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LARGE LOSSES—PROBABILITY MINIMIZING APPROACH

Abstract. The probability minimizing problem for large losses of portfolio in discrete and continuous time models is studied. This gives a generalization of quantile hedging presented in [3].

1. Introduction. Let (S_t) be a d-dimensional semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which represents the stock prices. We denote by $\mathcal{Q}$ the set of all martingale measures, i.e. $Q \in \mathcal{Q}$ if $Q \sim P$ and (S_t) is a martingale with respect to Q. Let H be an $\mathcal{F}$-measurable random variable called a *contingent claim*. It is known that on the market we have two prices: the buyer's price $u_b = \inf_{Q\in\mathcal{Q}} \mathbf{E}^Q[H]$ and the seller's price $u_s = \sup_{Q\in\mathcal{Q}} \mathbf{E}^Q[H]$, which are usually different. A natural question arises: what price from the so called arbitrage-free interval $[u_b, u_s]$ should be chosen? This problem was a motivation for introducing risk measures on financial markets. Various approaches to this question have been presented (see for instance [1]–[4], [6]).

In [3] Föllmer and Leukert study the quantile hedging problem. They define a random variable $\varphi_{x,\pi}$ connected with the strategy (x, π) by

$$\varphi_{x,\pi} = \mathbf{1}_{\{X_T^{x,\pi} \ge H\}} + \frac{X_T^{x,\pi}}{H}\,\mathbf{1}_{\{X_T^{x,\pi} < H\}},$$

where $X_T^{x,\pi}$ is the terminal value of the portfolio connected with the strategy π starting from the initial endowment x. If $x \ge u_s$ then for the hedging strategy $\widetilde{\pi}$ we have $\mathbf{E}[\varphi_{x,\widetilde{\pi}}] = 1$, otherwise $\mathbf{E}[\varphi_{x,\pi}] < 1$ for each π. The aim of the trader is to maximize $\mathbf{E}[\varphi_{x,\pi}]$ over π from the set of all admissible strategies. Actually, the motivation of quantile hedging was a slightly

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different problem, namely

$$P(X_T^{x,\pi} \geq H) \to \max_{\pi}.$$

This problem has been solved by the above approach only in a particular case.

Now assume that the investor has a loss function $u : [0,\infty) \to [0,\infty)$, $u(0) = 0$, which is assumed to be continuous and strictly increasing, and he accepts small losses of the portfolio. This means he has no objections to losses such that $u((H - X_T^{x,\pi})^+) \leq \alpha$, where $\alpha \geq 0$ is a level of acceptable losses fixed by the investor. He wants to avoid losses which exceed α. As the optimality criterion we take maximizing the probability that losses are small. More precisely, the problem is

$$P[u((H - X_T^{x,\pi})^+) \leq \alpha] \to \max_{\pi},$$

where π is an admissible strategy. Notice that for $\alpha = 0$ we obtain the original problem of quantile hedging.

The paper is organized as follows. In Section 2 we precisely formulate the problem. It turns out that the solution on complete markets has a clear economic interpretation. It is presented in Section 3. Sections 4 and 5 provide examples of the Black–Scholes model and the CRR model. For the B-S model an explicit solution is found while for the CRR model existence is clear, but solutions are found for some particular cases. In Section 6 a result for incomplete markets is proved and presented in a one-step trinomial model.

2. Problem formulation. We consider financial markets with either discrete or continuous time and with finite horizon T. Let S_t be a d-dimensional semimartingale describing evolution of stock prices on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. Let $X_t^{x,\pi}$ be the wealth process connected with the pair (x,π), where π is a predictable process describing a self-financing strategy and x is an initial endowment. Thus the wealth process is defined by $X_0^{x,\pi} = x$, $X_t^{x,\pi} = \pi_t \cdot S_t$ and the self-financing condition means that

