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OPTIMAL RISK SHARING AS A COOPERATIVE GAME

Abstract. The problem of choosing an optimal insurance policy for an individual has recently been better understood, particularly due to the papers by Gajek and Zagrodny (2000, 2004a,b,c). In this paper we study its multi-agent version: we assume that insureds cooperate with one another to maximize their utility function. They create coalitions by bringing their risks to the pool and purchasing a common insurance contract. The resulting outcome is divided according to a certain rule called strategy. We address the fundamental questions of profitability of cooperation and existence of strategies not rejected by any of the coalitions. These issues are closely related to the notion of Pareto optimality and the core of a game. We give a characterization of the former and prove the nonemptiness of the latter. Moreover, assuming that the pricing rule used by the insurance company is linear, we formulate necessary and sufficient conditions for the profitability of cooperation.

1. Introduction. A classical problem in insurance theory is to find an optimal insurance policy for an individual (or entity) exposed to risk. To be more precise, let X, a random variable, denote a possible loss the individual can suffer during the period of time under consideration. Let us further assume that he is willing to spend on an insurance policy no more than an amount p. Such a policy is described by a function I, where $I(X)$ denotes the portion of X that will be covered by the insurer. For such a contract, the insurance company charges a premium $\pi(I(X))$, where π is a functional which is called the pricing rule. Assuming that the individual acts according to a certain gain functional ψ, the goal is to find I which solves the following problem:

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$$\psi(I) \to \max!, \tag{1.1}$$

$$\pi(I(X)) \le p, \tag{1.2}$$

$$0 \le I(X) \le X. \tag{1.3}$$

The solution of (1.1)–(1.3) obviously depends heavily on the choice of the optimization criterion ψ and the pricing rule π. One of the earliest works on the subject is Arrow (1963). He considered a similar problem with π being the expected value principle and $\psi(I) = \mathbb{E}U(-X + I(X) - \pi(I(X)))$ for some utility function U. It should be pointed out that the crucial feature of the problem (1.1)–(1.3) is the inequality (1.2). Replacing it with equality not only simplifies the problem (the functional π does not appear in the value function), but, what is more important, leads to irrational decisions. Surprisingly, many authors prefer simplicity to economic soundness. A detailed discussion of this phenomenon can be found in Gajek and Zagrodny (2004b). Another natural criterion is the probability of solvency, i.e. $\psi(I) = \mathbb{E}\mathbb{1}(V - X + I(X) - \pi(I(X)) > 0)$, where V stands for the buyer's initial capital. A very elegant solution was given in Gajek and Zagrodny (2004c). The next appealing approach is to consider the deviations of the outcome from its mean, that is, $\psi(I) = \mathbb{E}[-\rho(X - I(X) - \mathbb{E}(X - I(X)))]$, ρ being a certain loss function. This case was treated in Gajek and Zagrodny (2004a); see also Gajek and Zagrodny (2000). In the remainder of this paper we will always assume that

$$\psi(I) = \mathbb{E}U(-X + I(X) - \pi(I(X))), \tag{1.4}$$

where U is a utility function.

The situation becomes more complex when we consider a multi-agent framework, that is, we allow individuals who buy insurance to cooperate with one another. The first economic models that tried to capture these interactions appeared in the papers by Borch (1960, 1962). In this classical setup the agents were allowed to exchange risks between themselves in hope of increasing their welfare, measured by the utility functions. The allocation of risks was considered optimal if it was Pareto optimal, i.e. there was no other allocation which would improve the outcome of all insureds and insurers. However, this approach had some disadvantages, most notably it implied side payments even when no transfer of risk took place or ruled out popular contracts, such as stop loss, as a solution (see Aase (2002) for an excellent review on the subject). Some well known extensions of this problem can be found in Gerber (1978) and Bühlmann and Jewell (1979). Later Baton and Lemaire (1981) observed that "Pareto optimality [...] do not preclude the possibility that a coalition of companies might be better off by seceding from the whole group". As a consequence they introduced the concept of collective rationality, which corresponds to the notion of the core of a game.

Sujis et al. (1998) took a somewhat different approach: they divided agents into two groups: individuals who could not exchange risks between one another, and insurers who could. Under many assumptions (most notably exponential form of utility functions, distribution of loss being a combination of exponential random variables and admissible exchange of risk being of a proportional form) they established theorems on nonemptiness of the core and characterization of Pareto optimal strategies.

Here we assume that individuals cannot exchange risks between themselves, but may create groups (called coalitions) and insure jointly at some insurance company. After a given period of time they divide their final outcome among themselves. One natural example of such a situation is group insurance.

We want our model to capture three basic features. First, the individuals' decisions should depend on the preferences of the insurer only through the pricing rule π. This is natural, since the insured is usually only concerned with the price he has to pay. Moreover, we will not assume that the insurance company acts according to some utility function, thus π need not to be a zero-utility based principle. Secondly, following the observation of Baton and Lemaire (1981), we want to go beyond Pareto optimality and consider the core of a game. Finally, we want our model to be consistent with the problem (1.1)–(1.4), as opposed to the previously mentioned approaches.

It is noteworthy that in the classical Borch setup the pricing rule emerges as a part of the solution (in the form of side payments). Vaguely speaking, this is the price agreed by the group participants and at which the transactions are carried out. This is not the case in our model, since π is not a matter of negotiations: it is given upfront (imposed by the insurance company) and not affected by the insureds' actions. Furthermore, the same pricing rule applies to all agents whether they cooperate or not.

Our model serves to answer the following two questions: is cooperation profitable (when compared to individual insurance) and does it pay off to create one grand coalition? To deal with the former question we start by considering the problem (1.1)–(1.4) for each $k \in \{1, \ldots, n\}$ separately. Denote by $L^*_{\{k\}}$ the optimal outcome of the kth individual (assuming the optimal solution exists). Now saying that cooperation is not profitable means that the strategy $(L^*_{\{1\}}, \ldots, L^*_{\{n\}})$ is Pareto optimal (assuming that it satisfies the definition of strategy from Section 2.2). As a consequence, it is important to get an insight into the structure of Pareto optimal strategies. Theorem 2.3.5 provides a full characterization of such strategies, while Theorem 2.3.7 gives necessary and sufficient conditions for Pareto optimality of $(L^*_{\{1\}}, \ldots, L^*_{\{n\}})$. The latter question is related to the notion of the core of a game. Vaguely speaking, this is the set of strategies which are not rejected by any subcoalition. The problem with the core is that in a large class of games it can be empty. However, under certain assumptions, we show that this is not the case in our situation (see Theorem 2.4.2).

The impetus for our considerations was given by the paper of Xia (2004), who considered optimal wealth distribution among agents who cooperate and invest in an incomplete financial market. The financial setup in that paper, however, differs from ours. First of all, the set of coalitions' outcomes is different. Secondly, in the cooperative investment problem there is one source of uncertainty, namely the asset price process, while in cooperative insurance it is necessary to consider as many sources of uncertainty as there are individuals (their possible future losses). Thirdly, we assume that the pricing functional is specified exogenously, which is typical of insurance business. Finally, Xia (2004) showed that there is a qualitative difference between complete and incomplete markets. Due to the lack of similar distinction in our model, there is no such phenomenon. Nevertheless, we use some techniques used by Xia (2004).

This paper is organized as follows: in Section 2 we introduce an insurance cooperative game, characterize Pareto optimal strategies and show that the core is nonempty. In Section 3 we present some examples. In Section 4 we elaborate on some potential practical problems (when the agents have to make some agreement specifying how the overall premium and contingent claim should be shared). We conclude the paper in Section 5. Section 6 provides all necessary proofs. Finally, in the Appendix we gather some auxiliary results and definitions that are used in the text.

2. Main results

2.1. Assumptions. Consider the problem (1.1)–(1.4) for each k separately. Thus we have X_k, U_k and p_k which correspond to the future possible loss, utility function and the budget constraint of the kth individual. Assume that $X_1, \ldots, X_n$ are nonnegative random variables, $\mathbb{P}(X_k > 0) > 0$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All identities concerning random variables will be assumed to hold almost surely. Let the pricing rule π be a functional (possibly equal to $+\infty$), defined on the space of nonnegative random variables, that satisfies the following conditions:

$$\pi(0) = 0, \tag{2.1}$$

$$\pi(tX + (1-t)Y) \le t\pi(X) + (1-t)\pi(Y) \text{ for all } X, Y \ge 0,\ t \in (0,1), \tag{2.2}$$

$$\begin{aligned} &\pi(Y_m) \to \pi(Y) \quad \text{if } Y_m \to Y \text{ a.s.,} \\ &0 \le Y_m \le Z \quad \text{and} \quad \pi(Z) < \infty, \quad m \to \infty. \end{aligned} \tag{2.3}$$

These conditions can be interpreted as follows. The first two say that the cost of not transferring the risk is zero and that diversification of risk should not increase the price. The last one essentially says that π is continuous, i.e. if two risks are similar, then their prices are similar. All three conditions are satisfied by basic pricing rules, such as expected value principle, standard deviation principle or variation principle (see e.g. Bühlmann (1970) or Deprez and Gerber (1985)). We assume that every loss X_k has a finite positive price, i.e. $0 < \pi(X_k) < +\infty$. Moreover, we suppose that it does not pay off for the agents to insure their whole risk, meaning that $0 \le p_k < \pi(X_k)$.

