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THREE ADDITIVE SOLUTIONS OF COOPERATIVE GAMES WITH A PRIORI UNIONS

Abstract. We analyze axiomatic properties of three types of additive solutions of cooperative games with a priori unions structure. One of these is the Banzhaf value with a priori unions introduced by G. Owen (1981), which has not been axiomatically characterized as yet. Generalizing Owen's approach and the constructions discussed by J. Deegan and E. W. Packel (1979) and L. M. Ruiz, F. Valenciano and J. M. Zarzuelo (1996) we define and study two other solutions. These are the Deegan–Packel value with a priori unions and the least square prenucleolus with a priori unions.

Each of known cooperative game solutions is usually constructed by means of different methods with specific assumptions. In this paper we investigate a modification of three types of such solutions.

The first of these solutions, the Banzhaf value of a player, was introduced by J. F. Banzhaf III (1965). It describes the average profit for a coalition after co-opting the player. Numerous applications of this concept are now known in the social and economic practice, because the relevant formulas represent a good instrument to investigate the power of participants in collective decision processes. In 1981 G. Owen constructed a modification of this notion—the Banzhaf value with a priori unions. The main assumption of this model is a partition of the set of players into nonempty disjoint subsets called a priori unions or precoalitions. The Banzhaf value with a priori unions was constructed on the basis of the "normal" Banzhaf value. E. Lehrer (1988) suggested the first axiomatization of the Banzhaf value. It is the unique solution with the following properties: dummy player, equal treatment, amalgamation and additivity. An axiomatization theorem for the

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Banzhaf value with a priori unions has not been formulated yet. In this paper we try to fill this gap. It is worth noting that this solution also satisfies the balance axiom.

J. Deegan and E. W. Packel (1979) constructed another solution of a cooperative game. The Deegan–Packel value can be interpreted as the sum of average values of every coalition (containing a player) per its member. That is, the larger the Deegan–Packel value of a given player, the greater the worth of the player in the game. In the above-mentioned article an axiomatization theorem (similar to the case of the Shapley value (L. S. Shapley (1953)) was proved. The only difference between these solutions is the fact that the sum of the Deegan–Packel values of all players is not equal to the value of the full coalition (i.e. the one which contains all players of the game). Instead, it has a special property—this sum is equal to the sum of the characteristic function values of all coalitions of the game.

Recently several types of solutions of cooperative games based on the excess vector (for example the prenucleolus and nucleolus (A. Sobolev (1975) and D. Schmeidler (1969) respectively) were constructed. But the basis of the considerations presented in this article is another solution introduced by L. M. Ruiz, F. Valenciano and J. M. Zarzuelo (1996) by application of the notion of excess vector, i.e. the least square prenucleolus. It assigns to a transfer utility cooperative game some preimputation, which is minimal with respect to the least square order relation. The authors of the cited paper have proved that the least square prenucleolus is the unique solution which satisfies the conditions of efficiency, additivity, inessential game and average marginal contribution monotonicity. The least square prenucleolus is also an additive normalization of the Banzhaf value.

In the present article Owen's construction of the modified Banzhaf value is extended to any solution of a cooperative game. By means of this idea, we define a Deegan–Packel value and a least square prenucleolus, both with a priori unions. We study the fundamental properties of these solutions and prove their axiomatizations.

I. Definitions and fundamental facts. Let n be a natural number. An n-person transferable utility cooperative game is uniquely defined by the set of players $N=\{1,\ldots,n\}$ and a function $v:2^N\to\mathbb{R}$ with $v(\emptyset)=0$, called the *characteristic function* of the game. Therefore we will next write briefly "n-person game v". An n-person game v is called *additive* if $v(S\cup K) = v(S)+v(K)$ for any disjoint nonempty sets $S,K\subseteq N$.

An n-dimensional vector $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ is called a *preimputation* if

$$\sum_{i=1}^{n} x_i = v(N).$$

The set of all preimputations of v will be denoted by $\mathbb{P}$, and the (2^n-1)-dimensional vector space of all n-person games will be denoted by G_N.

A *solution* is defined to be a function $\varphi : G_N \to \mathbb{R}^n$, which assigns to each game v a vector from $\mathbb{R}^n$.

The common feature of all solutions considered below is *additivity*:

$$\varphi(v+w) = \varphi(v) + \varphi(w)$$

for any two n-person cooperative games v, w, where $(v+w)(S) = v(S)+w(S)$ for any $S \subseteq N$.

Define

$$x(S) = \sum_{i \in S} x_i \quad \text{for any } S \subseteq N.$$

Denote by $S_1, S_2, \ldots, S_{2^n}$ all the subsets of N. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. A vector $r(v,x) \in \mathbb{R}^{2^n}$ with coordinates

$$r_i(v,x) = v(S_i) - x(S_i)$$

for $i = 1, 2, \ldots, 2^n$ is called the *excess vector*.

DEFINITION 1 (J. F. Banzhaf III (1965)). The *Banzhaf value* of player i of game v is defined as

$$B_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N} [v(S \cup \{i\}) - v(S)] \quad \text{for any } i \in N.$$

DEFINITION 2 (E. Lehrer (1988)). *Amalgamation* of any two different players a, b of an n-person game v is a transformation from game v into game $v_{(ab)}$ with the set of players $(N \setminus \{a,b\}) \cup \{p\}$, where p denotes a player representing the coalition $\{a,b\}$. The characteristic function of this game is defined to be

$$v_{(ab)}(K) = \begin{cases} v(K) & \text{if } p \notin K, \\ v((K \setminus \{p\}) \cup \{a,b\}) & \text{if } p \in K, \end{cases}$$

for any set $K \subseteq (N \setminus \{a,b\}) \cup \{p\}$.

Here is the first axiomatization of the Banzhaf value, given in 1988.

THEOREM 1 (E. Lehrer (1988)). *The solution*

$$\varphi(v) = B(v) = (B_1(v), \ldots, B_n(v))$$

is the unique solution on G_N which satisfies the following conditions:

(i) *If $i \in N$ and $v(S \cup \{i\}) = v(S) + v(\{i\})$ for any $S \subseteq N \setminus \{i\}$ then* $\varphi_i(v) = v(\{i\})$ (dummy player axiom).

(ii) *If $i, j \in N$, $i \neq j$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subseteq N \setminus \{i,j\}$ then* $\varphi_i(v) = \varphi_j(v)$ (equal treatment property).

(iii) *For any two different players* $i, j \in N$,

$$\varphi_p(v_{(ij)}) = \varphi_i(v) + \varphi_j(v)$$

(amalgamation axiom).