$$\pi_t \cdot S_t = \pi_{t+1} \cdot S_t$$

in the case of a discrete time model, and

$$dX_t^{x,\pi} = \pi_t dS_t, \qquad \pi \in L(S),$$

in the case of a continuous time model; $L(S)$ is the set of predictable processes integrable with respect to S. For simplicity assume that the interest rate is equal to zero and that the set $\mathcal{Q}$ of all martingale measures Q such that S_t is a martingale with respect to Q and $Q \sim P$, is not empty. Among all self-financing strategies we distinguish the set $\mathcal{A}$ of all admissible strategies which satisfy two additional conditions: $X_t^{x,\pi} \geq 0$ for all t and $X_t^{x,\pi}$ is a supermartingale with respect to each $Q \in \mathcal{Q}$. If $X_t^{x,\pi} \geq 0$ then the second requirement is automatically satisfied for S being a continuous semimartingale, since the wealth process is then a Q-local martingale bounded from below, so by Fatou's lemma it is a supermartingale. In discrete time $X_t^{x,\pi}$ is even a martingale (see [5, Th. 2]). Let H be a nonnegative, $\mathcal{F}_T$-measurable random variable, called a *contingent claim*, which satisfies $H \in L^1(Q)$ for each $Q \in \mathcal{Q}$. Its price at time 0 is given by $v_0 = \sup_{Q\in\mathcal{Q}} \mathbf{E}^Q[H]$. This means that there exists a strategy $\widetilde{\pi} \in \mathcal{A}$ such that $X_T^{v_0,\widetilde{\pi}} \geq H$. Such $\widetilde{\pi}$ is called a *hedging strategy*. Now assume that we are given an initial capital $0 \leq x_0 < v_0$. The question arises: what is an optimal strategy for such an endowment? As an optimality criterion we take minimizing probability of a large loss. Let $u : [0,\infty) \to [0,\infty)$ be a strictly increasing, continuous function such that $u(0) = 0$. Such a function will be called a *loss function*. Let $\alpha \geq 0$ be a level of acceptable losses. We are searching for a pair (x,π) such that

$$P[u((H - X_T^{x,\pi})^+) \leq \alpha] \to \max_{\pi\in\mathcal{A}}, \qquad x \leq x_0.$$

If there exists a solution (x,π) of the above problem, then it will be called *optimal*.

3. Complete models. Let $\mathcal{Q} = \{Q\}$, so the martingale measure is unique. Recall that in this case each nonnegative Q-integrable contingent claim X can be replicated. This means that there exists $\widetilde{\pi}$ such that $X_T^{v_0,\widetilde{\pi}} = X$, where $v_0 = \mathbf{E}^Q[X]$. In the complete case the solution of our problem has a clear economic interpretation. Let us start with the basic theorem describing the solution.

THEOREM 3.1. *If there exists $\widetilde{X} \in L^0_+$ which is a solution of the problem*

$$P[u((H - X)^+) \leq \alpha] \to \max, \qquad \mathbf{E}^Q[X] \leq x_0,$$

then the replicating strategy for $\widetilde{X}$ is optimal.

Proof. Recall that for (x,π) with $\pi \in \mathcal{A}$ the wealth process $X_t^{x,\pi}$ is a supermartingale with respect to Q. Thus we have $\mathbf{E}^Q[X_T^{x,\pi}] \leq x \leq x_0$ and

$$P[u((H - X_T^{x,\pi})^+) \leq \alpha] \leq P[u((H - \widetilde{X})^+) \leq \alpha]. \ \blacksquare$$

The main difficulty in this theorem is that we do not have an existence result for $\widetilde{X}$ nor any method of construction which could be used for practical applications. However, we show that the problem can be reduced to a simpler one by considering a narrower class of random variables than L_0^+ and for this class in some situations the problem can be explicitly solved. This is the idea behind considering strategies of class $\mathcal{S}$ which we describe below.