Suppose that the preferences of the kth agent are described by a utility function U_k. We assume that it is a finite, strictly increasing, strictly concave, continuously differentiable function defined on $(-\infty, 0]$ and satisfying

$$\lim_{x\to-\infty} U_k'(x) = +\infty, \qquad \lim_{x\to 0-} U_k'(x) = 0. \tag{2.4}$$

Remark 2.1.1.

1. The second condition in (2.4) means that the graph of U_k is almost parallel to the x-axis near zero. The first condition in (2.4) is not very restrictive in the sense that it does not impose any particular rate of convergence to $+\infty$ on U_k'.
2. Examples of functions satisfying all the above requirements are $U_\beta(x) = -e^{|x|^\beta}$, $\beta \ge 2$, and $U_\alpha(x) = -|x|^\alpha$, $\alpha \ge 2$. However, one has to assume the existence of higher order moments to use any of them. This issue can be solved by cutting U into two parts: the one responsible for the behaviour near zero and the other responsible for the behaviour at infinity.
3. The classical utility function $U(x) = -e^{-x}$ violates (2.4) so it is excluded from our considerations.
4. Suppose that U satisfies all the requirements. Define a function $V : \mathbb{R}_+ \to \mathbb{R}_+$ by the formula $V(x) = -U^{-1}(U(0) - x)$. Then V is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions: $V'(0) = \infty$ and $V'(+\infty) = 0$. Such functions V are often used in finance to measure the utility of the investor's terminal wealth.
5. We note that smoothness of the functions U_k is not required for all results in the paper to hold. However, for the sake of clarity we use the unified assumptions.

To make all the expectations finite we assume $\mathbb{E}U_k(-\sum_{k=1}^n (X_k - p_k)) > -\infty$ for all $k = 1, \dots, n$. Moreover, the problem (1.1)–(1.4) has a solution, say I_k^*, for every $k = 1, \dots, n$ (see Lemma 7.6) and we denote the corresponding optimal losses by $L_{\{1\}}^*, \dots, L_{\{n\}}^*$, where

$$L_{\{k\}}^* = -X_k + I_k^*(X_k) - \pi(I_k^*(X_k)).$$

2.2. Cooperative insurance game. Denote $[n] = \{1, \dots, n\}$ and suppose that a coalition can be formed by any group of agents and is described by a subset $\tau \subset [n]$. Its participants jointly insure their losses, which means they can choose an insurance policy that depends on the whole random vector $(X_k, k \in \tau)$. Since each of them has a capital restriction they can spend no more than the amount $\sum_{k\in\tau} p_k$. The outcome, dependent on the choice of insurance policy, has to be divided amongst the members of τ. The set of all dividing rules is the set of strategies available for the coalition τ.

To be more precise let us introduce some notation: $\mathcal{F}(\tau) = \sigma(X_k, k \in \tau)$, $L_0(\tau)$ denotes the set of all $\mathcal{F}(\tau)$-measurable random variables and $\mathbb{R}^\tau = \{(x_k, k \in \tau) \mid x_k \in \mathbb{R} \text{ for all } k \in \tau\}$. Now the admissible set of insurance policies for a coalition τ can be defined as

$$\mathcal{J}(\tau) = \Big\{ I : \mathbb{R}^\tau \to \mathbb{R} \;\Big|\; 0 \le I(x_k, k \in \tau) \le \sum_{k\in\tau} x_k,\ \pi(I(X_k, k \in \tau)) \le \sum_{k\in\tau} p_k \Big\}.$$

The outcome of the coalition τ depends obviously on the choice of insurance policy. Therefore, the set of all possible terminal losses of the whole coalition τ equals

$$\mathcal{L}(\tau) = \Big\{ L \in L_0(\tau) \;\Big|\; L = -\sum_{k\in\tau} X_k + I(X_k, k \in \tau) - \pi(I(X_k, k \in \tau)),\quad I \in \mathcal{J}(\tau) \Big\}.$$

Suppose that the outcome of the coalition τ was $L \in \mathcal{L}(\tau)$. The rule that specifies how to divide L amongst all members of τ is called a *strategy* and is described by a random vector $(\xi_k, k \in \tau)$. Here ξ_k stands for the part of L that goes to the kth agent. Because ξ_k should depend only on the values of $(X_k, k \in \tau)$, we assume that ξ_k is an $\mathcal{F}(\tau)$-measurable random variable. Moreover, it is reasonable to assume that ξ_k cannot exceed zero and that the whole loss L has to be distributed, i.e. $\sum_{k\in\tau} \xi_k = L$. Thus the set of all strategies may be defined by

$$\mathcal{A}(\tau) = \Big\{ (\xi_k, k \in \tau) \;\Big|\; \forall_{k\in\tau}\ \xi_k \in L_0(\tau),\ \xi_k \le 0,\ \sum_{k\in\tau} \xi_k = L,\ L \in \mathcal{L}(\tau) \Big\}.$$

Although the sets $\mathcal{A}(\tau)$ have a clear interpretation, they are not the best objects to work with. Firstly, notice that, in general, they may not be convex (as π need not be linear). Secondly, the vectors $(L_{\{1\}}, \dots, L_{\{n\}})$, where $L_{\{k\}} \in \mathcal{A}(\{k\})$, need not be elements of $\mathcal{A}([n])$. This means that the current definition of strategy is not broad enough to cover the case of agents acting on their own.

Therefore, we define a richer family of admissible strategies $\{\mathcal{A}_C(\tau)\}_{\tau\subset[n]}$ by

$$\mathcal{A}_C(\tau) = \Big\{(\xi_k, k \in \tau) : \Omega \to \mathbb{R}^\tau \;\Big|\; \forall_{k\in\tau}\ \xi_k \in L_0(\tau),\ \xi_k \le 0, \\ -\sum_{k\in\tau}(X_k + p_k) \le \sum_{k\in\tau}\xi_k \le L,\ L \in \mathcal{L}(\tau)\Big\},$$

which, at least for the case of subaddivite π, does not exhibit the aforementioned flaws. It is not clear, however, that the problem remains unchanged. Luckily this is the case: see Lemma 2.3.4 and Corollary 2.4.4.

Preferences of the coalition τ concerning a strategy $(\xi_k, k \in \tau) \in \mathcal{A}_C(\tau)$ depend solely on the utility functions $(U_k, k \in \tau)$. Therefore we give the following definition:

Definition 2.2.1. The tuple $(\{\mathcal{A}_C(\tau)\}_{\tau\subset N}, (U_k, k \in [n]))$ will be called a *cooperative insurance game.*

2.3. Pareto optimal strategies

Definition 2.3.1. We say that the strategy $(\xi_1, \ldots, \xi_n) \in \mathcal{A}_C([n])$ *dominates* the strategy $(\zeta_1, \ldots, \zeta_n) \in \mathcal{A}_C([n])$ if

$$\mathbb{E}U_k(\zeta_k) \le \mathbb{E}U_k(\xi_k), \qquad k = 1, \ldots, n,$$

with at least one strict inequality.

Definition 2.3.2. The strategy $(\xi_1^*, \ldots, \xi_n^*) \in \mathcal{A}_C([n])$ is called *Pareto optimal* if no other strategy $(\xi_1, \ldots, \xi_n) \in \mathcal{A}_C([n])$ dominates $(\xi_1^*, \ldots, \xi_n^*)$.

This definition says that the Pareto optimal strategy cannot be improved in the sense that no other strategy for all individuals can grant at least one of them a better result and not decrease the results of the others. The following proposition shows the intuitively obvious fact that the Pareto optimal strategies are the only strategies worth considering.

Proposition 2.3.3. *Every strategy $(\xi_1, \ldots, \xi_n) \in \mathcal{A}_C([n])$ that is not Pareto optimal is dominated by a certain Pareto optimal strategy $(\xi_1^*, \ldots, \xi_n^*) \in \mathcal{A}_C([n])$.*

Since we are actually interested in the strategies from $\mathcal{A}([n])$, we would like to know whether this set is large enough to contain all Pareto optimal strategies. The following lemma says that this is indeed the case and thus justifies the introduction of $\mathcal{A}_C([n])$.

Lemma 2.3.4. *The strategy $(\xi_1^*, \ldots, \xi_n^*) \in \mathcal{A}_C([n])$ is Pareto optimal if and only if $(\xi_1^*, \ldots, \xi_n^*) \in \mathcal{A}([n])$ and no other strategy $(\xi_1, \ldots, \xi_n) \in \mathcal{A}([n])$ dominates $(\xi_1^*, \ldots, \xi_n^*)$.*

We now use heuristic arguments to get an insight into the structure of Pareto optimal strategies. Loosely speaking, a strategy $(\xi_1^*, \ldots, \xi_n^*)$ is Pareto optimal if and only if the corresponding vector $(\mathbb{E}U_1(\xi_1^*),\dots,\mathbb{E}U_n(\xi_n^*))$ belongs to the "north-eastern part" of the boundary of the set

$$H([n]) = \{(\mathbb{E}U_k(\xi_k),\dots,\mathbb{E}U_n(\xi_n)) \mid (\xi_1,\dots,\xi_n)\in\mathcal{A}_C([n])\}.$$

Suppose for a moment that $H([n])$ is convex. Then there exists a correspondence between supporting hyperplanes and boundary points of $H([n])$. Consequently, we may choose a supporting hyperplane, say S, that corresponds to some Pareto optimal strategy. This strategy can be "recovered" from S by means of convex optimization.

