 3(2 - \theta)) \quad \text{and} \quad L_0 = \xi(2|\lambda| \ln 2 + 3(3 - \theta)).$$

It follows from Theorem 2.3 that if condition (3.3) holds, then problem (3.8) has a unique solution near u_0 . This assumption is weaker than the one given before using the Newton–Kantorovich hypothesis (3.1).

Note also that $L_0 < L$ for all $\theta \in [0, 1]$.

EXAMPLE 3.4. Consider the following nonlinear boundary value problem [4]:

$$\begin{cases} u'' = -u^3 - \rho u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$(3.10) \quad u(s) = s + \int_0^1 Q(s, t)(u^3(t) + \rho u^2(t)) dt$$

where Q is the Green function

$$Q(s, t) = \begin{cases} t(1 - s), & t \leq s, \\ s(1 - t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \leq s \leq 1} \int_0^1 |Q(s, t)| = \frac{1}{8}.$$

Let $\mathcal{X} = \mathcal{Y} = \mathcal{C}[0, 1]$ with the norm

$$\|x\| = \max_{0 \leq s \leq 1} |x(s)|.$$

Then problem (3.10) is in the form (1.1), where $F : \mathcal{D} \rightarrow \mathcal{Y}$ is defined as

$$[F(x)](s) = x(s) - s - \int_0^1 Q(s, t)(x^3(t) + \rho x^2(t)) dt,$$

and

$$G(x)(s) = 0.$$

It is easy to verify that the Fréchet derivative of F is

$$[F'(x)v](s) = v(s) - \int_0^1 Q(s, t)(3x^2(t) + 2\rho x(t))v(t) dt.$$

If we set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R)$, then since $\|u_0\| = 1$, it is easy to verify that $U(u_0, R) \subset U(0, R + 1)$. It follows that if $2\rho < 5$, then

$$\begin{aligned} \|I - F'(u_0)\| &\leq \frac{3\|u_0\|^2 + 2\rho\|u_0\|}{8} = \frac{3 + 2\rho}{8}, \\ \|F'(u_0)^{-1}\| &\leq \frac{1}{1 - \frac{3+2\rho}{8}} = \frac{8}{5 - 2\rho}, \\ \|F(u_0)\| &\leq \frac{\|u_0\|^3 + \rho\|u_0\|^2}{8} = \frac{1 + \rho}{8}, \\ \|F(u_0)^{-1}F(u_0)\| &\leq \frac{1 + \rho}{5 - 2\rho}. \end{aligned}$$

On the other hand, for $x, y \in \mathcal{D}$, we have

$$[(F'(x) - F'(y))v](s) = - \int_0^1 Q(s, t)(3x^2(t) - 3y^2(t) + 2\rho(x(t) - y(t)))v(t) dt.$$

Consequently,

$$\begin{aligned} \|F'(x) - F'(y)\| &\leq \frac{\|x - y\|(2\rho + 3(\|x\| + \|y\|))}{8} \\ &\leq \frac{\|x - y\|(2\rho + 6R + 6\|u_0\|)}{8} \\ &= \frac{\rho + 6R + 3}{4}\|x - y\|, \\ \|F'(x) - F'(u_0)\| &\leq \frac{\|x - u_0\|(2\rho + 3(\|x\| + \|u_0\|))}{8} \\ &\leq \frac{\|x - u_0\|(2\rho + 3R + 6\|u_0\|)}{8} \\ &= \frac{2\rho + 3R + 6}{8}\|x - u_0\|. \end{aligned}$$

Therefore, the conditions of Theorem 2.3 hold with

$$\eta = \frac{1 + \rho}{5 - 2\rho}, \quad L = \frac{\rho + 6R + 3}{4}, \quad L_0 = \frac{2\rho + 3R + 6}{8}.$$

Note also that $L_0 < L$.

Finally, we provide an example to show that method (1.6) is faster than method (1.4).

EXAMPLE 3.5 ([9]). Let $\mathcal{X} = \mathcal{Y} = (\mathbb{R}^2, \|\cdot\|_\infty)$. Consider the system

$$(3.11) \quad \begin{aligned} 3x^2y + y^2 - 1 + |x - 1| &= 0, \\ x^4 + xy^3 - 1 + |y| &= 0. \end{aligned}$$

For $v = (v_1, v_2)$ set

$$\|v\|_\infty = \|(v_1, v_2)\|_\infty = \max\{|v_1|, |v_2|\}, \quad F = (F_1, F_2), \quad G = (G_1, G_2),$$

where

$$\begin{aligned} F_1(v) &= 3v_1^2v_2 + v_2^2 - 1, & F_2(v) &= v_1^4 + v_1v_2^3 - 1, \\ G_1(v) &= |v_1 - 1|, & G_2(v) &= |v_2|. \end{aligned}$$