(iv) *Additivity.*

DEFINITION 3. The *Deegan–Packel value* of any player $i \in N$ of game v is defined by the formula

$$\mathrm{DP}_i(v) = \sum_{\substack{K \subseteq N \\ i \in K}} \frac{v(K)}{k}$$

where $k = \mathrm{card}(K)$ for any $K \subseteq N$.

Let σ be a permutation of the set N. Then we define $\sigma v(\sigma K) = v(K)$ for any $K \subseteq N$, where $\sigma(K) = \{\sigma(i) : i \in K\}$. A player $i \in N$ is called a *zero-player* if for any $K \subseteq N$, $v(K) = 0$ whenever $i \in K$. Below we give an axiomatization of solution described in Definition 3.

THEOREM 2 (J. Deegan and E. W. Packel (1979)). *The solution* $\varphi(v) = \mathrm{DP}(v) = (\mathrm{DP}_1(v), \ldots, \mathrm{DP}_n(v))$ *is the unique solution on* G_N *which satisfies the following conditions*:

(i) $\varphi_i(v) = 0 \Leftrightarrow i$ *is a zero-player* (zero-player axiom).

(ii) *For any permutation* σ *of the set* N *and every* $i \in N$ *we have* $\varphi_{\sigma(i)}(\sigma v) = \varphi_i(v)$ (symmetry).

(iii) $\sum_{i \in N} \varphi_i(v) = \sum_{S \subseteq N} v(S)$.

(iv) *Additivity.*

DEFINITION 4 (L. M. Ruiz, F. Valenciano and J. M. Zarzuelo (1996)). The *least square prenucleolus* of game v is a preimputation x of this game such that

$$\sum_{i=1}^{2^n} (r_i(v,x) - \overline{r(v,x)})^2 = \min_{y \in \mathbb{P}} \sum_{i=1}^{2^n} (r_i(v,y) - \overline{r(v,y)})^2$$

where $\overline{r(v,x)}$ denotes the arithmetic mean of the coordinates of $r(v,x)$. The ith coordinate of this solution is said to be the *least square prenucleolity value* and is denoted by $L_i(v)$, $i = 1, \ldots, n$.

L. M. Ruiz *et al.* (1996) proved that for any n-person game v and any player $i \in N$ the least square prenucleolus can be expressed by the formula

$$L_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}} \Big(n g_i(v) - \sum_{d \in N} g_d(v) \Big), \tag{1}$$

where $g_i(v) = \sum_{K\subseteq N,\, i\in K} v(K)$, or equivalently

$$\text{(1a)} \qquad L_i(v) = \frac{v(N)}{n} + \frac{1}{n2^{n-2}}\Big(\sum_{\substack{K\subseteq N \\ i\in K}} (n-k)v(K) - \sum_{\substack{K\subseteq N \\ i\notin K}} kv(K)\Big)$$

as well as

$$L_i(v) = B_i(v) + \frac{v(N) - \sum_{d=1}^{n} B_d(v)}{n}.$$

This last equality shows that the least square prenucleolity value is an additive normalization of the Banzhaf value.

In L. M. Ruiz, F. Valenciano and J. M. Zarzuelo (1996) the following axiomatic theorem was proved:

THEOREM 3. *The solution $\varphi(v) = L(v) = (L_1(v), \ldots, L_n(v))$ is the unique solution on G_N which satisfies the following axioms*:

(i) $\varphi(v)$ *is a preimputation of the game v* (efficiency).

(ii) *Additivity.*

(iii) *For any additive game v and any player i of this game*, $\varphi_i(v) = v(\{i\})$ (inessential game property).

(iv) *For any players i, j of a game v, $g_i(v) \le g_j(v)$ implies $\varphi_i(v) \le \varphi_j(v)$* (average marginal contribution monotonicity).

Now we will present a generalization of the construction of solution with a priori unions proposed by G. Owen (1977 and 1981) and applied to define the Shapley and the Banzhaf values with a priori unions, respectively. But first we define the a priori unions structure.

DEFINITION 5. Let m be a natural number not greater than n. A system of m subsets of N, $T = (T_1, \ldots, T_m)$, is called an *a priori unions structure* if it is a division of the set N, i.e. $\emptyset \neq T_i \subseteq N$ for any $i \in M = \{1, \ldots, m\}$, $T_i \cap T_j = \emptyset$ whenever $i \neq j$, $i, j \in M$ and $\bigcup_{j=1}^{m} T_j = N$. The sets T_i, $i = 1, \ldots, m$, are said to be *a priori unions.*

DEFINITION 6. The *division game* based on the cooperative game v is a game v^* whose players are the a priori unions T_i of the structure T (or, equivalently, elements $i \in M$), and its characteristic function is given as

$$v^*(S) = v\Big(\bigcup_{c\in S} T_c\Big) \quad \text{for any } S \subseteq M.$$

CONSTRUCTION 1. Let φ be a solution on G_N for any natural number n. Suppose that for the n-person cooperative game v an a priori unions structure T is defined. A solution $\varphi(v, T)$ with a priori unions can be constructed in two steps:

Step 1. Let $j \in M$, let K be a subset of T_j, and set $K' = T_j \setminus K$. Consider the game $v_{T,K}$ whose players are the a priori unions of T and

$$v_{T,K}(S) = v\Big(\bigcup_{c\in S} T_c \setminus K'\Big) \quad \text{for any } S \subseteq M.$$

Let v_j be a game with the set of players T_j such that $v_j(K) = \varphi_j(v_{T,K})$ for any set $K \subseteq T_j$.

Step 2. We construct a solution $\varphi(v_j)$ of the game v_j.

The solution $\varphi(v,T) = (\varphi_1(v,T),\ldots,\varphi_n(v,T))$, where $\varphi_i(v,T) = \varphi_i(v_j)$ for every $i \in T_j$ and $j \in M$, is called the *solution φ with a priori unions*.

Note that if $\varphi(v)$ is a preimputation of any n-person game v, then $\varphi(v,T)$ is also a preimputation of this game. Indeed, in this case, $\varphi(v_j)$ is a preimputation of v_j for any $j \in M$ and hence

$$\sum_{i\in T_j} \varphi_i(v,T) = \sum_{i\in T_j} \varphi_i(v_j) = v_j(T_j) = \varphi_j(v_{T,T_j}) = \varphi_j(v^*).$$

Thus

$$\sum_{i=1}^{n} \varphi_i(v,T) = \sum_{j\in M}\sum_{i\in T_j} \varphi_i(v,T) = \sum_{j\in M} \varphi_j(v^*) = v^*(M) = v(N).$$

Definition 7 (G. Owen (1981)). The ith coordinate of the solution with a priori unions (see Construction 1) for

$$\varphi(v) = B(v) = (B_1(v),\ldots,B_n(v))$$

is said to be the *Banzhaf value with a priori unions* of player $i \in T_j$ of game v. It is denoted by $B_i(v,T)$ and can be expressed by the formula

$$B_i(v,T) = \frac{1}{2^{m-1}}\,\frac{1}{2^{t_j-1}} \sum_{S\subseteq M}\sum_{K\subseteq T_j} [v(Q_S \cup K \cup \{i\}) - v(Q_S \cup K)] \tag{2}$$

where $Q_S = \bigcup_{c\in S} T_c$ for any $S \subseteq M$.