In the light of these considerations we introduce a function U_λ defined on $(-\infty,0]$ by the formula

$$U_\lambda(x) = \sup\Big\{\sum_{k:\,\lambda_k>0}\lambda_k U_k(y_k)\ \Big|\ \sum_{k:\,\lambda_k>0} y_k = x,\ y_k\le 0\Big\},$$

where

$$\lambda = (\lambda_1,\dots,\lambda_n),\quad \lambda_k\ge 0 \quad\text{and}\quad \sum_{k=1}^n \lambda_k = 1. \tag{2.5}$$

Compare this with Lemma 6.3.2.

Theorem 2.3.5 below provides a characterization of Pareto optimal strategies and gives a nice interpretation of U_λ as a common utility function. It is also an analogue of the famous Borch theorem (see Borch (1960, 1962)). Similar results were established by many authors (see e.g. Wilson (1968) or Laurent (1972)).

THEOREM 2.3.5. *Let $(\xi_1^*,\dots,\xi_n^*)\in\mathcal{A}_C([n])$ and $\Lambda = \{k\in[n] \mid \mathbb{P}(\xi_k^*<0) > 0\}$. Then the strategy $(\xi_1^*,\dots,\xi_n^*)$ is Pareto optimal if and only if there exists λ satisfying* (2.5), *$\lambda_k>0$ for $k\in\Lambda$ and $\sum_{k\in\Lambda}\lambda_k = 1$ such that*

$$\lambda_k U_k'(\xi_k^*) = U_\lambda'(L^*) \quad \textit{for all } k\in\Lambda,$$

where L^* is the unique solution to the optimization problem

$$\sup_{L\in\mathcal{L}([n])} \mathbb{E}U_\lambda(L). \tag{2.6}$$

REMARK 2.3.6.

1. There exists a unique solution to (2.6) for any λ satisfying (2.5) (see the proof of Theorem 2.3.5).
2. The above theorem parametrizes the family of Pareto optimal strategies. In this sense it is similar to the classical theorem of Borch (1962). However, given λ, one has to solve the optimization problem (2.6). Consequently, the Pareto optimal strategy is at least as difficult to find as the solution to (2.6).
3. Since the solution of (2.6) depends on the pricing rule π so does the Pareto optimal strategy.

Recall that the strategy $(L^*_{\{1\}}, \ldots, L^*_{\{n\}})$ is of special interest, as it represents the best possible performance of the agents when they act on their own. As already mentioned, the cooperation is not profitable when this strategy is Pareto optimal (provided that it belongs to $\mathcal{A}_C([n])$). In a special case, when the insurer uses the expected value principle, we characterize this situation in the following theorem.

THEOREM 2.3.7. *Suppose that* $\pi(X) = (1+\theta)\mathbb{E}X$ *for some* $\theta > 0$. *Assume that for some* $a > 1$, $X_k \in L^a(\mathbb{P})$ *and* $U'_k(-X_k - p_k) \in L^b(\mathbb{P})$ *for all* $k = 1, \ldots, n$, *where* $1/a + 1/b = 1$. *Moreover, let* $U'_\lambda(-\sum_{k=1}^n (X_k + p_k)) \in L^b(\mathbb{P})$ *for all* λ *satisfying* (2.5) *and* $\lambda_k > 0$. *Then* $(L^*_{\{1\}}, \ldots, L^*_{\{n\}})$ *is Pareto optimal if and only if the random variables* $U'_k(L^*_{\{k\}})$ *are identical up to (nonnegative) multiplicative constants.*

The space $L^a(\mathbb{P})$ is the usual space of random variables satisfying $\mathbb{E}|X|^a < \infty$.

REMARK 2.3.8.

1. The necessary conditions in the above theorem are often not satisfied (see Example 3.1). This leads to the conclusion that even in the situation when pooling risks does not decrease their price, the cooperation is still profitable.
2. For example if $U_k(x) = -\alpha_k x^2$ and $\mathbb{E}X^2 < \infty$ then the assumptions of Theorem 2.3.7 are satisfied with $a = b = 2$.

2.4. Core of the cooperative insurance game. From the previous section we know that Pareto optimal strategies are the only ones worth considering. We would like to impose some other condition which a "decent" Pareto optimal strategy should satisfy. In order to do so we introduce the following

DEFINITION 2.4.1. The set of all strategies $(\xi_1, \ldots, \xi_n) \in \mathcal{A}_C([n])$ such that there does not exist any subset $\tau \subset [n]$ and $(\zeta_k, k \in \tau) \in \mathcal{A}_C(\tau)$ satisfying

$$\mathbb{E}[U_k(\xi_k)] < \mathbb{E}[U_k(\zeta_k)] \quad \text{for all } k \in \tau$$

is called the *core* of the cooperative insurance game.

Thus the core is the set of strategies (for all agents) that are not rejected by any coalition. The following theorem is the main result of this section.

THEOREM 2.4.2. *Suppose that* π *is positively homogenous, i.e.* $\pi(tX) = t\pi(X)$ *for* $X \geq 0$, $t \geq 0$. *Then the core of the cooperative insurance game is nonempty.*

Remark 2.4.3.

1. Convexity together with positive homogeneity imply that π is subadditive, i.e. $\pi(X+Y) \leq \pi(X) + \pi(Y)$. Actually the converse is also true: see Deprez and Gerber (1985). One can verify that the expected value and standard deviation principles are positively homogeneous.
2. The convexity of π is not sufficient for Theorem 2.4.2 to hold. A suitable counterexample is provided in Example 3.3.

As an immediate consequence of Theorem 2.4.2 and Proposition 2.3.3 we get

Corollary 2.4.4. *Suppose that π is positively homogenous. Then there exists at least one Pareto optimal strategy that belongs to the core of the cooperative insurance game.*

Remark 2.4.5. Assume that the core is nonempty.

1. If $(L^*_{\{1\}}, \dots, L^*_{\{n\}})$ is Pareto optimal, then it belongs to the core of the cooperative insurance game. Indeed, if $(\xi^*_1, \dots, \xi^*_n)$ is a Pareto optimal strategy that belongs to the core, then by the definition (applied to $\tau = \{k\}$ and $\zeta_k = L^*_{\{k\}}$)
$$\mathbb{E}U_k(\xi^*_k) \geq \mathbb{E}U_k(L^*_{\{k\}})$$
for each $k \in [n]$. But since $(L^*_{\{1\}}, \dots, L^*_{\{n\}})$ is Pareto optimal there must be equalities everywhere, and the result follows.
2. If $(L^*_{\{1\}}, \dots, L^*_{\{n\}})$ is not Pareto optimal, then there exists a strategy $(\xi^*_1, \dots, \xi^*_n)$ that is Pareto optimal, belongs to the core and dominates $(L^*_{\{1\}}, \dots, L^*_{\{n\}})$.

3. Examples

Example 3.1. Suppose that U_k, p_k, π are all the same and satisfy the assumptions of Theorem 2.3.7. Let $Y_1, \dots, Y_n$ be i.i.d. random variables independent of some random variable X_0, each of them with the support equal to the positive halfline. Let $X_k = \alpha X_0/n + (1-\alpha)Y_k$, where $\alpha \in [0,1]$, that is, each individual is exposed to the combination of a common risk and an independent factor. It is known that the solution to (1.1)–(1.4) is of the form $I^*(x) = (x - m^*)^+$ for some m^* (see Arrow (1963)). Because $X_1, \dots, X_n$ are identically distributed it follows that
$$L^*_{\{k\}} = X_k \wedge m^* - \pi,$$
where $\pi = \pi(I^*(X_k))$. Thus $U^*_k(L^*_{\{k\}})$ are identical up to (nonnegative) multiplicative constants if and only if $\alpha = 1$. Consequently, by Theorem 2.3.7 and Remark 2.4.5, the cooperation is not profitable (i.e. the strategy

$(L^*_{\{1\}}, \ldots, L^*_{\{n\}})$ is Pareto optimal) if only if they have the same risk in their portfolio.

EXAMPLE 3.2. Assume that $n = 3$ and X_1, X_2, X_3 are independent and follow the gamma distribution given by the density function

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \mathbb{1}_{(0,\infty)}(x),$$

with different shape parameters a_1, a_2, a_3 and the same rate parameter b. Assume that the preferences of individuals are described by the same type of utility function $U_k(x) = -\alpha_k |x|^\beta$, $i = 1, 2, 3$, for some $\beta \geq 2$ and $\alpha_k > 0$. Let λ satisfy (2.5). Then $U_\lambda(x) = -\alpha|x|^\beta$, where $\alpha = (\sum_{\lambda_k > 0} (\lambda_k \alpha_k)^{1/(1-\beta)})^{1-\beta}$. We assume that the parameters have the following values: $b = 10$, $a_1 = 3$, $a_2 = 5$, $a_3 = 10$, $p_1 = 0.1$, $p_2 = 0.5$, $p_3 = 0.9$, $\alpha_1 = 0.2$, $\alpha_2 = 0.1$, $\alpha_3 = 0.6$ and $\beta = 2$. Finally we put $\pi(X) = 1.5\mathbb{E}X$. It follows that

$$\mathbb{E}U_1(L^*_{\{1\}}) = -0.0339, \quad \mathbb{E}U_2(L^*_{\{2\}}) = -0.0293, \quad \mathbb{E}U_3(L^*_{\{3\}}) = -0.6569.$$

Take $\lambda = (0.35, 0.6, 0.05)$. Then Theorem 2.3.5 implies that for the corresponding Pareto optimal strategy $(\xi^*_1, \xi^*_2, \xi^*_3)$ we have

$$\mathbb{E}U_1(\xi^*_1) = -0.0262, \quad \mathbb{E}U_2(\xi^*_2) = -0.0268, \quad \mathbb{E}U_3(\xi^*_3) = -0.6430.$$

We conclude that the cooperation is profitable and the Pareto optimal strategy corresponding to $\lambda = (0.35, 0.6, 0.05)$ dominates the benchmark strategy $(L^*_{\{1\}}, L^*_{\{2\}}, L^*_{\{3\}})$.