Economic motivation for introducing strategies of class $\mathcal{S}$. For (x,π) with $\pi \in \mathcal{A}$ consider two sets: $A = \{\omega \in \Omega : u((H - X_T^{x,\pi})^+) \leq \alpha\}$ and its complement A^{c}. Basing on (x,π) let us build a modified strategy $(\widetilde{x},\widetilde{\pi})$ in the following way. On A the investor's loss is smaller than α. However, from our point of view it can be as large as possible, but not larger than α. Therefore let $(\widetilde{x},\widetilde{\pi})$ be such that on A we have $u((H-X_T^{\widetilde{x},\widetilde{\pi}})^+)=\alpha$. On A^{c} the investor did not manage to hedge large loss, so the portfolio value can be 0 as well. Such a $(\widetilde{x},\widetilde{\pi})$ we will regard as a *strategy of class* $\mathcal{S}$. What is the advantage of such a modification? It turns out that $\widetilde{\pi}\in\mathcal{A}$ and

$$P[u((H-X_T^{\widetilde{x},\widetilde{\pi}})^+)\le\alpha]=P[u((H-X_T^{x,\pi})^+)\le\alpha],\qquad \widetilde{x}\le x.$$

This motivates searching a solution of the problem only among strategies of class $\mathcal{S}$. Below we present this idea in a more precise way.

Definition 3.2. A random variable $X\in L_+^0$ is of *class* $\mathcal{S}$ if there exists $A\in\mathcal{F}$ containing $\{u(H)\le\alpha\}$ such that

(1) on A we have

 (a) if $u(H)\le\alpha$ then $X=0$,
 (b) if $u(H)>\alpha$ then $u(H-X)=\alpha$,

(2) on A^{c} we have $X=0$.

Notice that on the set A we have $X=0$ if $H\le u^{-1}(\alpha)$ and $X=H-u^{-1}(\alpha)$ if $H>u^{-1}(\alpha)$. Thus on A we have $X=(H-u^{-1}(\alpha))^+$. Since $X=0$ on A^{c} we obtain $X=\mathbf{1}_A(H-u^{-1}(\alpha))^+$. In other words $X\in\mathcal{S}$ if it is of the form $X=\mathbf{1}_A(H-u^{-1}(\alpha))^+$ for some $A\in\mathcal{F}$ such that $A\supseteq\{u(H)\le\alpha\}$.

Lemma 3.3. *For each $X\in L_+^0$ such that $\mathbf{E}^Q[X]\le x_0$ there exists a random variable $Z\in\mathcal{S}$ such that $\mathbf{E}^Q[Z]\le\mathbf{E}^Q[X]$ and*

$$P[u((H-X)^+)\le\alpha]=P[u((H-Z)^+)\le\alpha].$$

Proof. Define $A:=\{\omega:u(H-X)^+\le\alpha\}$. Then $Z:=\mathbf{1}_A(H-u^{-1}(\alpha))^+\in\mathcal{S}$ and we have

$$\begin{aligned}
P[u((H-Z)^+)\le\alpha]&=P[u((H-\mathbf{1}_A(H-u^{-1}(\alpha))^+)^+)\le\alpha]\\
&=P[\omega\in A:u((H-(H-u^{-1}(\alpha))^+)^+)\le\alpha]+P[\omega\in A^{\mathrm{c}}:u(H)\le\alpha]\\
&=P[\omega\in A\cap\{u(H)\le\alpha\}:u(H)\le\alpha]\\
&\quad+P[\omega\in A\cap\{u(H)>\alpha\}:u(u^{-1}(\alpha)))\le\alpha]\\
&=P(A).
\end{aligned}$$

On the set A^{c} we have $Z=0\le X$. On A if $u(H)\le\alpha$ then $Z=0\le X$ and if $u(H)>\alpha$ then $Z=H-u^{-1}(\alpha)\le X$. Thus $Z\le X$ and $\mathbf{E}^Q[Z]\le\mathbf{E}^Q[X]$. ■

Remark 3.4. The above calculations show that for any random variable $X=\mathbf{1}_B(H-u^{-1}(\alpha))^+\in\mathcal{S}$ we have

$$P(u(H-X)^+\le\alpha)=P(B).$$

Using Lemma 3.3 and Remark 3.4 we can reformulate Theorem 3.1 as follows:

THEOREM 3.5. *If there exists a set* $\widetilde{A} \supseteq \{u(H) \leq \alpha\}$ *which is a solution of the problem*

$$P(A) \to \max, \tag{3.5.1}$$

$$E^Q[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0, \tag{3.5.2}$$

then the replicating strategy for $\mathbf{1}_{\widetilde{A}}(H - u^{-1}(\alpha))^+$ *is optimal.*