The divided differences of order one $[x, y; F], [x, y; G] \in \mathcal{M}_{2 \times 2}(\mathbb{R})$ are for $w = (w_1, w_2)$,

$$[v, w, F]_{i,1} = \frac{F_i(w_1, w_2) - F_i(v_1, w_2)}{w_1 - v_1}, \quad [v, w, F]_{i,2} = \frac{F_i(v_1, w_2) - F_i(v_1, v_2)}{w_2 - v_2}$$

provided that $w_1 \neq v_1$ and $w_2 \neq v_2$. If $w_1 = v_1$ or $w_2 = v_2$, replace $[x, y; F]$ by F' . Similarly we define

$$[v, w; G]_{i,1} = \frac{G_i(w_1, w_2) - G_i(v_1, w_2)}{w_1 - v_1}, \quad [v, w; G]_{i,2} = \frac{G_i(v_1, w_2) - G_i(v_1, v_2)}{w_2 - v_2}$$

for $w_1 \neq v_1$ and $w_2 \neq v_2$. If $w_1 = v_1$ or $w_2 = v_2$, replace $[x, y; G]$ by the zero 2×2 matrix in $\mathcal{M}_{2 \times 2}(\mathbb{R})$.

We consider three interesting choices for the operator A :

$$(3.12) \quad A(v, w) = F(v) + G(v) + F'(v)(w - v),$$

$$(3.13) \quad A(u, v, w) = F(v) + G(v) + ([u, v; F] + [u, v; G])(w - v),$$

$$(3.14) \quad A(u, v, w) = F(v) + G(v) + (F'(v) + [u, v; G])(w - v).$$

Using the method (1.4), based on the operator (3.12) for $y_0 = (1, 0)^T$, and the methods based on the operators (3.13) and (3.14) (the latter is (1.6)) for $y_{-1} = (5, 5)^T, y_0 = (1, 0)^T$, we obtain the following three tables respectively:

n	$y_n^{(1)}$	$y_n^{(2)}$	$\ y_n - y_{n-1}\ $
0	1	0	
1	1	0.3333333333333333	3.333E-1
2	0.906550218340611	0.354002911208151	9.344E-2
3	0.885328400663412	0.338027276361322	2.122E-2
4	0.891329556832800	0.326613976593566	1.141E-2
5	0.895238815463844	0.326406852843625	3.909E-3
6	0.895154671372635	0.327730334045043	1.323E-3
7	0.894673743471137	0.327979154372032	4.809E-4
8	0.894598908977448	0.327865059348755	1.140E-4
9	0.894643228355865	0.327815039208286	5.002E-5
10	0.894659993615645	0.327819889264891	1.676E-5
11	0.894657640195329	0.327826728208560	6.838E-6
12	0.894655219565091	0.327827351826856	2.420E-6
13	0.894655074977661	0.327826643198819	7.086E-7
...			
39	0.894655373334687	0.327826521746298	5.149E-19

n	$y_n^{(1)}$	$y_n^{(2)}$	$\ y_n - y_{n-1}\ $
-1	5	5	
0	1	0	5.000E+00
1	0.989800874210782	0.012627489072365	1.262E-02
2	0.921814765493287	0.307939916152262	2.953E-01
3	0.900073765669214	0.325927010697792	2.174E-02
4	0.894939851625105	0.327725437396226	5.133E-03
5	0.894658420586013	0.327825363500783	2.814E-04
6	0.894655375077418	0.327826521051833	3.045E-04
7	0.894655373334698	0.327826521746293	1.742E-09
8	0.894655373334687	0.327826521746298	1.076E-14
9	0.894655373334687	0.327826521746298	5.421E-20

n	$y_n^{(1)}$	$y_n^{(2)}$	$\ y_n - y_{n-1}\ $
-1	5	5	
0	1	0	5
1	0.909090909090909	0.363636363636364	3.636E-01
2	0.894886945874111	0.329098638203090	3.453E-02
3	0.894655531991499	0.327827544745569	1.271E-03
4	0.894655373334793	0.327826521746906	1.022E-06
5	0.894655373334687	0.327826521746298	6.089E-13
6	0.894655373334687	0.327826521746298	2.710E-20

We have not verified the hypotheses of Theorem 2.3 for the above starting points. However, it is clear that the hypotheses of Theorem 2.3 are satisfied for all three methods for starting points closer to the solution

$$x^* = (.894655373334687, .327826521746298)^T$$

chosen from the lists of the tables displayed above.

Hence method (1.6) for choice (3.14) converges faster than (1.4) suggested in Chen and Yamamoto [11], Zabrejko and Nguen [26] in this case and the method of chord [4], [6], [27].

Conclusion. Using our new idea of recurrent functions, and Zabrejko–Zinčenko-type conditions, we provided a semilocal convergence analysis for (NTM) in order to approximate a locally unique solution of an equation in a Banach space. Our analysis has the following advantages over the work in [11]: weaker sufficient convergence conditions and larger convergence domain. Note that in the case of Newton’s method, these advantages are obtained under the same computational cost, since in practice the computation of the Lipschitz constant L requires the computation of L_0 . Numerical examples further validating the results are also provided in this study.

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Received on 26.1.2010;
revised version on 23.6.2010

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