EXAMPLE 3.3. Let $n = 2$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\}) = \mathbb{P}(\{\omega_3\}) = 1/3$. Consequently, we can treat random variables as vectors in $\mathbb{R}^3$. Assume that $x \in (0, 1)$ and define $X_1 = (x, x, 0)$ and $X_2 = (0, x, x)$. Furthermore, let $p_1 = p_2 = p = x^2$, $U_1(y) = U_2(y) = U(y) = -100y^2$ and $\pi(X) = 3\mathbb{E}X^2$. Plainly π is not positively homogeneous (thus not subadditive), but satisfies (2.1)–(2.3). We will show that the core is empty. One can compute $I^*_{\{1\}} = (i, i, 0)$ and $I^*_{\{2\}} = (0, i, i)$, where $i = \sqrt{p/2}$ and show that $(L^*_{\{1\}}, L^*_{\{2\}}) \notin \mathcal{A}([n])$. By Example 3.2 we know that $U_\lambda(y) = -100\lambda(1 - \lambda)y^2$. Assume that $x = 0.1$. Then the best joint insurance they can get is

$$I^* = \left(a, \frac{2x + 2p}{x + 2p} a, a\right), \quad \text{where} \quad a = \sqrt{\frac{2p}{2 + (\frac{2x+2p}{x+2p})^2}}.$$

Then $\xi^*_1 = (1-\lambda)L^*$ and $\xi^*_2 = \lambda L^*$. It can be seen that the point $(\mathbb{E}U(L^*_{\{1\}}), \mathbb{E}U(L^*_{\{2\}}))$ does not belong to the area below the curve

$$\{(\mathbb{E}U((1 - \lambda)L^*), \mathbb{E}U(\lambda L^*)) \mid \lambda \in (0, 1)\},$$

and no Pareto optimal strategy can dominate $(L^*_{\{1\}}, L^*_{\{2\}})$. Thus the core has to be empty.

4. Final remarks. In practice the agents will have to make some kind of agreement specifying how to share the overall premium and claims. Suppose that the agents join forces and purchase an optimal policy, I^*, which is a solution to the problem

$$(4.1)\qquad \begin{cases} \mathbb{E}U_\lambda\Big(-\sum X_i + I(X_1,\dots,X_n) - \pi(I)\Big) \to \max!,\\ \pi(I) \le \sum p_i,\\ 0 \le I(X_1,\dots,X_n) \le \sum X_i \end{cases}$$

(which is just a slightly different way of writing (2.6)). Denote $\pi^* = \pi(I^*)$. Since the premium for the policy I^* has to be paid beforehand, each agent has to pay some part of π^* ex ante. If $\pi^* = \sum_{k=1}^n p_k$ then there is no problem, since everyone has to pay the maximum declared amount. However, it may happen that $\pi^* < \sum_{k=1}^n p_k$ (see Gajek and Zagrodny (2004b)). Suppose for a moment that a rule of dividing π^* has been chosen, resulting in a vector $p^* = (p_1^*,\dots,p_n^*)$ such that $\sum_{k=1}^n p_k^* = \pi^*$ and $p_k^* \in [0,p_k]$. Let $\xi^* = (\xi_1^*,\dots,\xi_n^*)$ be an optimal sharing rule. Then the actual protection R_k that agent k is buying amounts to $R_k = \xi_k^* + X_k + p_k^*$ and possibly depends on all $X_1,\dots,X_n$. Putting things differently, the random variable $-X_k + R_k$ is the risk carried by the kth agent.

Unfortunately, the problem of dividing π^* seems to have no one, clear-cut solution. We suggest a simple and commonly understood method, that is, of least squares. Namely, the kth agent has to pay p_k^*, where $p_1^*,\dots,p_n^*$ solves the problem

$$(4.2)\qquad \begin{cases} \sum_{k=1}^{n} (p_k^* - p_k)^2 \to \min!,\\ \sum_{k=1}^{n} p_k^* = \pi^*,\\ 0 \le p_k^* \le p_k. \end{cases}$$

With this method, the more the agent k has declared to pay, the more he will actually spend. On the other hand, he improves his outcome, while staying within the budget and with a chance of paying less than he would on his own.

One can also improve this method by including the risk aversion as well as the magnitude of risk of particular agents. Define w_k as the deterministic equivalent of the loss X_k for the kth agent, i.e. w_k satisfies $U_k(0) = \mathbb{E}U_k(-X_k + w_k)$. Note that the more the agent is risk averse, or the greater the risk X_k, the larger the w_k. Thus we can introduce weights

$w_k / \sum w_k$ and consider (4.2) with the modified objective function

$$\sum_{k=1}^{n} \frac{w_k}{\sum w_k} (p_k^* - p_k)^2.$$

This method forces the agents with greater risk or greater risk aversion to pay more for their share.

We now address the question of determining the optimal strategies. Suppose that a solution to the problem (2.6), or equivalently (4.1), is given. Then Theorem 2.3.5 gives an explicit form of all Pareto optimal strategies. Additionally, it shows that this is a parameterized family, which makes it possible to choose an element belonging to the core (see Corollary 2.4.4). As a result, finding optimal strategies is as difficult as solving the problem (2.6).

Finally, there are some open questions left. Firstly, under what conditions I^*, the solution of (4.1), is of the form $f(\sum_{k=1}^{n} X_k)$, for a certain function f. Secondly, whether the presented setup can be extended to other, not concave, gain functionals, in particular the one corresponding to survival probability. Lastly, many interesting questions concerning cooperative game theory can be found in the "Concluding Remarks" section of Xia (2004).

5. Conclusions. We constructed a model describing possible interactions between utility maximizing individuals who are exposed to risk and wish to buy an insurance policy. We assumed that they cannot exchange risk between one another, but may cooperate by purchasing a joint insurance arrangement. The model was designed to capture certain desired properties and answer two questions on profitability of cooperation. We showed that any decent strategy should be Pareto optimal (Proposition 2.3.3) and we gave a characterization of all Pareto optimal strategies (Theorem 2.3.5), which resembles the famous Borch theorem (see Borch (1960, 1962)). Additionally, under expected value principle and some integrability conditions, we presented the necessary and sufficient conditions for the optimality of cooperation (Theorem 2.3.7). We also showed that the core is nonempty, if only the pricing rule is homogeneous (Theorem 2.4.2). Finally some matters of practical importance and open questions were addressed.

6. Proofs. Let $\tau \subset [n]$. We will be using the following notation:

$$\begin{aligned} &\mathbb{R}^\tau = \{(x_k, k \in \tau) \mid x_k \in \mathbb{R}\}, \quad \mathbb{R}_+^\tau = \{(x_k, k \in \tau) \mid x_k \in \mathbb{R}_+\}, \\ &\mathring{\mathbb{R}}_+^\tau = \mathbb{R}_+^\tau \setminus \{0\}. \end{aligned} \tag{6.1}$$

6.1. Proof of Proposition 2.3.3. Let $(\xi_1, \ldots, \xi_n) \in \mathcal{A}_C([n])$ be a non-Pareto optimal strategy. Such a strategy can be refined iteratively over each coordinate (using a "greedy algorithm") in order to achieve Pareto optimality within at most n steps. Hence we will focus only on one step of this procedure.

By our assumption there exist $m \in [n]$, $\varepsilon > 0$ and a strategy $(\xi_1^\varepsilon, \dots, \xi_n^\varepsilon) \in \mathcal{A}_C([n])$ such that

$$\text{(6.2)} \qquad \begin{aligned} \mathbb{E}[U_m(\xi_m)] + \varepsilon &\le \mathbb{E}[U_m(\xi_m^\varepsilon)], \\ \mathbb{E}[U_k(\xi_k)] &\le \mathbb{E}[U_k(\xi_k^\varepsilon)] \quad \forall k \in [n] \setminus \{m\}. \end{aligned}$$

Let ε^* be the supremum of all $\varepsilon > 0$ for which there exists $(\xi_1^\varepsilon, \dots, \xi_n^\varepsilon) \in \mathcal{A}_C([n])$ satisfying (6.2). Obviously $\varepsilon^* > 0$ and it is sufficient to show that (6.2) is satisfied with ε^* and some strategy $(\xi_1^{\varepsilon^*}, \dots, \xi_n^{\varepsilon^*}) \in \mathcal{A}_C([n])$.