Definition 8. The ith coordinate of the solution with a priori unions for $\varphi(v) = \mathrm{DP}(v) = (\mathrm{DP}_1(v),\ldots,\mathrm{DP}_n(v))$ is said to be the *Deegan–Packel value with a priori unions* of player $i \in T_j$ of game v and is denoted by $\mathrm{DP}_i(v,T)$.

Definition 9. The ith coordinate of the solution with a priori unions for $\varphi(v) = L(v) = (L_1(v),\ldots,L_n(v))$, i.e. for the least square prenucleolus, is said to be the *least square prenucleolity value with a priori unions* of player $i \in T_j$ of game v and is denoted by $L_i(v,T)$. The vector $L(v,T) = (L_1(v,T),\ldots,L_n(v,T))$ is called the *least square prenucleolus with a priori unions* of game v.

II. An axiomatization of the Banzhaf value with a priori unions. By application of the axioms in Theorem 1 (after some modification) and introduction of a new important balance property one can formulate a consistent system of axioms which characterizes the Banzhaf value with a priori unions.

In the case of the solution with a priori unions the additivity axiom is formulated as follows:

Additivity with a priori unions:

$$\varphi(v,T)+\varphi(w,T)=\varphi(v+w,T)$$

for any n-person games v, w with the same a priori unions structure T.

Set $T_{(i)}=(T_1,\dots,T_{j-1},\,T_j\setminus\{i\},\,T_{j+1},\dots,T_m,\{i\})$ for every $i\in T_j$ and $j\in M$.

THEOREM 4. *The solution φ with a priori unions on G_N with given structure $T=(T_1,\dots,T_m)$ satisfies*

$$\varphi(v,T)=B(v,T)=(B_1(v,T),\dots,B_n(v,T))$$

if and only if the following conditions hold for any $v\in G_N$:

(i) *If $i\in N$ and $v(S\cup\{i\})=v(S)+v(\{i\})$ for any $S\subseteq N\setminus\{i\}$ then $\varphi_i(v,T)=v(\{i\})$.*

(ii) *If $j\in M=\{1,\dots,m\}$, $a,b\in T_j$, $a\neq b$ and $v(S\cup\{a\})=v(S\cup\{b\})$ for every $S\subseteq N\setminus\{a,b\}$ then $\varphi_a(v,T)=\varphi_b(v,T)$.*

(iii) *For any $j\in M$ and $a,b\in T_j$,*

$$\varphi_a(v,T)-\varphi_a(v,T_{(b)})=\varphi_b(v,T)-\varphi_b(v,T_{(a)}).$$

(iv) *For any $j\in M$ and two different players $a,b\in T_j$,*

$$\varphi_p(v_{(ab)},T_{[ab]})=\varphi_a(v,T)+\varphi_b(v,T),$$

where $T_{[ab]}=(T_1,\dots,T_{j-1},T_{j[ab]},T_{j+1},\dots,T_m)$, $T_{j[ab]}=T_j\setminus\{a,b\}\cup\{\{a,b\}\}$.

(v) *Additivity with a priori unions.*

Axiom (i) (*dummy player axiom*) is identical as in the case of the "normal" Banzhaf value (see Theorem 2); *equal treatment* and *amalgamation* properties ((ii) and (iv)) are here restricted to the members of the same a priori union. A new condition is (iii), called the *balance property.* This axiom reflects the fact that breaking up cooperation between two individuals should affect both individuals equally.

The balance axiom was formulated by M. Vázquez-Brage, A. van den Nouweland and I. García-Jurado (1997), who used it to axiomatize the Shapley value in "airport" games (a special case of cooperative games).

Proof of Theorem 4. Let v be any n-person cooperative game with the a priori unions structure T. The solution $B(v,T) = (B_1(v,T),\dots,B_n(v,T))$ satisfies axioms (i)–(v). The proof of the dummy player, equilibrium, balance and additivity properties follows directly from formula (2).

Suppose that $j \in M$ and two different players $a, b \in T_j$ have been amalgamated. Then from Theorem 1 and the construction of the Banzhaf value with a priori unions we obtain

$$B_a(v,T) + B_b(v,T) = B_a(v_j) + B_b(v_j) = B_p(v_{j(ab)}).$$

Note that $B_p(v_{j(ab)}) = B_p(v_{(ab)j})$, because the characteristic functions of both games are identical. Thus

$$B_a(v,T) + B_b(v,T) = B_p(v_{j(ab)}) = B_p(v_{(ab)j}) = B_p(v_{(ab)}, T_{[ab]}).$$

In this way we proved that the amalgamation axiom is satisfied by the Banzhaf value with a priori unions.

Now we must prove the sufficiency of properties (i)–(v). Let T_0 and $T_\#$ be the trivial a priori unions structures, i.e. T_0 consists only of one-player a priori unions and in $T_\#$ the only a priori union is the set of all players. Then $\gamma(v,T_0) = \gamma(v,T_\#) = \gamma(v)$ for any solution γ of an n-person game v with a priori unions structure T.

Suppose that two different solutions $\varphi^{(1)}(v,T)$ and $\varphi^{(2)}(v,T)$ of game v with structure T satisfy axioms (i)–(v). Backward induction on the number of a priori unions (i.e. on m) and the forward induction on the number of members of the given a priori union $T_j \in T$ (i.e. on t_j) will be used. From properties (i), (ii), (iv), (v) and Theorem 1 it follows that $\varphi^{(1)}(v,T_0) = \varphi^{(2)}(v,T_0) = B(v,T_0) = B(v)$.

Suppose that for any a priori unions structure T with $m+1$ unions we have the equality $\varphi^{(1)}(v,T) = \varphi^{(2)}(v,T)$. We will prove that if the number of a priori unions is m then the solutions $\varphi^{(1)}(v,T)$ and $\varphi^{(2)}(v,T)$ are also identical.