Proof. Indeed, by Lemma 3.3 the problem

$$P[u(H - X)^+ \leq \alpha] \to \max, \quad \mathbf{E}^Q[X] \leq x_0, \quad X \in L^0_+,$$

can be replaced by

$$P[u(H - X)^+ \leq \alpha] \to \max, \quad \mathbf{E}^Q[X] \leq x_0, \quad X \in \mathcal{S}.$$

However, by Remark 3.4, for $X = \mathbf{1}_A(H - u^{-1}(\alpha))^+ \in \mathcal{S}$ we have $P[u(H - X)^+ \leq \alpha] = P(A)$ and the required formulation is obtained. ∎

REMARK 3.6. Consider the optimization problem of Theorem 3.5 given by (3.5.1) and (3.5.2) but without the requirement that $\widetilde{A} \supseteq \{u(H) \leq \alpha\}$. Notice that if $P(u(H) \leq \alpha) > 0$ then the solution $\widetilde{A}$ must contain $\{u(H) \leq \alpha\}$. Suppose the contrary and define $\widetilde{\widetilde{A}} := \widetilde{A} \cup \{u(H) \leq \alpha\}$. Then $\mathbf{E}^Q[\mathbf{1}_{\widetilde{\widetilde{A}}}(H - u^{-1}(\alpha))^+] = \mathbf{E}^Q[\mathbf{1}_{\widetilde{A}}(H - u^{-1}(\alpha))^+] \leq x_0$ and $P(\widetilde{\widetilde{A}}) > P(\widetilde{A})$, which is a contradiction. This shows that that the requirement $\widetilde{A} \supseteq \{u(H) \leq \alpha\}$ in Theorem 3.5 can be dropped.

In some particular cases the existence and construction of the set $\widetilde{A}$ can be obtained by using the Neyman–Pearson lemma. To this end let us introduce a measure $\overline{Q}$ which is absolutely continuous with respect to Q by

$$\frac{d\overline{Q}}{dQ} = \frac{(H - u^{-1}(\alpha))^+}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}.$$

Then the set $\widetilde{A}$ solves the following problem:

$$P(A) \to \max, \quad \overline{Q}(A) \leq \frac{x_0}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}.$$

To make the paper self-contained we present a part of the Neyman–Pearson lemma. Let P_1 and P_2 be two probability measures such that the density dP_1/dP_2 exists.

LEMMA 3.7. *If there exists a constant* β *such that* $P_2\{dP_1/dP_2 \geq \beta\} = \gamma$ *then*

$$P_1\left\{\frac{dP_1}{dP_2} \geq \beta\right\} \geq P_1(B)$$

for any set B *satisfying* $P_2(B) \leq \gamma$.

Proof. Let B satisfy $P_2(B) \le \gamma$ and set $\widetilde{B} := \{dP_1/dP_2 \ge \beta\}$. Then

$$\begin{aligned} P_1(\widetilde{B}) - P(B) &= \int_{\Omega} (\mathbf{1}_{\widetilde{B}} - \mathbf{1}_B)\, dP_1 \\ &= \int_{dP_1/dP_2 \ge \beta} (\mathbf{1}_{\widetilde{B}} - \mathbf{1}_B)\, dP_1 + \int_{dP_1/dP_2 < \beta} (\mathbf{1}_{\widetilde{B}} - \mathbf{1}_B)\, dP_1 \\ &\ge \int_{dP_1/dP_2 \ge \beta} (\mathbf{1}_{\widetilde{B}} - \mathbf{1}_B)\beta\, dP_2 - \int_{dP_1/dP_2 < \beta} \mathbf{1}_B \beta\, dP_2 \\ &= \beta\Big(\int_{\widetilde{B}} dP_2 - \int_B dP_2\Big) = \beta(\gamma - P_2(B)) \ge 0. \ \blacksquare \end{aligned}$$

This lemma is useful for the Black–Scholes model since there the condition

$$\overline{Q}\left\{\frac{dP}{d\overline{Q}} \ge \beta\right\} = \frac{x_0}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}$$

is satisfied. However, for discrete Ω this condition no longer holds. This will be shown in the example of the CRR model.