Obviously one can choose a sequence $\{\varepsilon_k\}_{k \ge 1}$ such that $\varepsilon_k > 0$, $\varepsilon_k \nearrow \varepsilon^*$ as $k \to \infty$ and ε_k satisfies (6.2) for some $(\xi_1^{\varepsilon_k}, \dots, \xi_n^{\varepsilon_k}) \in \mathcal{A}_C([n])$. Then by Lemma 7.5 below there exists a sequence

$$(\zeta_1^{(k)}, \dots, \zeta_n^{(k)}) \in \operatorname{conv}((\xi_1^{\varepsilon_j}, \dots, \xi_n^{\varepsilon_j}) : j \ge k)$$

such that $(\zeta_1^{(k)}, \dots, \zeta_n^{(k)})$ converges to some $(\zeta_1, \dots, \zeta_n)$ almost surely. In what follows we show that $(\zeta_1, \dots, \zeta_n)$ plays the role of $(\xi_1^{\varepsilon^*}, \dots, \xi_n^{\varepsilon^*})$. The convexity of the principle functional π implies that $\mathcal{A}_C([n])$ is convex and thus $(\zeta_1^{(k)}, \dots, \zeta_n^{(k)}) \in \mathcal{A}_C([n])$. Since $\zeta_j^{(k)}$ is bounded, we can apply the Lebesgue theorem to get $\mathbb{E}U_j^+(\zeta_j^{(k)}) \to \mathbb{E}U_j^+(\zeta_j)$ as $k \to \infty$. On the other hand, from the Fatou lemma,

$$-\mathbb{E}U_j^-(\zeta_j) \ge \limsup_{k \to \infty} [-\mathbb{E}U_j^-(\zeta_j^{(k)})].$$

Let $\{\alpha_j^{(k)}\}$ denote the corresponding vector of weights for $(\zeta_1^{(k)}, \dots, \zeta_n^{(k)})$ (see Lemma 7.5). Then

$$\begin{aligned} \mathbb{E}U_m(\zeta_m) &= \mathbb{E}U_m^+(\zeta_m) - \mathbb{E}U_m^-(\zeta_m) \\ &\ge \lim_{k \to \infty} \mathbb{E}U_m^+(\zeta_m^{(k)}) + \limsup_{k \to \infty} [-EU_m^-(\zeta_m^{(k)})] \\ &= \limsup_{k \to \infty} \mathbb{E}U_m(\zeta_m^{(k)}) \ge \limsup_{k \to \infty} \sum_{j \ge 0} \alpha_j^{(k)} \mathbb{E}U_m(\xi_m^{\varepsilon_{k+j}}) \\ &\ge \limsup_{k \to \infty} (\mathbb{E}U_m(\xi_m) + \varepsilon_k) \\ &= \mathbb{E}U_m(\xi_m) + \varepsilon^* \end{aligned}$$

and by similar arguments $\mathbb{E}[U_k(\xi_k)] \le \mathbb{E}[U_k(\zeta_k)]$ for $k \in [n] \setminus \{m\}$.

It remains to show that $\zeta \in \mathcal{A}_C([n])$. Let $I^{(k)} \in \mathcal{J}([n])$ be such that

$$\sum_{i \in [n]} \zeta_i^{(k)} \le -\sum_{i \in [n]} X_i + I^{(k)}(X_1, \dots, X_n) - \pi(I^{(k)}(X_1, \dots, X_n)).$$

Then again by Lemma 7.5 there exists a sequence

$$\widetilde{I}^{(k)}(X_1, \dots, X_n) \in \operatorname{conv}(I^{(j)}(X_1, \dots, X_n) : j \ge k)$$

that converges almost surely to some $I(X_1,\dots,X_n)$. By (2.2)–(2.3) it follows that $I\in\mathcal{J}([n])$. Assume that

$$\widetilde{I}^{(k)}(X_1,\dots,X_n)=\sum_{j\geq 0}\beta_j^{(k)} I^{(k+j)}(X_1,\dots,X_n),$$

where $\{\beta_j^{(k)}\}_{j\geq 0}$ are the corresponding weights. Then by (2.2) we have

$$\sum_{j\geq 0}\beta_j^{(k)}\sum_{i\in[n]}\zeta_i^{(k+j)}\leq -\sum_{i\in[n]}X_i+\widetilde{I}^{(k)}(X_1,\dots,X_n)-\pi(\widetilde{I}^{(k)}(X_1,\dots,X_n)).$$

Taking the limit $k\to\infty$ and using (2.3) we get

$$\sum_{i\in[n]}\zeta_i\leq -\sum_{i\in[n]}X_i+I(X_1,\dots,X_n)-\pi(I(X_1,\dots,X_n)),$$

which completes the proof. ∎

6.2. Proof of Lemma 2.3.4. Let $\xi\in\mathcal{A}_C([n])\backslash\mathcal{A}([n])$. It is enough to prove that there exists $\widetilde{\xi}\in\mathcal{A}([n])$ that dominates ξ. Indeed, there exists $L\in\mathcal{L}([n])$ such that $\sum_{k=1}^n\xi_k\leq L$ almost surely and $\sum_{k=1}^n\xi_k<L$ with positive probability. Define $\widetilde{\xi}_k=\xi_k\left(\sum_{i=1}^n\xi_i\right)^{-1}L$, with the convention $0/0=0$. Then $\widetilde{\xi}_k\leq 0$ and $\sum_{k=1}^n\widetilde{\xi}_k=L$, that is, $(\widetilde{\xi}_1,\dots,\widetilde{\xi}_n)\in\mathcal{A}([n])$. Moreover, $\widetilde{\xi}_k\geq\xi_k$ for all $k\in[n]$ and there exists k such $\mathbb{P}(\widetilde{\xi}_k>\xi_k)>0$. Since U_k are strictly increasing, the conclusion follows. ∎

6.3. Proof of Theorem 2.3.5. The proof consists of several steps. We replace the problem of finding Pareto optimal strategies by a scalar optimization problem (Lemma 6.3.1) and we solve it in Lemmas 6.3.2 and Corollary 6.3.3. Next we prove that U_λ is a "nice" utility function (Lemma 6.3.4). Finally we bring everything together. Before proceeding let us introduce the following functions:

$$F_k(y)=(U_k')^{-1}(y),\qquad F_\lambda(y)=\sum_{k\colon\lambda_k>0}F_k(\lambda_k^{-1}y),\qquad G_\lambda(x)=(F_\lambda)^{-1}(x).$$

LEMMA 6.3.1. *Let $(\xi_1^*,\dots,\xi_n^*)\in\mathcal{A}_C([n])$ and $\Lambda=\{k\in[n]\mid\mathbb{P}(\xi_k^*<0)>0\}$. Then $(\xi_1^*,\dots,\xi_n^*)$ is Pareto optimal if and only if there exist constants $\lambda_k>0$, $k\in\Lambda$, such that $\sum_{k\in\Lambda}\lambda_k=1$ and*

$$\sum_{k\in\Lambda}\lambda_k\mathbb{E}U_k(\xi_k^*)=\sup_{(\xi_k\mathbb{1}_\Lambda(k),\,k\in[n])\in\mathcal{A}_C([n])}\sum_{k\in\Lambda}\lambda_k\mathbb{E}U_k(\xi_k).$$

Proof. Sufficiency. Suppose that the strategy $(\xi_1^*,\dots,\xi_n^*)$ is not Pareto optimal. Then from the definition there exists $(\widetilde{\xi}_1,\dots,\widetilde{\xi}_n)\in\mathcal{A}_C([n])$ such that

$$\mathbb{E}U_k(\xi_k^*)\leq\mathbb{E}U_k(\widetilde{\xi}_k),\qquad k=1,\dots,n,$$

with strict inequality for some $k \in \Lambda$ (for $k \notin \Lambda$ we have $\widetilde{\xi}_k = 0$ a.s.). Multiplying both sides of the above inequalities by the corresponding $\lambda_k > 0$ and summing up over $k \in \Lambda$ we arrive at a contradiction.

Necessity. Define

$$H = \{(\mathbb{E}U_k(\xi_k), k \in \Lambda) \mid (\xi_k \mathbb{1}_\Lambda(k), k \in [n]) \in \mathcal{A}_C([n])\} \subset \mathbb{R}^\Lambda.$$

From the definition $(\xi_1^*, \dots, \xi_n^*) \in \mathcal{A}_C([n])$ is Pareto optimal if

$$(\mathbb{E}U_k(\xi_k^*), k \in \Lambda) \notin H - \mathring{\mathbb{R}}_+^\Lambda.$$

It can be easily deduced that $H - \mathring{\mathbb{R}}_+^\Lambda$ is a convex set. Thus by the separation theorem there exist constants λ_k, $k \in \Lambda$, not all zero, such that

$$\sup_{\substack{(\xi_k \mathbb{1}_\Lambda(k), k\in[n]) \in \mathcal{A}_C([n]) \\ r \in \mathring{\mathbb{R}}_+^\Lambda}} \Big(\sum_{k\in\Lambda} \lambda_k \mathbb{E}(U_k(\xi_k)) - \sum_{k\in\Lambda} \lambda_k r_k\Big) \le \sum_{k\in\Lambda} \lambda_k \mathbb{E}(U_k(\xi_k^*)). \tag{6.3}$$

The right-hand side is finite, which implies that $\lambda_k \ge 0$. Moreover, $\lambda_k \ne 0$ for all $k \in \Lambda$. Indeed, suppose that there exists j such that $\lambda_j = 0$. Then

$$\sup_{\substack{(\xi_k \mathbb{1}_\Lambda(k), k\in[n]) \in \mathcal{A}_C([n]) \\ r \in \mathring{\mathbb{R}}_+^\Lambda}} \Big(\sum_{k\in\Lambda} \lambda_k \mathbb{E}(U_k(\xi_k)) - \sum_{k\in\Lambda} \lambda_k r_k\Big) = \sum_{k\in\Lambda} \lambda_k \mathbb{E}U_k(0),$$

and we have a contradiction with (6.3). Finally we normalize the constants λ_k and the proof is finished. ∎

LEMMA 6.3.2. *Assume that $\Lambda \subset [n]$ and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be such that $\lambda_k = 0$ for $k \notin \Lambda$, $\lambda_k > 0$ for $k \in \Lambda$, $\sum_{k\in\Lambda} \lambda_k = 1$. Let $z \le 0$ be some constant. Then the unique solution to the deterministic problem*

$$\begin{cases} \sum_{k\in\Lambda} \lambda_k U_k(x_k) \to \max!, \\ x_k \le 0, \quad k \in \Lambda, \\ \sum_{k\in\Lambda} x_k = z, \end{cases}$$

is the vector $(F_k(\lambda_k^{-1} G_\lambda(z)), k \in \Lambda)$.

