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Youcef Berkoun and Hocine Fellag (Tizi-Ouzou)

HURWICZ'S ESTIMATOR OF THE AUTOREGRESSIVE MODEL WITH NON-NORMAL INNOVATIONS

Abstract. Using the Bahadur representation of a sample quantile for m-dependent and strong mixing random variables, we establish the asymptotic distribution of the Hurwicz estimator for the coefficient of autoregression in a linear process with innovations belonging to the domain of attraction of an α-stable law $(1 < \alpha < 2)$. The present paper extends Hurwicz's result to the autoregressive model.

1. Introduction. Let $(X_t,\, t \in \mathbb{Z})$, be a stationary linear process defined on a probability space (Ω, A, P) of the form

$$X_t = \sum_{i\ge 0} \rho^i \varepsilon_{t-i}, \qquad t \in \mathbb{Z}, \tag{1.1}$$

where $0 \le |\rho| < 1$ and $(\varepsilon_t)_t$ is a sequence of i.i.d. symmetric random variables in the domain of attraction of an α-stable law, $1 < \alpha < 2$, i.e. $P(|\varepsilon_t| > x) \sim x^{-\alpha}L(x)$ where L is a slowly varying function at infinity in the sense that $L(tx)/L(x) \to 1$ as $x \to \infty$ for all $t > 0$, and with the tails balance

$$\frac{P(\varepsilon_t > x)}{P(|\varepsilon_t| > x)} \to p_1 \quad \text{and} \quad \frac{P(\varepsilon_t < -x)}{P(|\varepsilon_t| > x)} \to q_1 \qquad \text{as } x \to \infty$$

with

$$p_1 + q_1 = 1, \qquad 0 < p_1 < 1.$$

The assumption that the ε_t's are in the domain of attraction of an α-stable law is more general than assuming that they are α-stable distributed.

Time series with infinite variance innovations are very useful in applications in diverse areas such as hydrology, finance, telecommunication and

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others. This comes from the fact that the observed data cannot be accurately modeled by a probability distribution with finite variance, but are better described by heavy tailed distributions.

In what follows, we are interested in testing $H_0 : \rho = 0$ against the alternative $H_1 : \rho \neq 0$ with signifiance level γ.

In the literature, there are many different tests of the hypothesis of independence. For background on these tests see Berkoun et al. (2003). Here, we restrict ourselves to the statistical test based on the Hurwicz (1950) estimator

$$T_\rho = \mathrm{Med}\left(\frac{X_2}{X_1}, \frac{X_3}{X_2}, \ldots, \frac{X_{n+1}}{X_n}\right) \tag{1.2}$$

where $\mathrm{Med}(\cdot,\cdot)$ is the sample median. Zieliński (1999) proved that the Hurwicz estimator is median-unbiased for every distribution of innovations symmetric around zero. Notice that

$$E(|\varepsilon_t|^\vartheta < \infty) \quad \forall \vartheta < \alpha \quad \text{and} \quad E(\varepsilon_t) = 0, \tag{1.3}$$

$$\int_{\mathbb{R}} |f_\varepsilon(x+y) - f_\varepsilon(x)|\, dx \le c|y| \qquad \forall y, \tag{1.4}$$

where c is a positive constant and f_ε is the probability density function of ε_t. One can remark that, according to (1.3), each quotient X_t/X_{t-1} is an unbiased estimator of ρ. We reject the null hypothesis if $T_\rho^2 \ge c$, where c is the constant given by $P(T_\rho \ge \sqrt{c}) = \gamma/2$.

The restriction to Hurwicz's estimator is motivated by the fact that, among several other estimators of the autoregression coefficient, the test in question has a normal asymptotic distribution.

For a linear process with infinite variance innovations, most estimators (as the least square estimator) for the autoregression coefficient suffer from complex asymptotic distribution. This makes statistical inference for such models difficult.

To perform the test, we need to derive the asymptotic distribution of the test under study. The key tool is the Bahadur representation of sample quantiles obtained in a certain dependence framework.

The paper is organized as follows. Section 2 contains some definitions and notations. In Section 3, we present our main results. Section 4 should be viewed as an appendix, and contains most of the results used in this paper.

2. Definitions and notation. Set $Z_i = X_{i+1}/X_i$, $i = 1, \ldots, n$, and let F be their common distribution function, with density function f. Let $Z_{(1)}, \ldots, Z_{(n)}$ be the corresponding ordered random variables. Let z_p be the

pth quantile of F defined by

$$z_p = F^{-1}(p) = \inf\{z : F(z) \geq p\}.$$

We define the pth sample quantile by $F_n^{-1}(p) = \inf\{z : F_n(z) \geq p\}$. We have

$$F_n^{-1}(p) = \widehat{Z}_{(np)} = \begin{cases} Z_{(np)} & \text{if } np \text{ is an integer,} \\ Z_{([np]+1)} & \text{if not,} \end{cases}$$

where $[np]$ denotes the integer part of np and F_n is the empirical distribution of the random variables $Z_1, \ldots, Z_n$. In particular, $T_\rho = F_n^{-1}(1/2)$, which we denote simply by $\widehat{Z}_{(np)}$.