Let T_j be an a priori union chosen from T for some $j \in M$. Two cases are possible:

1) $\operatorname{card}(T_j) = t_j = 1$. From the above it follows that $\varphi^{(1)}(v,T) = \varphi^{(1)}(v^*,T_0) = \varphi^{(2)}(v^*,T_0) = \varphi^{(2)}(v,T)$.

2) $t_j \geq 2$. Let a, b be any two different players of T_j. Then by property (iii) for $k = 1, 2$ we have

$$\varphi_a^{(k)}(v,T) - \varphi_a^{(k)}(v,T_{(b)}) = \varphi_b^{(k)}(v,T) - \varphi_b^{(k)}(v,T_{(a)}).$$

By the induction hypothesis,

$$\varphi_a^{(1)}(v,T) - \varphi_a^{(2)}(v,T) = \varphi_b^{(1)}(v,T) - \varphi_b^{(2)}(v,T).$$

This means that there exists a constant λ such that for any $a \in T_j$,

(3) $$\varphi_a^{(1)}(v,T) - \varphi_a^{(2)}(v,T) = \lambda.$$

Now we argue by induction on t_j. When $t_j = 2$, then as a result of amalgamation of players a, b we obtain

$$\varphi_p^{(1)}(v_{(ab)}, T_{[ab]}) = \varphi_j^{(1)}(v^*_{(ab)}, T_0) = \varphi_j^{(2)}(v^*_{(ab)}, T_0) = \varphi_p^{(2)}(v_{(ab)}, T_{[ab]});$$

and by (ii),

$$\varphi_a^{(1)}(v,T) - \varphi_a^{(2)}(v,T) + \varphi_b^{(1)}(v,T) - \varphi_b^{(2)}(v,T) = 0,$$

hence $\lambda = 0$. It follows that in this case $\varphi^{(1)}(v,T) = \varphi^{(2)}(v,T)$.

3) If the cardinality of T_j is $t_j - 1$, then for any $a \in T_j$,

$$\varphi_a^{(1)}(v,T) = \varphi_a^{(2)}(v,T).$$

Suppose that $\operatorname{card}(T_j) = t_j \geq 3$ and a, b, c are three different players which belong to T_j. Thus, by (iii) (and for any $k = 1, 2$),

$$\varphi_p^{(k)}(v_{(ab)}, T_{[ab]}) - \varphi_p^{(k)}(v_{(ab)}, T_{[ab](c)}) = \varphi_c^{(k)}(v_{(ab)}, T_{[ab]}) - \varphi_c^{(k)}(v_{(ab)}, T_{[ab](p)}).$$

Therefore, by the induction hypothesis,

$$\varphi_p^{(1)}(v_{(ab)}, T_{[ab]}) - \varphi_p^{(2)}(v_{(ab)}, T_{[ab]}) = \varphi_c^{(1)}(v_{(ab)}, T_{[ab]}) - \varphi_c^{(2)}(v_{(ab)}, T_{[ab]}).$$

Hence there exists a constant μ such that for every player c in $T_j \setminus \{a,b\} \cup \{p\}$ we have

$$\varphi_c^{(1)}(v_{(ab)}, T_{[ab]}) - \varphi_c^{(2)}(v_{(ab)}, T_{[ab]}) = \mu.$$

The induction hypothesis implies that $\mu = 0$. Hence and from (iv) we have

$$\begin{aligned}\varphi_a^{(1)}(v,T) + \varphi_b^{(1)}(v,T) &= \varphi_p^{(1)}(v_{(ab)}, T_{[ab]}) = \varphi_p^{(2)}(v_{(ab)}, T_{[ab]}) \\ &= \varphi_a^{(2)}(v,T) + \varphi_b^{(2)}(v,T)\end{aligned}$$

and hence $\lambda = 0$ by (3). Finally, $\varphi^{(1)}(v,T) = \varphi^{(2)}(v,T)$. Thus there exists a unique solution of game v with the a priori unions structure T, which satisfies conditions (i)–(v). By the beginning of this proof this solution is represented by the n-dimensional vector whose coordinates are the Banzhaf values with a priori unions of the particular players of game v, i.e. $\varphi^{(1)}(v,T) = \varphi^{(2)}(v,T) = B(v,T)$. In this way the proof of Theorem 4 is complete. ∎

III. Properties of the Deegan–Packel value with a priori unions. We adopt the convention that the cardinality of a set is denoted by the corresponding lower case letter, e.g. $\operatorname{card}(K) = k$ etc.

According to Construction 1, the Deegan–Packel value with a priori unions of any player i which belongs to the a priori union T_j, $j \in M$, can be expressed by the formula

$$\mathrm{DP}_i(v,T)=\mathrm{DP}_i(v_j)=\sum_{\substack{K\subseteq T_j\\ i\in K}}\frac{v_j(K)}{k}=\sum_{\substack{K\subseteq T_j\\ i\in K}}\sum_{\substack{S\subseteq M\\ j\in S}}\frac{v(Q_S\setminus T_j\cup K)}{ks}.$$

In our axiomatization of this solution (similarly to the case of the Shapley value with a priori unions, cf. G. Owen (1977)) symmetry axioms of two types are formulated: symmetry under permutation of the players of N and under permutation of the a priori unions structure.

THEOREM 5. *A solution φ with a priori unions on G_N with given structure $T=(T_1,\dots,T_m)$ satisfies*

$$\varphi(v,T)=\mathrm{DP}(v,T)=(\mathrm{DP}_1(v,T),\dots,\mathrm{DP}_n(v,T))$$

if and only if the following conditions hold for any $v\in G_N$:

(i) *If $i\in N$ is a zero-player then $\varphi_i(v,T)=0$* (zero-player axiom).

(ii) *For every constant $c\in\mathbb{R}$ and for any $i\in N$, $\varphi_i(cv,T)=c\varphi_i(v,T)$, where $(cv)(K)=cv(K)$ for any $K\subseteq N$.*

(iii) *For every permutation σ of N and for any $i\in N$, $\varphi_i(v,T)=\varphi_{\sigma(i)}(\sigma v,\sigma T)$, where $\sigma T=(\sigma T_1,\dots,\sigma T_m)$* (symmetry with respect to players).

(iv) *For every permutation ϱ of M and for any $i\in N$, $\varphi_i(v,T)=\varphi_i(v,T_\varrho)$, where $T_\varrho=(T_{\varrho(1)},\dots,T_{\varrho(m)})$* (symmetry with respect to a priori unions).