Proof. Let $\alpha = (\alpha_k, k \in \Lambda \cup \{0\})$, $x = (x_k, k \in \Lambda)$ and define the Lagrange function

$$L_\alpha(x) = \sum_{k\in\Lambda} \lambda_k U(x_k) - \alpha_0 \Big(\sum_{k\in\Lambda} x_k - z\Big) - \sum_{k\in\Lambda} \alpha_k x_k.$$

Then $x^* \in \mathbb{R}^\Lambda$ is optimal if $x_k \le 0$ for $k \in \Lambda$, $\sum_{k\in\Lambda} x_k = z$ and there exist $\alpha_0 \in \mathbb{R}$ and $\alpha_k \ge 0$, $k \in \Lambda$, such that $\nabla L_\alpha(x^*) = 0$ (due to the classical Kuhn–Karush–Tucker theorem). Take $\alpha_k = 0$ for all $k \in \Lambda$ and $\alpha_0 = G_\lambda(z)$. It is easy to check that the vector $x^* = (F_k(\lambda_k^{-1} G_\lambda(z)), k \in \Lambda)$

satisfies $\lambda_k U_k'(x_k^*) = G_\lambda(z)$, $x_k^* \le 0$ for $k \in \Lambda$ and $\sum_{k\in\Lambda} x_k^* = z$. Thus x^* is optimal. ∎

COROLLARY 6.3.3. *Let Λ, λ be as in Lemma 6.3.2 and fix a random variable g satisfying $-\sum_{k=1}^n (X_k + p_k) \le g \le L$ for some $L \in \mathcal{L}([n])$. Then the unique (up to sets of measure zero) solution of the problem*

$$\begin{cases} \sum_{k\in\Lambda} \lambda_k U_k(\xi_k) \to \max!, \\ \xi_k \le 0, \quad k \in \Lambda, \\ \sum_{k\in\Lambda} \xi_k = g, \end{cases} \tag{6.4}$$

is the vector $(F_k(\lambda_k^{-1} G_\lambda(g)), k \in \Lambda)$.

Proof. Fix ω and apply Lemma 6.3.2. ∎

LEMMA 6.3.4. *The function U_λ is strictly concave, strictly increasing and twice continuously differentiable. Moreover, $U_\lambda' = G_\lambda$.*

Proof. For the proof see Xia (2004, Lemma 4.4). ∎

We now proceed with the proof of Theorem 2.3.5. Let $\xi^* = (\xi_1^*, \ldots, \xi_n^*)$ and Λ be as in the statement. By Lemma 6.3.1 and Corollary 6.3.3 the strategy ξ^* is optimal if and only if there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_k = 0$ for $k \notin \Lambda$, $\lambda_k > 0$ for $k \in \Lambda$, $\sum_{k\in\Lambda} \lambda_k = 1$ and

$$\begin{aligned}\sum_{k\in\Lambda} \lambda_k \mathbb{E} U_k(\xi_k^*) &= \sup_{(\xi_k \mathbb{1}_\Lambda(k),\, k\in[n]) \in \mathcal{A}_C([n])} \sum_{k\in\Lambda} \lambda_k \mathbb{E} U_k(\xi_k) \\ &= \sup_{(\xi_k \mathbb{1}_\Lambda(k),\, k\in[n]) \in \mathcal{A}_C([n])} \mathbb{E} U_\lambda\Big(\sum_{k\in\Lambda} \xi_k\Big) = \sup_{L\in\mathcal{L}([n])} \mathbb{E} U_\lambda(L).\end{aligned}$$

From Lemma 6.3.4 and (a slightly modified version of) Lemma 7.6 it follows that there exists a unique L^* that maximizes $\mathbb{E} U_\lambda(L)$ amongst all $L \in \mathcal{L}([n])$. The claim now follows from the uniqueness of the solution to (6.4). ∎

6.4. Proof of Theorem 2.3.7. As $\mathbb{P}(X_k > 0) > 0$ it follows that $\mathbb{P}(L_{\{k\}}^* < 0) > 0$ for all $1 \le k \le n$.

Necessity. This follows directly from Theorem 2.3.5.

Sufficiency. By our assumptions there exists a random variable η taking values in $[0, \infty)$ and λ satisfying (2.5), $\lambda_k > 0$ for $k = 1, \ldots, n$, such that

$$U_k'(L_{\{k\}}^*) = \lambda_k^{-1} \eta.$$

This implies that $L_{\{k\}}^* = F_k(\lambda_k^{-1} \eta)$. Denote

$$L^* = \sum_{k=1}^n L_{\{k\}}^* = \sum_{k=1}^n F_k(\lambda_k^{-1} \eta).$$

Then $\eta = G_\lambda(L^*)$ and

$$U_k'(L^*_{\{k\}}) = \lambda_k^{-1}\eta = \lambda_k^{-1} G_\lambda(L^*) = \lambda_k^{-1} U_\lambda'(L^*). \tag{6.5}$$

By Theorem 2.3.5 it is enough to prove that L^* is a solution to the problem

$$\sup_{L\in\mathcal{L}([n])} \mathbb{E} U_\lambda(L).$$

First consider the problem (1.1)–(1.4) with U_k, X_k and p_k instead of U, X, p respectively. Denote the distribution of X_k by μ_k, i.e. $\mu_k(dx) = \mathbb{P}(X_k \in dx)$. Since $X_k \in L^a(\mathbb{P})$ we seek a function $I \in L^a(\mu_k)$ satisfying (1.2)–(1.3) that maximizes the corresponding ψ_k (see (1.4)). The directional derivative of ψ_k at the point I in the direction $I_0 \in L^a(\mu_k)$ is defined as

$$d\psi_k(I)(I_0) = \frac{d}{dt}\mathbb{E}U_k(-X_k + I_k(X_k) + tI_0(X_k) - \pi(I(X_k) + tI_0(X_k)))\bigg|_{t=0}.$$

Because $U_k'(-X_k - p_k) \in (L^a(\mathbb{P}))^* = L^b(\mathbb{P})$ it follows that ψ_k is Gateaux differentiable (see e.g. Ioffe and Tikhomirov (1979, page 22) for definition) and its Gateaux derivative at I is

$$d\psi_k(I) = U_k'(-X_k + I(X_k) - \pi(I(X_k))) - (1+\theta)\mathbb{E}U_k'(-X_k + I(X_k) - \pi(I(X_k))).$$

Now the necessary Kuhn–Tucker conditions (see Ioffe and Tikhomirov (1979, Theorem 2′, page 69); compare also Zagrodny (2003)) state that if I_k^* is optimal then there exists $\alpha_k \geq 0$ such that

$$0 \in -\partial\psi_k(I_k^*) + \alpha_k \partial(\pi(\cdot) - p_k)(I_k^*) + N_k(I_k^*|D_k), \tag{6.6}$$

$$\alpha_k(\pi(I_k^*(X_k)) - p_k) = 0, \tag{6.7}$$

where $D_k = \{f \in L^a(\mu_k) \mid \mu_k(\{x \in \mathbb{R}_+ \mid 0 \leq f(x) \leq x\}) = 1\}$ and

$$N_k(I_k^*|D_k) = \Big\{g \in L^b(\mu_k) \;\Big|\; \int_{\mathbb{R}_+} g(x)(I(x) - I_k^*(x))\,\mu_k(dx) \leq 0,\ \forall I \in D_k\Big\}.$$

Since π is also Gateaux differentiable (as a linear functional) it follows that the subdifferentials $\partial\psi_k$, $\partial(p_k - \pi(I_k^*))$ consist of one point, namely the Gateaux derivatives. Moreover, $g \in N_k(I_k^*|D_k)$ if and only if the following three conditions hold:

$$g \leq 0 \quad \mu_k\text{-a.s. on } A_k = \{x \in \mathbb{R}_+ \mid I_k^*(x) = 0\}, \tag{6.8}$$

$$g = 0 \quad \mu_k\text{-a.s. on } B_k = \{x \in \mathbb{R}_+ \mid 0 < I_k^*(x) < x\}, \tag{6.9}$$

$$g \geq 0 \quad \mu_k\text{-a.s. on } C_k = \{x \in \mathbb{R}_+ \mid I_k^*(x) = x \neq 0\}. \tag{6.10}$$

(Recall that we say that $g \in B$ μ-a.s. on $A \subset \mathbb{R}_+$ if $\mu(\{g \in B\} \cap A) = \mu(A)$.) The sufficiency of these conditions is obvious. To prove the necessity we only consider (6.8), as (6.9)–(6.10) follow by similar arguments. Assume that $\mu(\{x \in \mathbb{R}_+ \mid I_k^*(x) = 0\}) > 0$ and take $I(x) = x\mathbb{1}_{A_k\cap\{g>0\}}(x) +$

$I_k^*(x)\mathbb{1}_{B_k\cup C_k}(x) \in D_k$. Then

$$\int_{\mathbb{R}_+} g(x)(I(x) - I_k^*(x))\,\mu_k(dx) = \int_{A_k\cap\{g>0\}} xg(x)\,\mu_k(dx) \le 0,$$

which implies that $g \le 0$ μ_k-a.s. on A_k.