4. Black–Scholes model. Here we follow an example presented in [3]. The stock price S_t is given by

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = s,$$

where μ and $\sigma > 0$ are constants and W_t is a standard Brownian motion. For this model

$$S_t = s e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}$$

and the unique martingale measure Q is given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T\right).$$

Moreover, the process $W_t^* = W_t + \frac{\mu}{\sigma} t$ is a Brownian motion with respect to Q. Notice that the density of the martingale measure can be expressed in terms of S_T, namely

$$\frac{dQ}{dP} = c S_T^{-\mu/\sigma^2}, \quad \text{where } c \text{ is some constant.}$$

We study a risk minimizing problem for a European call option with strike K. Recall that the problem is reduced to constructing a set $\widetilde{A}$ which solves

$$P(A) \to \max, \quad \overline{Q}(A) \le \frac{x_0}{\mathbf{E}^Q[(S_T - K - u^{-1}(\alpha))^+]},$$

where the measure $\overline{Q}$ is as in the previous section:

$$\frac{d\overline{Q}}{dQ} = \frac{(S_T - K - u^{-1}(\alpha))^+}{\mathbf{E}^Q[(S_T - K - u^{-1}(\alpha))^+]}.$$

Notice the that the superscript "+" above can be dropped since $((a–b)^+–c)^+ = (a-b-c)^+$ for any $a, b, c \geq 0$. According to the Neyman–Pearson lemma we are searching for the set $\widetilde{A}$ of the form

$$\begin{aligned}\left\{\frac{dP}{d\overline{Q}} \geq c_1\right\} &= \left\{\frac{dP}{dQ} \geq c_2(S_T - K - u^{-1}(\alpha))^+\right\} \\ &= \{S_T^{\mu/\sigma^2} \geq cc_2(S_T - K - u^{-1}(\alpha))^+\},\end{aligned}$$

where c_1, c_2 are nonnegative constants such that

$$E^Q[\mathbf{1}_{\widetilde{A}}(S_T - K - u^{-1}(\alpha))^+] = x_0. \tag{4.1}$$

Let us consider two cases:

Case 1: $\mu \leq \sigma^2$. Then the function $x \mapsto x^{\mu/\sigma^2}$ is concave and has value 0 at 0 and thus the solution is given by $\widetilde{A} = \{S_T \leq c_3\} = \{W_T^* \leq c_4\}$, where c_3 and c_4 are such that $c_3 = se^{\sigma c_4 - \frac{1}{2}\sigma^2 T}$ are constant numbers satisfying (4.1). The optimal strategy is a strategy which replicates the following contingent claim:

$$\begin{aligned}&\mathbf{1}_{\widetilde{A}}(S_T - K - u^{-1}(\alpha))^+ = \mathbf{1}_{\{S_T \leq c_3\}}(S_T - K - u^{-1}(\alpha))^+ \\ &\quad= (S_T - K - u^{-1}(\alpha))^+ - (S_T - c_3)^+ - (c_3 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_3\}}\end{aligned}$$

and the corresponding probability is

$$P(\widetilde{A}) = P(W_T^* \leq c_4) = \Phi\left(\frac{c_4 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right).$$

To calculate the constants c_3 and c_4 from (4.1) we use the formula for pricing a European call option:

$$\begin{aligned}&\mathbf{E}^Q[(S_T - K - u^{-1}(\alpha))^+ - (S_T - c_3)^+ - (c_3 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_3\}}] \\ &\qquad= s\Phi(\overline{d}_+) - (K + u^{-1}(\alpha))\Phi(\overline{d}_-) - s\Phi\left(\frac{-c_4 + \sigma T}{\sqrt{T}}\right) + c_3\Phi\left(-\frac{c_4}{\sqrt{T}}\right) \\ &\qquad\quad - (c_3 - K - u^{-1}(\alpha))Q\{W_T^* > c_4\} \\ &\qquad= s\Phi(\overline{d}_+) - (K + u^{-1}(\alpha))\Phi(\overline{d}_-) - s\Phi\left(\frac{-c_4 + \sigma T}{\sqrt{T}}\right) \\ &\qquad\quad + (K + u^{-1}(\alpha))\Phi\left(-\frac{c_4}{\sqrt{T}}\right) = x_0,\end{aligned}$$

where

$$\overline{d}_\pm = -\frac{1}{\sigma\sqrt{T}}\ln\left(\frac{K + u^{-1}(\alpha)}{s}\right) \pm \frac{1}{2}\sigma\sqrt{T}$$

and Φ stands for the distribution function of the $N(0,1)$ distribution.