(v) *If ω is a game with the set of players N such that $\omega(Q_S\setminus T_j\cup K)=sk\cdot v(Q_S\setminus T_j\cup K)$ for all $\emptyset\neq S\subseteq M$, $\emptyset\neq K\subseteq T_j$, and $j\in M$ then*

$$\varphi_i(\omega,T)=\sum_{\substack{S\subseteq M\\ j\in S}}\sum_{\substack{K\subseteq T_j\\ i\in K}}v(Q_S\setminus T_j\cup K)\quad \textit{for every } i\in T_j.$$

(vi) *Additivity with a priori unions.*

This system of axioms is similar to the set of axioms for the Shapley value with a priori unions (cf. G. Owen (1977)). The main difference is axiom (iv). Because in Construction 1 the coalitions of non-trivial subsets of different a priori unions (i.e. $P_1\cup\ldots\cup P_m$, where $P_j\subsetneq T_j$, $j=1,\dots,m$, and at least two sets of $P_1,\dots,P_m$ are non-empty) are ignored, the Deegan–Packel value with a priori unions for all players must also depend only on the worth of the sets $Q_S\setminus K'$, $S\subseteq M$, $K\subseteq T_j$, $j\in M$. Axiom (v) also shows that in the game in which the worth of a coalition depends only on its cardinality, the analyzed solution depends only on the real value of each player and the a priori union which includes him.

Proof of Theorem 5. The solution $\mathrm{DP}(v,T)$ satisfies all the above-mentioned axioms. Conversely, suppose that $\varphi(v,T)$ is a solution of an n-person game v with a priori unions structure T which satisfies axioms (i)–(v). Note (cf. G. Owen (1977)) that any game v can be uniquely represented as a linear combination of the n-person games w^R $(R \subseteq N)$ with the same a priori unions structure T and the characteristic function defined as

$$w^R(K) = \begin{cases} 1 & \text{if } K = R, \\ 0 & \text{if } K \neq R, \end{cases} \qquad \text{for any } K \subseteq N.$$

We can restrict ourselves to those games w^R in which $R = Q_S \setminus T_j \cup K$ for any $\emptyset \neq S \subseteq M$, $\emptyset \neq K \subseteq T_j$, and $j \in M$. In other cases we have $\varphi_i(w^R, T) = 0$ for any $i \in N$, by (i)–(iv).

So, let $j \in S$ and $R = Q_S \setminus T_j \cup K$ for any fixed $\emptyset \neq S \subseteq M$, $\emptyset \neq K \subseteq T_j$, and $j \in M$ (if $j \notin S$ then $R = Q_{S\cup\{j\}} \setminus T_j \cup K$). Then by (i), (iii) and (iv) there exist constants α_K, β_S such that

$$\varphi_i(w^R, T) = \begin{cases} \alpha_K \beta_S & \text{if } i \in K, \\ 0 & \text{if } i \notin K, \end{cases} \qquad \text{for any } i \in T_j,$$

and $\varphi_i(w^R, T) = 0$ for any $i \in N \setminus T_j$.

Consider a game ω with the set of players N and with the a priori unions structure T, whose characteristic function is defined as follows:

$$\omega(U) = \begin{cases} z \cdot p \cdot w^R(U) & \text{if } U = Q_Z \setminus T_j \cup P,\ \emptyset \neq Z \subseteq M,\ j \in Z,\ P \subseteq T_j, \\ 0 & \text{otherwise.} \end{cases}$$

By (v),

$$\varphi_i(\omega, T) = \sum_{\substack{Z\subseteq M \\ j\in Z}} \sum_{\substack{P\subseteq T_j \\ i\in P}} w^R(Q_Z \setminus T_j \cup P) = 1.$$

On the other hand, we have $\omega(U) = sk \cdot w^R(U)$ for any $U \subseteq N$ and by (ii),

$$\varphi_i(\omega, T) = \varphi_i(sk \cdot w^R, T) = sk \cdot \varphi_i(w^R, T) = sk \cdot \alpha_K \beta_S$$

for any $i \in T_j$, $j \in M$. Therefore

$$\alpha_K \beta_S = \frac{1}{sk}$$

and for any constant c and game cw^R, where

$$cw^R(U) = \begin{cases} c & \text{if } U = R, \\ 0 & \text{if } U \neq R, \end{cases} \qquad \text{for any } U \subseteq N,$$

we have

$$\varphi_i(cw^R, T) = \begin{cases} c/(sk) & \text{if } i \in K \text{ and } j \in S, \\ 0 & \text{if } i \notin K \text{ or } j \notin S, \end{cases} \qquad \text{for any } i \in T_j,$$

$$\varphi_i(cw^R, T) = 0 \qquad \text{for any } i \in N \setminus T_j.$$

In the other cases $\varphi_i(cw^R, T) = 0$ for any $i \in N$.

By the additivity axiom (vi) we obtain finally

$$\begin{aligned}\varphi_i(v,T) &= \varphi_i\Big(\sum_{R\subseteq N} v(R)w^R, T\Big)\\ &= \sum_{\substack{S\subseteq M\\ j\in S}} \sum_{\substack{K\subseteq T_j\\ i\in K}} v(Q_S\setminus T_j\cup K)\varphi_i(w^{Q_S\setminus T_j\cup K}, T)\\ &= \sum_{\substack{S\subseteq M\\ j\in S}} \sum_{\substack{K\subseteq T_j\\ i\in K}} \frac{v(Q_S\setminus T_j\cup K)}{sk} = \mathrm{DP}_i(v,T)\end{aligned}$$

for any $i\in T_j$ and $j\in M$. ■

On the basis of the above-mentioned results we can formulate two remarks. First, the solution $\mathrm{DP}(v,T)$ satisfies the balance axiom. Namely, for any $a,b\in T_j$ and $j\in M$ we have