Definition 2.1. The sequence $(Z_n)_n$ of random variables is said to be *m-dependent* if $(Z_i, Z_{i+1}, \ldots, Z_r)$ and $(Z_s, Z_{s+1}, \ldots)$ are independent whenever $s - r > m$.

For example, let $(Z_t)_t$ be a sequence of i.i.d. random variables. Define $Y_n = f(Z_n, Z_{n+1}, \ldots, Z_{n+m})$ for a real Borel measurable function on $\mathbb{R}^{m+1}$; then $(Y_n)_n$ are stationary and m-dependent.

Definition 2.2. Let $(Z_t)_t$ be a strictly stationary process defined on the probability space $(\Omega, \mathcal{A}, P)$. Let $\mathcal{F}_a^{a+b}$ denote the σ-algebra generated by $Z_a, Z_{a+1}, \ldots, Z_{a+b}$ where $-\infty \leq a \leq b \leq +\infty$. For $n \geq 1$, we define

$$\alpha(n) = \sup_{m\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^{m},\, B\in\mathcal{F}_{-\infty}^{n+m}} |P(A\cap B) - P(A)P(B)|.$$

The process $(Z_t)_t$ is said to be *strong mixing* or *α-mixing* if $\alpha(n) \to 0$ as $n \to \infty$.

For example, a finite-state, irreducible, aperiodic Markov chain is α-mixing. The strong mixing condition introduced by Rosenblatt (1956) is a tool of great interest to derive the asymptotic limit for dependent random variables. The dependence described by α-mixing is the weakest, as it is implied by other types of mixing. There are many results on the central limit theorem in the context of random variables satisfying strong mixing conditions; see, e.g., Doukhan (1994).

Remark 2.1. Let $Y_t = g(X_t, X_{t-1}, \ldots, X_{t-k})$ be a measurable function for finite k. If $(X_t)_t$ is α-mixing, then $(Y_t)_t$ is also α-mixing. Moreover, if $(X_t)_t$ is $O(n^{-k})$ then $(Y_t)_t$ is also $O(n^{-k})$. Note that stationarity, m-dependence and mixing properties are preserved by any Borel measurable transformation.

3. Main results

3.1. Asymptotic distribution of Hurwicz's estimator under the null hypothesis. Under the null hypothesis, set $Z_i = \varepsilon_{i+1}/\varepsilon_i$, $i = 1, \ldots, n$, and $p = P(Z_i \leq z_p) = F(z_p)$. Let $Y_i = p - I_{(Z_i\leq z_p)}$, $S_n = \sum_{i=1}^{n} Y_i$ and

$\sigma_n^2 = V(n^{-1/2}S_n)$. Let f be the distribution function of the Z_i's. We will assume that

$$(\mathrm{A})\quad \begin{cases} f = dF \text{ is bounded in some neighborhood } V_0 \text{ of } z_p \text{ with } 0 < z_p < \infty \\ \qquad \text{and } 0 < f(z_p) < \infty, \\ f' \text{ is bounded in } V_0. \end{cases}$$

THEOREM 3.1. *If the assumption* (A) *is satisfied, then*

$$\frac{n^{1/2}(\widehat{Z}_{(np)} - z_p)f(z_p)}{\sigma_n} \xrightarrow{L} N(0,1)$$

where

$$\sigma_n^2 = E(Y_1^2) + 2(1 - 1/n)E(Y_1Y_2).$$

Proof. The random variables $Z_i = \varepsilon_{i+1}/\varepsilon_i$, $i = 1, \ldots, n$, are 1-dependent. Applying Lemma 4.1 (see Appendix), we obtain

$$(\widehat{Z}_{np} - z_p)f(z_p) = (p - F_n(z)) + O(n^{-3/4}\log n)$$

with $n^{1/2}O(n^{-3/4}\log n) \xrightarrow{P} 0$. We can write

$$\begin{aligned}(\widehat{Z}_{(np)} - z_p)f(z_p) &= \frac{1}{n}\sum_{i=1}^{n}(p - I_{(Z_i \le z_p)}) + O(n^{-3/4}\log n) \\ &= \frac{1}{n}\sum_{i=1}^{n} Y_i + O(n^{-3/4}\log n).\end{aligned}$$