CASE 2: $\mu > \sigma^2$. In this case the function $x \mapsto x^{\mu/\sigma^2}$ is convex and therefore our solution is of the form

$$\widetilde{A} = \{S_T < c_5\} \cup \{S_T > c_6\} = \{W_T^* < c_7\} \cup \{W_T^* > c_8\},$$

where $c_5 < c_6$ are two solutions of the equation $x^{\mu/\sigma^2} = \overline{c}(x - K - u^{-1}(\alpha))^+$, where $\overline{c}$ is a constant number such that (4.1) holds. The constants c_7, c_8 are given by $c_5 = se^{\sigma c_7 - \frac{1}{2}\sigma^2 T}, c_6 = se^{\sigma c_8 - \frac{1}{2}\sigma^2 T}$. The optimal strategy is a strategy which replicates the following contingent claim:

$$\begin{aligned}
&\mathbf{1}_{\widetilde{A}}(S_T - K - u^{-1}(\alpha))^+ \\
&\quad = (S_T - K - u^{-1}(\alpha))^+ - (S_T - c_5)^+ - (c_5 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_5\}} \\
&\qquad + (S_T - c_6)^+ + (c_6 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_6\}}
\end{aligned}$$

and the corresponding probability is

$$P(\widetilde{A}) = P(W_T^* < c_7) + P(W_T^* > c_8) = \Phi\biggl(\frac{c_7 - \frac{\mu}{\sigma}T}{\sqrt{T}}\biggr) + \Phi\biggl(-\frac{c_8 - \frac{\mu}{\sigma}T}{\sqrt{T}}\biggr).$$

Now we need to determine all necessary constants. Using the same methods as in the previous case we obtain

$$\begin{aligned}
\text{(4.2)} \qquad \mathbf{E}^Q[\mathbf{1}_{\widetilde{A}}(S_T - K - u^{-1}(\alpha))^+] &= s\Phi(\overline{d}_+) - (K + u^{-1}(\alpha))\Phi(\overline{d}_-) \\
&\quad - s\Phi\biggl(-\frac{c_7}{\sqrt{T}} + \sigma\sqrt{T}\biggr) + s\Phi\biggl(-\frac{c_8}{\sqrt{T}} + \sigma\sqrt{T}\biggr) \\
&\quad + (K + u^{-1}(\alpha))\biggl(\Phi\biggl(-\frac{c_7}{\sqrt{T}}\biggr) - \Phi\biggl(-\frac{c_8}{\sqrt{T}}\biggr)\biggr) = x_0.
\end{aligned}$$

Summarizing, the constants are determined by (4.2) and by the fact that c_5, c_6 are solutions of the equation $x^{\mu/\sigma^2} = \overline{c}(x - K - u^{-1}(\alpha))^+$, where $\overline{c}$ is a positive constant.

5. CRR model. Let $(S_n)_{n=0,1,\dots,N}$ be a stock price given by

$$S_{n+1} = S_n(1 + \varrho_n), \quad S_0 = S,$$

where (ϱ_n) is a sequence of independent random variables such that $p := P(\varrho_n = u) = 1 - P(\varrho_n = d)$, where $u > d$, $u > 0$, $d < 0$. This means that at any time the price process S_n can increase to the value $S_n(1 + u)$ or decrease to $S_n(1 + d)$. We assume that $p \in (0, 1)$. It is known that the unique martingale measure for this model is given by $p^* := -d/(u - d)$.