$$\begin{aligned}&\mathrm{DP}_a(v,T)-\mathrm{DP}_a(v,T_{(b)})\\ &\quad= \sum_{\substack{K\subseteq T_j\\ a\in K}} \sum_{\substack{S\subseteq M\\ j\in S}} \frac{v(Q_S\setminus T_j\cup K)}{sk} - \sum_{\substack{K\subseteq T_j\setminus\{b\}\\ a\in K}} \sum_{\substack{S\subseteq M_\bullet\\ j\in S}} \frac{v(Q_S\setminus (T_j\setminus\{b\})\cup K)}{sk}\\ &\quad= \sum_{\substack{K\subseteq T_j\\ a,b\in K}} \sum_{\substack{S\subseteq M\\ j\in S}} \frac{v(Q_S\setminus T_j\cup K)}{sk} - \sum_{\substack{K\subseteq T_j\\ a,b\in K}} \sum_{\substack{S\subseteq M\\ j\in S}} \frac{v(Q_S\setminus T_j\cup K)}{(s+1)(k-1)}\\ &\quad= \sum_{\substack{K\subseteq T_j\\ a,b\in K}} \sum_{\substack{S\subseteq M\\ j\in S}} v(Q_S\setminus T_j\cup K)\cdot\left(\frac{1}{sk}-\frac{1}{(s+1)(k-1)}\right)\\ &\quad= \sum_{\substack{K\subseteq T_j\\ b\in K}} \sum_{\substack{S\subseteq M\\ j\in S}} \frac{v(Q_S\setminus T_j\cup K)}{sk} - \sum_{\substack{K\subseteq T_j\setminus\{a\}\\ b\in K}} \sum_{\substack{S\subseteq M_\bullet\\ j\in S}} \frac{v(Q_S\setminus (T_j\setminus\{a\})\cup K)}{sk}\\ &\quad= \mathrm{DP}_b(v,T)-\mathrm{DP}_b(v,T_{(a)}),\end{aligned}$$

where $T_{(i)}=(T_1,\ldots,T_{j-1},T_{j(i)},T_{j+1},\ldots,T_m,T_{m+1})$, $T_{j(i)}=T_j\setminus\{i\}$, $T_{m+1}=\{i\}$, $i=a,b$, $M_\bullet=\{1,\ldots,m+1\}$.

In the case of the "normal" Deegan–Packel value axiom (iv) is equivalent to the following property: if ϑ is a game with the set of players N such that $\vartheta(S)=s\cdot v(S)$ for any $S\subseteq N$, then

$$\varphi_i(\vartheta)=\sum_{\substack{S\subseteq N\\ i\in S}} v(S)\quad \text{for any } i\in N.$$

This property seems weaker than axiom (iii) of Theorem 2, because it concerns only some specifically defined type of games.

IV. An axiomatic characterization of the least square prenucleolus with a priori unions. Now we will present the most important properties of the least square prenucleolus with a priori unions, which are necessary to formulate the axiomatic theorem regarding this solution.

LEMMA 1. *For any a priori union T_j which belongs to the structure T and any $i \in T_j$ the least prenucleolity value of player i can be expressed as*:

$$
\begin{aligned}
(4)\qquad L_i(v,T) &= \frac{v(N)}{mt_j} + \frac{1}{mt_j 2^{m-2}}\Big[mg_j(v^*) - \sum_{c\in M} g_c(v^*)\Big] \\
&+ \frac{1}{t_j 2^{t_j-2}} \\
&\times \Big[t_j \sum_{\substack{K\subseteq T_j\\ i\in K}} \Big(\frac{v(N\setminus T_j\cup K)}{m} + \frac{1}{m2^{m-2}}\Big(mg_j(v_{T,K}) - \sum_{c\in M} g_c(v_{T,K})\Big)\Big) \\
&- \sum_{i\in T_j} \sum_{\substack{K\subseteq T_j\\ i\in K}} \Big(\frac{v(N\setminus T_j\cup K)}{m} + \frac{1}{m2^{m-2}}\Big(mg_j(v_{T,K}) - \sum_{c\in M} g_c(v_{T,K})\Big)\Big)\Big]
\end{aligned}
$$

where $t_j = \operatorname{card}(T_j)$ for any $j \in M$.

Proof. The above formula can be obtained from formula (1) applied to game v_j:

$$L_i(v_j) = \frac{v_j(T_j)}{t_j} + \frac{1}{t_j 2^{t_j-2}}\Big(t_j g_i(v_j) - \sum_{i\in T_j} g_i(v_j)\Big),$$

where

$$g_i(v_j) = \sum_{\substack{K\subseteq T_j\\ i\in K}} v_j(K).$$

Next we replace the values $v_j(K)$ for any $K \subseteq T_j$ with the formula calculated also on the basis of (1):

$$L_j(v_{T,K}) = \frac{v_{T,K}(M)}{m} + \frac{1}{m2^{m-2}}\Big(mg_j(v_{T,K}) - \sum_{j\in M} g_j(v_{T,K})\Big).$$

This is the least square prenucleolity value of player j of game $v_{T,K}$. ■

LEMMA 2. *For any player $i \in N$ of an additive game v with a priori unions structure T, $L_i(v,T) = v(\{i\})$.*

Proof. If v is an additive game, then so is $v_{T,K}$ (for any $K \subseteq T_j$ and any $j \in M$). Hence from Theorem 3, $L_j(v_{T,K}) = v_{T,K}(\{j\}) = v(K)$. Thus $v_j(K) = v(K)$, and v_j is also an additive game. Finally, it follows that $L_i(v,T) = v_j(\{i\}) = v(\{i\})$. ∎

THEOREM 6. *The least square prenucleolus with a priori unions is the unique solution φ with a priori unions on G_N with given structure $T = (T_1, \ldots, T_m)$ which satisfies the following conditions for every $v \in G_N$:*

(i) $\varphi(v,T)$ *is a preimputation of game* v (efficiency).
(ii) *Additivity with a priori unions.*
(iii) *For any permutation σ of N and all $i \in N$, $\varphi_i(v,T) = \varphi_{\sigma(i)}(\sigma v, \sigma T)$.*
(iv) *For any permutation ϱ of M and for any $i \in N$, $\varphi_i(v,T) = \varphi_i(v,T_\varrho)$.*
(v) *If v is an additive game then $\varphi_i(v,T) = v(\{i\})$ for any $i \in N$.*

Proof. The least square prenucleolus with a priori unions satisfies axioms (i)–(v). This is a consequence of the relevant properties of the "normal" least square prenucleolus as well as of Construction 1 and Lemmas 1 and 2.

Suppose that a solution $\varphi(v,T)$ satisfies (i)–(v). Define

$$\psi_j = \sum_{i \in T_j} \varphi_i(v,T) \quad \text{for any } j \in M.$$

Because by (i), $\psi = (\psi_1, \ldots, \psi_m)$ is a preimputation of v^*, it can be understood as a solution of game v^*, and denoted by $\psi(v^*) = (\psi_1(v^*), \ldots, \psi_m(v^*))$.

For any $R \subseteq N$ consider the n-person game w^R with a priori unions structure T whose characteristic function is defined as follows:

$$w^R(K) = \begin{cases} 1 & \text{if } R = K, \\ 0 & \text{if } R \neq K, \end{cases} \qquad \text{for any } K \subseteq N.$$

As mentioned in the proof of Theorem 5, every n-person game v with a priori unions structure T can be uniquely represented as a linear combination of the games w^R.