In what follows we abuse notation by writing

$$A_k = \{I_k^*(X_k) = 0\}, \qquad B_k = \{0 < I_k^*(X_k) < X_k\},$$
$$C_k = \{I_k^*(X_k) = X_k,\, X_k \ne 0\}.$$

Recall that $L^*_{\{k\}} = -X_k + I_k^*(X_k) - \pi(I_k^*(X_k))$. Then (6.6)–(6.10) imply

$$U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \le 0 \qquad \mathbb{P}\text{-a.s. on } A_k,$$
$$U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) = 0 \qquad \mathbb{P}\text{-a.s. on } B_k,$$
$$U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \ge 0 \qquad \mathbb{P}\text{-a.s. on } C_k,$$
$$\alpha_k(\pi(I_k^*(X_k) - p_k) = 0.$$

We show that in fact $\mathbb{P}(C_k) = 0$. Assume otherwise, i.e. $\mathbb{P}(C_k) > 0$. Since the policy $I_k(X_k) \equiv X_k$ is not optimal ($p_k < \pi(X_k)$), we have $\mathbb{P}(A_k \cup B_k) > 0$. Therefore,

$$\mathbb{E}[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \mid A_k \cup B_k] \le 0, \tag{6.11}$$
$$\mathbb{E}[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \mid C_k] \ge 0. \tag{6.12}$$

The left hand side in (6.12) is in fact equal to

$$U_k'(-\pi(I_k^*(X_k))) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k).$$

Thus subtracting (6.11) and (6.12) we get

$$U_k'(-\pi(I_k^*(X_k))) - \mathbb{E}(U_k'(L^*_{\{k\}}) \mid A_k \cup B_k) \ge 0$$

This is impossible, since U_k' is a decreasing function.

Moreover, as $I_k(X_k) \equiv 0$ is not optimal (there is no $\alpha_k \ge 0$ such that the necessary Kuhn–Tucker conditions are satisfied), we have $\mathbb{P}(B_k) > 0$.

Recall that $U_\lambda'(L^*) = \lambda_k U_k'(L^*_{\{k\}})$. We claim that the quantity $\lambda_k\alpha_k$ is constant (independent of k). To prove this, fix $k \ne j$ and assume that $\mathbb{P}(B_k \cap B_j) > 0$. Then

$$\lambda_k\mathbb{E}[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \mid B_k \cap B_j] = 0,$$
$$\lambda_j\mathbb{E}[U_j'(L^*_{\{j\}}) - (1+\theta)(\mathbb{E}U_j'(L^*_{\{j\}}) + \alpha_j) \mid B_k \cap B_j] = 0,$$

and the claim follows. Suppose now that $\mathbb{P}(B_k\cap B_j) = 0$. Then $\mathbb{P}(B_k\cap A_j) > 0$ (since $\mathbb{P}(A_j \cup B_j) = 1$ and $\mathbb{P}(B_k) > 0$) and thus

$$\lambda_k \mathbb{E}[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k) \mid B_k \cap A_j] = 0,$$
$$\lambda_j \mathbb{E}[U_j'(L^*_{\{j\}}) - (1+\theta)(\mathbb{E}U_j'(L^*_{\{j\}}) + \alpha_j) \mid B_k \cap A_j] \le 0.$$

As a consequence, $\lambda_k\alpha_k \le \lambda_j\alpha_j$. The claim follows by interchanging k and j.

We are now ready to finish the proof. For notational convenience we write I or I^* instead of $I(X_1,\dots,X_n)$ or $I^*(X_1,\dots,X_n)$. Denote the Lagrange functional by

$$L_\alpha(I) = \mathbb{E}U_\lambda(-X + I - \pi(I)) - \alpha(\pi(I) - p),$$

where $X = \sum_{k=1}^n X_k$, $p = \sum_{k=1}^k p_k$ and $\alpha = \lambda_1\alpha_1 \ge 0$. Let $I \in \mathcal{J}([n])$. Then by concavity of U_λ,

$$\begin{aligned}
(6.13)\quad & L_\alpha(I^*) - L_\alpha(I)\\
&\ge \mathbb{E}\{U_\lambda'(-X + I^* - \pi(I^*))[I^* - I - \pi(I^*) + \pi(I)]\} - \alpha(\pi(I^*) - \pi(I))\\
&= \mathbb{E}\{(I^* - I)[U_\lambda'(-X + I^* - \pi(I^*))\\
&\quad - (1+\theta)(\mathbb{E}U_\lambda'(-X + I^* - \pi(I^*)) + \alpha)]\}.
\end{aligned}$$

Since $\mathbb{P}(C_k) = 0$ we have $\mathbb{P}(I^*(X_1,\dots,X_n) = X,\, X \ne 0) = 0$. Let M_k, $k = 1,\dots,n$, be the disjoint sets that sum up to Ω and $M_k \subset A_k \cup B_k$. Consequently,

$$\begin{aligned}
&\mathbb{E}\{(I^* - I)[U_\lambda'(-X + I^* - \pi(I^*)) - (1+\theta)(\mathbb{E}U_\lambda'(-X + I^* - \pi(I^*)) + \alpha)]\}\\
&\quad = \sum_{k=1}^n \lambda_k \mathbb{E}\{(-I)[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k)]\mathbb{1}_{M_k\cap A_k}\}\\
&\qquad + \sum_{k=1}^n \lambda_k \mathbb{E}\{(I^* - I)[U_k'(L^*_{\{k\}}) - (1+\theta)(\mathbb{E}U_k'(L^*_{\{k\}}) + \alpha_k)]\mathbb{1}_{M_k\cap B_k}\}\\
&\quad \ge 0.
\end{aligned}$$

Therefore, as $\alpha(\pi(I^*) - p) = \sum_{k=1}^n \lambda_k\alpha_k(\pi(I_k^*(X_k)) - p_k) = 0$, we get

$$\mathbb{E}U_\lambda(L^*) = L_\alpha(I^*) \ge L_\alpha(I) \ge \mathbb{E}U_\lambda(L)$$

for any $L \in \mathcal{L}([n])$, which completes the proof. ∎

6.5. Proof of Theorem 2.4.2. Define $H(\tau) = \{(\mathbb{E}U_k(\zeta_k^\tau), k \in \tau) \mid \zeta^\tau \in \mathcal{A}_C(\tau)\}$ and $K(\tau) = H(\tau) - \mathbb{R}_+^\tau$ for every $\tau \subset [n]$. Then $\{K(\tau) \mid \tau \subset [n]\}$ is an n-person cooperative game (see Appendix). Denote by C_τ the natural projection of $\mathbb{R}^n$ to $\mathbb{R}^\tau$. We will prove the following two lemmas.

LEMMA 6.5.1. *The game $\{K(\tau) \mid \tau \subset [n]\}$ is balanced (see Appendix).*

Proof. Let $\mathcal{B}$ be a balanced family of sets (see Appendix). We have to show that for $r \in \bigcap_{\tau\in\mathcal{B}}(C_\tau)^{-1}(K(\tau))$ there exists $\zeta \in \mathcal{A}_C([n])$ such that $r_k \le \mathbb{E}U_k(\zeta_k)$. Because $r \in \bigcap_{\tau\in\mathcal{B}}(C_\tau)^{-1}(K(\tau))$, for all $\tau \in \mathcal{B}$ there exists

$\xi^\tau \in \mathcal{A}_C(\tau)$ such that

$$r_k \le \mathbb{E}U_k(\xi^\tau_k) \quad \text{for all } k \in \tau.$$

Since $\mathcal{B}$ is balanced there exist constants $m(\tau) > 0$ such that $\sum_{\tau \ni k,\, \tau \in \mathcal{B}} m(\tau) = 1$ for each $k \in [n]$. From concavity of U_k we get

$$\begin{aligned} r_k = \sum_{\tau \ni k,\, \tau\in\mathcal{B}} m(\tau) r_k &\le \sum_{\tau \ni k,\, \tau\in\mathcal{B}} m(\tau)\mathbb{E}U_k(\xi^\tau_k) \\ &\le \mathbb{E}U_k\Big(\sum_{\tau \ni k,\, \tau\in\mathcal{B}} m(\tau)\xi^\tau_k\Big) = \mathbb{E}U_k(\zeta_k), \end{aligned}$$

where the vector $(\zeta_1, \ldots, \zeta_n)$ is of the form $\zeta_k = \sum_{\tau\ni k,\, \tau\in\mathcal{B}} m(\tau)\xi^\tau_k$ for $k \in [n]$. It only remains to prove that $(\zeta_1, \ldots, \zeta_n) \in \mathcal{A}_C([n])$. Let us notice that $\zeta_k \le 0$ and

$$\sum_{k=1}^n \zeta_k = \sum_{k=1}^n \sum_{\tau\ni k,\, \tau\in\mathcal{B}} m(\tau)\xi^\tau_k = \sum_{\tau\in\mathcal{B}}\sum_{k\in\tau} m(\tau)\xi^\tau_k = \sum_{\tau\in\mathcal{B}} m(\tau)\sum_{k\in\tau}\xi^\tau_k.$$