The desired result follows if we prove that $n^{-1/2}S_n$ converges to a normal distribution. It is clear that the random variables $Y_1, Y_2, \ldots$ are 1-dependent with $E(Y_i) = 0$ and $E(|Y_i|^3) < R$ for all i. Therefore, by stationarity of the process, assuming $m = 1$ and using Lemma 3.2, the proof of Theorem 3.1 is completed. ∎

3.2. Asymptotic distribution of Hurwicz's estimator under the alternative hypothesis. Under the alternative hypothesis, the random variables Z_i are not m-dependent. Due to this difference, the asymptotic limit of the estimator T_ρ can be established under the assumption that the linear process (1.1) has the strong mixing property. Mixing is rather hard to verify and some regularity conditions are required. Necessary and sufficient conditions for a linear process to be strong mixing is developed in Chanda (1974), Gorodetskiĭ (1977), Withers (1981), and Andrews (1984).

As in Theorem 3.1, in order to obtain the limiting distribution of the Hurwicz estimator, the key ingredient is the Bahadur representation of sample quantiles for strong mixing random variables. Extension of Bahadur's result (1966) has been obtained by Sen (1972) for ϕ-mixing random variables.

Babu and Singh (1978), Yoshihara (1995), and Sun (2006) established the Bahadur representation for α-mixing sequences. For weakly dependent random variables, some recent contributions can be found in Wu (2005) and Kulik (2007) and for negatively associated sequence in Ling (2008).

Set $Z_i = X_{i+1}/X_i,\ i = 1, \dots, n$. Let F and f be the cumulative distribution function and the probability density function of the Z_i's respectively. Denote by z_p the pth quantile of F and let $Y_i = p - I_{(Z_i \le z_p)}$. As in the previous section, let $S_n = \sum_{i=1}^n Y_i$ and $\sigma_n^2 = V(n^{-1/2} S_n)$.

THEOREM 3.2. *Assume that condition* (A) *holds. Then*

$$(\widehat{Z}_{(np)} - z_p) f(z_p) = (p - F_n(z_p)) + O_{\text{a.s.}}(n^{-3/(4+\delta)} \log n).$$

In addition, if $\inf_n \sigma_n^2 > 0$, *then*

$$\frac{n^{1/2}(\widehat{Z}_{(np)} - z_p) f(z_p)}{\sigma_n} \xrightarrow{L} N(0,1)$$

where $\sigma_n^2 = E(Y_1^2) + 2\sum_{k=1}^{n-1}(1 - k/n) E(Y_1 Y_{1+k})$.

Proof. In order to prove the theorem, we first need to establish the strong mixing property of the linear process (1.1). To this end, we shall verify that the conditions of Gorodetskiĭ's theorem hold (see Appendix, Theorem 4.3). From the fact that the innovation ε_t is in the domain of attraction of an α-stable law, conditions (1) and (2) of Gorodetskiĭ's theorem are satisfied. It is easy to see that condition (3) is also satisfied. Let us verify condition (4). Observe that $c_j = \rho^j$, hence

$$\sum_{i \ge k} \Big(\sum_{j \ge i} |\rho^j|^\delta \Big)^{\delta/(1+\delta)} = C |\rho|^{k\delta/(1+\delta)}.$$

So the required assumptions for a linear process to be strong mixing are satisfied. Hence, the linear process (1.1) is strong mixing and the condition $\alpha(n) \le CO(n^{-\beta})$ for large n is fulfilled and $\sum_{j \ge 0} \alpha(j) < \infty$. In view of Remark 2.1, the process $(Z_t)_t$, where $Z_t = X_{t+1}/X_t$, is also α-mixing. Using the Bahadur representation of sample quantiles for strong mixing random variables (see Appendix Theorem 4.4), we have

$$\begin{aligned}(\widehat{Z}_{(np)} - z_p) f(z_p) &= \frac{1}{n} \sum_{i=1}^n (p - I_{(Z_i \le z_p)}) + O(n^{-3/(4+\delta)} \log n) \qquad \text{a.s.} \\ &= \frac{1}{n} \sum_{i=1}^n Y_i + O(n^{-3/(4+\delta)} \log n).\end{aligned}$$

Moreover, from the well-known Billingsley inequality (see Appendix, Lemma 4.3), we have

$$\sum_{j \ge 1} \operatorname{cov}(Y_1, Y_j) < \infty.$$

Note that

$$V(n^{-1/2}S_n) = n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(Y_i, Y_j) = n^{-1}\sum_{k=-n}^{n}(n-|k|)\mathrm{cov}(Y_1, Y_{1+k})$$

and we have

$$\lim_{n\to\infty}\sigma_n^2 = \sigma^2 = E(Y_1^2) + 2\sum_{j\geq 1}E(Y_1Y_{1+j}).$$

To complete the proof, it suffices to use the central limit theorem due to Ibragimov and Linnik (1971) (see Appendix, Theorem 4.3) applied to uniformly bounded strong mixing random variables.