In our case the considerations can be restricted to those games w^R for which $R = Q_S \setminus T_j \cup K$, $K \subseteq T_j$ and $S \subseteq M = \{1, \ldots, m\}$ (in the other cases, from (i), (iii) and (iv) we obtain $\varphi_i(w^R,T) = 0$ for any $i \in N$). Note that for any $R \subseteq N$ and any $i \in N \setminus T_j$ we have $\varphi_i(w^R,T) = 0$.

Thus from (i) and (iv) it follows that there exists a non-negative constant ξ_S such that if $R \neq N$ then

$$\psi_j((w^R)^*) = \begin{cases} \dfrac{\xi_S}{s} & \text{if } j \in S, \\[2ex] \dfrac{-\xi_S}{m-s} & \text{if } j \notin S, \end{cases}$$

and $\psi_j((w^N)^*) = 1/m$ for any $j \in M$. Note that $\xi_S = 0$ if $S = \emptyset$. Therefore from the additivity axiom we conclude that for any two disjoint subsets S_1

and S_2 of M we have the equality

$$\psi_j((w^{Q_{S_1}})^* + (w^{Q_{S_2}})^*) = \psi_j((w^{Q_{S_1}})^*) + \psi_j((w^{Q_{S_2}})^*)$$
$$= \begin{cases} \dfrac{\xi_{S_1}}{s_1} - \dfrac{\xi_{S_2}}{m-s_2} & \text{if } i \in S_1, \\ \dfrac{\xi_{S_2}}{s_2} - \dfrac{\xi_{S_1}}{m-s_1} & \text{if } i \in S_2. \end{cases}$$

By symmetry we have $\psi_{j_1}((w^{Q_{S_1}})^* + (w^{Q_{S_2}})^*) = \psi_{j_2}((w^{Q_{S_1}})^* + (w^{Q_{S_2}})^*)$ for all $j_p \in S_p$, $p = 1, 2$, so that $\xi_{S_1}/s_1 - \xi_{S_2}/(m-s_2) = \xi_{S_2}/s_2 - \xi_{S_1}/(m-s_1)$ and

$$\xi_{S_1}/(s_1(n-s_1)) = \xi_{S_2}/(s_2(m-s_2)). \tag{5}$$

This relation holds for any two nonempty coalitions different from M. If $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cup S_2 \neq M$ then (5) applies to S_1 and $M \setminus (S_1 \cup S_2)$ as well as to S_2 and $M \setminus (S_1 \cup S_2)$, and hence to S_1, S_2. In the case of $S_1 \cap S_2 \neq \emptyset$ and $S_1 \cup S_2 = M$, the relation (5) applies to disjoint sets S_1 and $M \setminus S_1$ as well as to S_2 and $M \setminus S_2$, and therefore to $M \setminus S_1$ and $M \setminus S_2$, as well as to S_1 and S_2. Define $\xi = \xi_S/(s(m-s))$. Thus

$$\begin{aligned} \psi_j(v^*) &= \psi_j\Big(\sum_{S\subseteq M} v^*(S)\cdot (w^S)^*\Big) = \sum_{S\subseteq M} v^*(S)\psi_j((w^S)^*) \\ &= v^*(M)\psi_j((w^N)^*) + \sum_{\substack{S\subset M\\ j\in S}} \frac{\xi_S}{s}\, v^*(S) - \sum_{\substack{S\subset M\\ j\notin S}} \frac{\xi_S}{m-s}\, v^*(S) \\ &= \frac{v^*(M)}{m} + \xi\Big(\sum_{\substack{S\subset M\\ j\in S}} (m-s)\cdot v^*(S) - \sum_{\substack{S\subset M\\ j\notin S}} s\cdot v^*(S)\Big). \end{aligned}$$

Consider the additive game v^*. Then from (v) we conclude that $\xi = 1/(m2^{m-2})$ and

$$\sum_{i\in T_j} \varphi_i(v,T) = L_j(v^*) \quad \text{for every } j \in M. \tag{6}$$

From (ii) and (iii) it follows that there exist positive constants α_K, β_S such that if $S, K \neq \emptyset$, $S \neq M$, $K \neq T_j$ then

$$\varphi_i(w^R, T) = \begin{cases} \dfrac{\alpha_K}{k}\,\dfrac{\beta_S}{s} & \text{if } i \in K,\ j \in S, \\ \dfrac{-\alpha_K}{t_j - k}\,\dfrac{\beta_S}{s} & \text{if } i \notin K,\ j \in S. \end{cases}$$

In particular, if $K \neq \emptyset$, T_j and $S = M$ then

$$\varphi_i(w^R, T) = \begin{cases} \dfrac{\alpha_K}{mt_j} & \text{if } i \in K, \\ \dfrac{-\alpha_K}{m(t_j - k)} & \text{if } i \notin K. \end{cases}$$

If $K = \emptyset$ and $S \setminus \{j\} \neq \emptyset$ or if $K = T_j$ and $S \neq M$ then

$$\varphi_i(w^R, T) = \begin{cases} \dfrac{\beta_S}{st_j} & \text{if } j \in S,\ K = T_j, \\ \dfrac{\beta_{S\cup\{j\}}}{(s+1)t_j} & \text{if } j \notin S,\ K = T_j, \\ \dfrac{-\beta_S}{(m-s)t_j} & \text{if } j \notin S,\ K = \emptyset, \\ \dfrac{-\beta_{S\setminus\{j\}}}{(m-s-1)t_j} & \text{if } j \in S,\ K = \emptyset. \end{cases} \tag{7}$$

Note moreover that if $S = M$ and $K = T_j$ then $\varphi_i(w^R, T) = 1/(mt_j)$. Of course, we have $\varphi_i(w^R, T) = 0$ if $R = \emptyset$.