From the definition there exists $I^\tau \in \mathcal{J}(\tau)$ such that

$$\sum_{k\in\tau}\xi^\tau_k \le -\sum_{k\in\tau} X_k + I^\tau(X_k, k\in\tau) - \pi(I^\tau).$$

Therefore, changing the order of summation gives

$$\sum_{\tau\in\mathcal{B}} m(\tau)\sum_{k\in\tau}\xi^\tau_k \le -\sum_{k=1}^n X_k + \sum_{\tau\in\mathcal{B}} m(\tau) I^\tau((X_k, k\in\tau)) - \sum_{\tau\in\mathcal{B}} m(\tau)\pi(I^\tau).$$

Denote $\widetilde{I}(x_1, \ldots, x_n) = \sum_{\tau\in\mathcal{B}} m(\tau) I^\tau(x_k, k \in \tau)$. It remains to show that $I \in \mathcal{J}([n])$. Observe that $0 \le \widetilde{I}(x_1, \ldots, x_n) \le \sum_{k=1}^n x_k$. From homogeneity (and equivalently subadditivity) of the premium functional π we get

$$\pi(\widetilde{I}) \le \sum_{\tau\in\mathcal{B}} m(\tau)\pi(I^\tau),$$

which implies that $\pi(\widetilde{I}) \le \sum_{k=1}^n p_k$. This completes the proof. ∎

LEMMA 6.5.2. *The set $K(\tau)$ is closed in the $\mathbb{R}^\tau$ topology and bounded from above for all $\tau \subset [n]$.*

Proof. One only has to prove closedness, as boundedness is obvious. Let $a^{(k)} \in K(\tau)$ be a sequence such that $a^{(k)} \to a \in \mathbb{R}^\tau$. Then there exists a sequence $\xi^{(k)} \in \mathcal{A}_C(\tau)$ such that $a^{(k)}_j \le \mathbb{E}U_j(\xi^{(k)}_j)$ for $j \in \tau$. From Lemma 7.5 there exists a sequence

$$\zeta^{(k)} \in \operatorname{conv}(\xi^{(j)} : j \ge k)$$

that converges almost surely to some ζ. By similar arguments to those in the proof of Proposition 2.3.3 we find that $\zeta \in \mathcal{A}_C(\tau)$ and

$$\mathbb{E}U_j(\zeta_j) \geq \limsup_{k\to\infty} \mathbb{E}U_j(\zeta_j^{(k)}) \geq \liminf_{k\to\infty}\Big(\sum_{j\geq 0}\beta_j^{(k)} a_j^{(k+j)}\Big) = a_j,$$

where $\{\beta_j^{(k)}\}$ are weights associated with $\zeta^{(k)}$. Therefore there exists $b \in \mathbb{R}_+^\tau$ such that $a_j = \mathbb{E}U_j(\zeta_j) - b_j$, which we needed to show. ∎

From the Scarf Theorem (see Appendix) the core of the game $\{K(\tau) \mid \tau \subset [n]\}$ is nonempty. Moreover, it has a nontrivial intersection with $H([n])$. Indeed, suppose that z is in the core and $z \in K([n])\backslash H([n])$. From the definition there exists $h \in H([n])$ such that $z_k \leq h_k$ for every $k = 1, \ldots, n$ (with at least one strict inequality). Therefore h is also in the core. This completes the proof. ∎

7. Appendix. The definitions and the theorem below come from Aubin (1993).

DEFINITION 7.1. A family of sets $\{K(\tau) \mid \tau \subset [n]\}$ such that $K(\tau) = K(\tau) - \mathbb{R}_+^\tau$ for all $\tau \subset [n]$ is called an *n-person cooperative game.*

DEFINITION 7.2. We say that a family $\mathcal{B}$ of sets $\tau \subset [n]$ is *balanced* if there exist constants $m(\tau) > 0$ such that

$$\sum_{\tau\ni k,\, \tau\in\mathcal{B}} m(\tau) = 1 \quad \text{for all } k \in [n].$$

DEFINITION 7.3. Let $\tau \subset [n]$ and denote by C_τ the natural projection of $\mathbb{R}^n$ on $\mathbb{R}^\tau$ (see (6.1) for the definition). We say that the cooperative game is *balanced* if for every balanced family $\mathcal{B}$ we have

$$\bigcap_{\tau\in\mathcal{B}} (C_\tau)^{-1}(K(\tau)) \subset K(N).$$

THEOREM 7.4 (Scarf). *Suppose that the cooperative game is balanced and the set $K(\tau)$ is bounded from above and closed in the topology of $\mathbb{R}^\tau$ for all $\tau \subset [n]$. Then the core is a nonempty set.*

The following lemma comes from Delbaen and Schachermayer (1994, Lemma A1.1).

LEMMA 7.5. *Let f_n be a sequence of nonpositive random variables. Then there exists a sequence of random variables $g_k \in \operatorname{conv}(f_j : j \geq k)$ and a nonpositive random variable g such that $g_k \to g$ almost surely as $k \to \infty$.*

Recall that $g_k \in \operatorname{conv}(f_j : j \geq k)$ if there exists a sequence $\{\alpha_j^{(k)}\}_{j\geq 0}$ satisfying $\alpha_j^{(k)} \geq 0$ and $\sum_{0\leq j\leq M} \alpha_j^{(k)} = 1$ for some $M = M(k) < \infty$ such that $g_k = \sum_{j\geq 0} \alpha_j^{(k)} f_{k+j}$.

LEMMA 7.6. *Suppose that* $U \in C^1((-\infty,0)) \cap C((-\infty,0])$ *is a finite, strictly increasing and strictly concave function and* $\mathbb{E}U(-X-p) > -\infty$. *Then the problem* (1.1)–(1.4) *has a unique solution, i.e. if* I_1 *and* I_2 *are two solutions then* $\mathbb{P}(I_1(X) = I_2(X)) = 1$.

Proof. Denote the supremum in the problem (1.1)–(1.4) by s^*. By definition it is possible to choose a sequence I^k such that

$$0 \le I^k(X) \le X, \qquad \pi(I^k) \le p,$$

and $\mathbb{E}U(-X + I^k(X) - \pi(I^k(X))) \to s^*$ as $k \to \infty$. By Lemma 7.5 there exists a sequence $Y^k \in \mathrm{conv}(I^j(X),\, j \ge k)$ and Y such that $Y^k \to Y$ almost surely. From properties (2.2) and (2.3) of pricing rule it follows that $\pi(Y) \le p$. Moreover, $0 \le Y \le X$. Thus there exists a function I^* satisfying $I^*(X) = Y$ and it is a solution to the problem. The uniqueness follows from the strict concavity of U. ■

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References

K. Aase (2002), *Perspectives of risk sharing*, Scand. Actuar. J. 2, 73–128.

K. Arrow (1963), *Uncertainty and the welfare economics of medical care*, Amer. Econom. Rev. 53, 941–973.

J.-P. Aubin (1993), *Optima and Equilibria*, Springer, Berlin.

B. Baton and J. Lemaire (1981), *The core of a reinsurance market*, Astin Bull. 12, 57–71.

K. Borch (1960), *The safety loading of reinsurance premiums*, Skand. Aktuarietidskrift 43, 163–184.

K. Borch (1962), *Equilibrium in reinsurance market*, Econometrica 30, 424–444.

H. Bühlmann (1970), *Mathematical Methods in Risk Theory.* New York, Springer.

H. Bühlmann and W. Jewell (1979), *Optimal risk exchanges*, Astin Bull. 10, 243–262.

F. Delbaen and Schachermayer (1994), *A general version of the fundamental theorem of asset pricing*, Math. Ann. 300, 463–520.

O. Deprez and H. Gerber (1985), *On convex principles of premium calculation*, Insurance Math. Econom. 4, 179–189.

L. Gajek and D. Zagrodny (2000), *Insurer's optimal reinsurance strategies*, ibid. 27, 105–112.

L. Gajek and D. Zagrodny (2004a), *Optimal reinsurance under general risk measures*, ibid. 34, 227–240.

L. Gajek and D. Zagrodny (2004b), *A price-utility paradox of the risk sharing*, unpublished manuscript.

L. Gajek and D. Zagrodny (2004c), *Reinsurance arrangements maximizing insurer's survival probability*, J. Risk Insurance 71, 421–435.

H. Gerber (1978), *Pareto-optimal risk exchanges and related decision problems*, Astin Bull. 10, 25–33.
A.-D. Ioffe and V. M. Tikhomirov (1979), *Theory of Extremal Problems*, North-Holland, Amsterdam.
P.-J. Laurent (1972), *Approximation et optimisation*, Hermann, Paris.
J. Sujis, A. De Waegenaere and P. Borm (1998), *Stochastic cooperative games in insurance*, Insurance Math. Econom. 22, 209–228.
R. Wilson (1968), *The theory of syndicates*, Econometrica 36, 119–132.
J. Xia (2004). *Multi-agent investment in incomplete markets*, Finance Stoch. 8, 241–259.
D. Zagrodny (2003), *An optimality of change loss type strategy*, Optimization 52, 757–772.

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