Finally, applying this theorem to $S_n = \sum_{i=1}^{n} Y_i$, the desired result follows immediately. ■

4. Appendix

Lemma 4.1 (Sen, 1968). *Let $(Z_t)_t$ be a sequence of m-dependent random variables. If condition* (A) *holds, then*

$$(\widehat{Z}_{(np)} - z_p)f(z_p) = p - F_n(z_p) + O(n^{-3/4}\log n) \quad \textit{with probability one.}$$

Lemma 4.2 (Sen, 1968). *Let $(Y_t)_t$ be a sequence of m-dependent random variables with $E(Y_i) = 0$ and $E(|Y_i|^3) < R < \infty$ for $i = 1, 2, \ldots$. Let $S_n = \sum_{i=1}^{n} Y_i$ and*

$$\sigma_n^2 = \frac{1}{n}\Big(\sum_{i=1}^{n}E(Y_i^2) + 2\sum_{h=1}^{m}\sum_{i=1}^{n-h}E(Y_iY_{i+h})\Big).$$

If $\inf_n \sigma_n^2 > 0$, *then*

$$\frac{n^{-1/2}S_n}{\sigma_n} \xrightarrow{L} N(0,1).$$

Note that in the above lemma, the sequence $(Y_t)_t$ is not necessarily stationary.

Theorem 4.1 (Gorodetskiĭ, 1977). *Let $(X_t)_t$ be a linear process such that $X_t = \sum_{j\geq 0} c_j\varepsilon_{t-j}$, where ε_t are i.i.d. centered random variables. Suppose that*

(1) *There exists $\delta > 0$ such that $E(|\varepsilon_t|^\delta) < \infty$.*
(2) *The density f_ε of ε_t satisfies the condition* (1.4).
(3) $\sum_{j\geq 0} c_j z^j \neq 0$ *for some* $|z| \leq 1$.
(4) $\sum_{i\geq k}(\sum_{j=i}^{\infty}|c_j|^\delta)^{1/(1+\delta)} < \infty$.

Then $(X_t)_t$ is strong mixing with coefficient not exceeding

$$M\sum_{i\geq k}\Big(\sum_{j=i}^{\infty}|c_j|^\delta\Big)^{1/(1+\delta)}$$

where M is constant.

THEOREM 4.2 (Sun, 2006). *Let $(X_t)_t$ be a strict stationary α-mixing process with common distribution F. Assume that:*

(1) *There exists $\varepsilon > 0$ such that $\alpha(n) \le CO(n^{-\beta})$ for some $\beta > 0$.*
(2) *F satisfies condition* (A).

Then, for any $\delta \in \left[\frac{11}{4(\beta+1}, \frac{1}{4}\right]$ and $\beta > 10$,

$$(\widehat{Z}_{(np)} - z_p) f(z_p) = (p - F_n(z_p)) + O(n^{-3/(4+\delta)} \log n) \qquad a.s.$$

with $F_n(z_p) = n^{-1} \sum I_{(X_i \le z_p)}$.

LEMMA 4.3 (Billingsley inequality). *Let $(X_t)_t$ be a stationary strong mixing process. If X is measurable with respect to $\mathcal{F}_{-\infty}^{t}$ and Y with respect to $\mathcal{F}_{t+k}^{\infty}$ and if $|X| \le C_1$, $|Y| \le C_2$, then*

$$|E(XY) - E(X)E(Y)| \le 4C_1C_2\alpha(k).$$

The above lemma gives a bound of the covariance for strong mixing sequences.

THEOREM 4.3 (Ibragimov, 1969). *Assume that $(X_t)_t$ is a strict stationary α-mixing process such that $E(X_t) = 0$, $P(|X_t| < c) = 1$ for some $c > 0$ and $\sum_{j \ge 1} \alpha(j) < \infty$ and $\sigma^2 = E({X_1}^2) + 2\sum_{j\ge 1} E(X_1 X_{1+j}) > 0$. Then*

$$S_n/\sqrt{n} \xrightarrow{L} N(0, \sigma^2).$$

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Youcef Berkoun, Hocine Fellag
Laboratoire de Mathématiques Pures et Appliquées
Université Mouloud Mammeri
Tizi-Ouzou, 15000 Algeria
E-mail: youberk@yahoo.com
hfellag@mail.ummto.dz

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