Consider two sets: $R_1 = Q_S \setminus T_j \cup K_1$, $R_2 = Q_S \setminus T_j \cup K_2$, where $S \neq \emptyset$, $K_i \neq \emptyset$, $i = 1, 2$, $K_1 \cap K_2 = \emptyset$. Thus, by additivity with a priori unions,

$$\varphi_i(w^{R_1} + w^{R_2}, T) = \varphi_i(w^{R_1}, T) + \varphi_i(w^{R_2}, T) \quad \text{for any } i \in T_j$$

and hence

$$\varphi_i(w^{R_1} + w^{R_2}, T) = \begin{cases} \dfrac{\alpha_{K_1}\beta_S}{k_1 s} + \dfrac{-\alpha_{K_2}\beta_S}{(t_j - k_2)s} & \text{if } i \in K_1,\ j \in S, \\ \dfrac{-\alpha_{K_1}\beta_S}{(t_j - k_1)s} + \dfrac{\alpha_{K_2}\beta_S}{k_2 s} & \text{if } i \in K_2,\ j \in S. \end{cases}$$

From (ii) and (iii) it follows that

$$\varphi_{i_1}(w^{R_1} + w^{R_2}, T) = \varphi_{i_2}(w^{R_1} + w^{R_2}, T) \quad \text{for } i_p \in K_p,\ p = 1, 2. \tag{8}$$

From (6) and (7) we conclude that

$$\beta_S = \frac{1}{m2^{m-2}}\, s(m-s) \quad \text{for any } S \subseteq M.$$

In this way

$$\beta_{S_1}/(s_1(m - s_1)) = \beta_{S_2}/(s_2(m - s_2)) \quad \text{for any } S_1, S_2 \subseteq M. \tag{9}$$

Note that from (8) and (9) we have

$$\frac{\beta_S}{s}\left(\frac{\alpha_{K_1}}{k_1} + \frac{\alpha_{K_1}}{t_j - k_1}\right) - \frac{\beta_S}{s}\left(\frac{\alpha_{K_2}}{t_j - k_2} + \frac{\alpha_{K_2}}{k_2}\right) = 0 \quad \text{for } j \in S.$$

We know from (9) that $\beta_S/(s(m-s))$ is independent of the set S. Define $\beta = \beta_S/(s(m-s))$. Since $\beta = \beta_S/(s(m-s))$, we obtain

$$\beta m\left(\frac{\alpha_{K_1}}{k_1} + \frac{\alpha_{K_1}}{t_j - k_1}\right) = \beta m\left(\frac{\alpha_{K_2}}{t_j - k_2} + \frac{\alpha_{K_2}}{k_2}\right)$$

and therefore

$$\alpha_{K_1}/(k_1(t_j - k_1)) = \alpha_{K_2}/(k_2(t_j - k_2)) \quad \text{for any disjoint } K_1, K_2 \subseteq T_j. \tag{10}$$

Hence, (10) holds for any subsets K_1, K_2 of T_j. Indeed, if $K_1 \cap K_2 \neq \emptyset$ and $K_1 \cup K_2 \neq T_j$ then (10) applies to K_1 and $T_j \setminus (K_1 \cup K_2)$, to K_2 and $T_j \setminus (K_1 \cup K_2)$, and hence to K_1, K_2. In the case of $K_1 \cap K_2 \neq \emptyset$ and $K_1 \cup K_2 = T_j$, the relation (10) holds for the disjoint sets K_1 and $T_j \setminus K_1$ as well as for K_2 and $T_j \setminus K_2$, and hence for $T_j \setminus K_1$ and $T_j \setminus K_2$, and for K_1 and K_2. Set $\alpha = \alpha_K/(k(t_j - k))$.

One can easily check that $\varphi(cw^R, T) = c\varphi(w^R, T)$ for any constant c.

Finally,

$$\begin{aligned}
\varphi_i(v,T) &= \varphi_i\Big(\sum_{R\subseteq N} v(R)w^R, T\Big) = \sum_{R\subseteq N} v(R)\varphi_i(w^R,T) \\
&= \frac{v(N)}{t_j m} + \frac{1}{t_j}\beta\Big(\sum_{\substack{S\subseteq M\\ j\in S}} (m-s)v(Q_S) - \sum_{\substack{S\subseteq M\\ j\notin S}} sv(Q_S)\Big) \\
&\quad + \alpha\Bigg[\sum_{\substack{K\subseteq T_j\\ i\in K}} (t_j - k)\Bigg(\frac{v(N\setminus T_j \cup K)}{m} \\
&\quad + \beta\Big(\sum_{\substack{S\subseteq M\\ j\in S}} (m-s)v(Q_S\setminus K') - \sum_{\substack{S\subseteq M\\ j\notin S}} sv(Q_S\setminus K')\Big)\Bigg) \\
&\quad - \sum_{\substack{K\subseteq T_j\\ i\notin K}} k\Bigg(\frac{v(N\setminus T_j \cup K)}{m} \\
&\quad + \beta\Big(\sum_{\substack{S\subseteq M\\ j\in S}} (m-s)v(Q_S\setminus K') - \sum_{\substack{S\subseteq M\\ j\notin S}} sv(Q_S\setminus K')\Big)\Bigg)\Bigg]
\end{aligned} \tag{11}$$

for any $i \in T_j$ and $j \in M$. We know that (cf. (6) and (9))

$$\beta = \frac{1}{m2^{m-2}}. \tag{12}$$

According to (11), (12) and condition (v) we can calculate that

$$\alpha = \frac{1}{t_j 2^{t_j - 2}}.$$

The proof of sufficiency of the conditions of the theorem is complete, because from (1a) it follows that for these coefficients α, β, (11) is equivalent to (4). ■

Conditions (iii) and (iv) can be easily replaced with the following two axioms connected with axiom (iv) of Theorem 3 (i.e. with average marginal contribution monotonicity):

1) $g_{i_1}(v,T) \geq g_{i_2}(v,T)$ implies

$$\varphi_{i_1}(v,T) \geq \varphi_{i_2}(v,T) \quad \text{for any } i_1, i_2 \in T_j \text{ and for any } j \in M,$$

2) $g_{j_1}(v^*) \geq g_{j_2}(v^*)$ implies

$$\sum_{i \in T_{j_1}} \varphi_i(v,T) \geq \sum_{i \in T_{j_2}} \varphi_i(v,T) \quad \text{for any } j_1, j_2 \in M.$$

G. Owen (1977) proved that $\mathrm{Sh}(v,T) = (\mathrm{Sh}_1(v,T), \ldots, \mathrm{Sh}_n(v,T))$ where $\mathrm{Sh}_i(v,T)$ is the Shapley value with a priori unions constructed by Construction 1 for $\varphi(v) = \mathrm{Sh}(v) = (\mathrm{Sh}_1(v), \ldots, \mathrm{Sh}_n(v))$ and

$$\mathrm{Sh}_i(v) = \sum_{K \subseteq N} \frac{k!(n-k-1)!}{n!} \left(v(K \cup \{i\}) - v(K)\right) \quad \text{for any } i \in N$$

is the unique solution which satisfies axioms (i)–(iv) in Theorem 6 and the dummy player axiom ((i) in Theorem 1). Hence we can conclude that the main difference between the Shapley value with a priori unions and the least square prenucleolus with a priori unions is that the former solution satisfies a more general dummy player axiom. Therefore, the least square prenucleolus with a priori unions is determined by a system of weaker conditions.

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