Let us study the risk minimizing problem for the call option with strike K. Set $(S_N - \overline{K})^+ := (S_N - K - u^{-1}(\alpha))^+$ and consider two measures: the objective measure

$$P(\omega_k) = p^k(1 - p)^{N-k}$$

and the measure $\overline{Q}$ (which is not necessarily a probability measure) given by

$$\overline{Q}(\omega_k) := (S(1+u)^k(1+d)^{N-k} - \overline{K})^+ p^{*k}(1-p^*)^{N-k}.$$

Here ω_k means an elementary event for which the number of jumps upwards is equal to k. Our aim is to find the set $\widetilde{A}$ which solves

$$P(A) \to \max, \qquad \overline{Q}(A) \le x_0.$$

For the CRR model the existence of $\widetilde{A}$ is clear since Ω is finite. However we want to find it explicitly. Unfortunately, the Neyman-Pearson lemma for the measures P and $\overline{Q}$ cannot be applied here since Ω is discrete and the condition

$$\overline{Q}\left\{\frac{dP}{dQ} \ge a(H - u^{-1}(\alpha))^+\right\} = \frac{x_0}{E[(H-u^{-1}(\alpha))^+]} \quad \text{for some } a > 0$$

is very rarely satisfied. The first way of constructing $\widetilde{A}$, which seems to be natural, is to find a constant $\overline{a}$ such that

$$\overline{a} = \inf\left\{a : \overline{Q}\left\{\frac{dP}{dQ} \ge a(H - u^{-1}(\alpha))^+\right\} \le \frac{x_0}{\mathbf{E}^Q[(H-u^{-1}(\alpha))^+]}\right\}$$

and then expect that

$$\overline{A} = \left\{\frac{dP}{dQ} \ge \overline{a}(H - u^{-1}(\alpha))^+\right\}$$

is a solution. Unfortunately, this is not a right construction as shown in the example below.

EXAMPLE. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let P and Q be measures given by $p_1 = 7/15$, $p_2 = 4/15$, $p_3 = 4/15$ and $q_1 = 4/10$, $q_2 = 3/10$, $q_3 = 3/10$. We want to maximize $P(A)$ subject to the condition $Q(A) \le x_0 = 6/10$. We have $p_1/q_1 = 63/54$, $p_2/q_2 = 48/54$, $p_3/q_3 = 48/54$ and the above construction gives $\widetilde{A} = \{\omega_1\}$. However $Q(\{\omega_2, \omega_3\}) = 6/10$ and $P(\{\omega_2,\omega_3\}) = 8/15 > 7/15 = P(\omega_1)$.

Below we present a lemma which provides a construction of $\widetilde{A}$ when the measures satisfy a certain condition. It turns out that this condition is satisfied by a significant number of cases in the hedging problem of call option.

LEMMA 5.1. *Let $\Omega = \{\omega_1, \ldots, \omega_n\}$, suppose the* (*not necessarily probability*) *measures P and Q satisfy*

$$p_1 \ge \cdots \ge p_n > 0 < q_1 \le \cdots \le q_n,$$

and let γ be a fixed constant. Let $\widetilde{A} = \{\omega_1, \ldots, \omega_k\}$, where the number k is such that $Q(\omega_1, \ldots, \omega_k) \le \gamma$ and $Q(\omega_1, \ldots, \omega_k, \omega_{k+1}) > \gamma$. Then $P(\widetilde{A}) \ge P(A)$ for any set A satisfying $Q(A) \le \gamma$.

Proof. Let $B \subseteq \Omega$ be such that $Q(B) \le \gamma$.

(1) First assume that $\widetilde{A} \cap B = \emptyset$. Then $|B| \le k$ and $P(\widetilde{\omega}) \ge P(\omega)$ for each $\widetilde{\omega} \in \widetilde{A}$ and $\omega \in B$. As a consequence,

$$P(\widetilde{A}) = \sum_{\omega \in \widetilde{A}} P(\omega) \ge \sum_{\omega \in B} P(\omega) = P(B).$$

(2) If $\widetilde{A} \cap B \neq \emptyset$ then by (1), $\widetilde{A} \setminus \{\widetilde{A} \cap B\}$ is a solution of

$$P(A) \to \max, \qquad Q(A) \le \gamma - Q(\{\widetilde{A} \cap B\}),$$

and so $P(\widetilde{A} \setminus \{\widetilde{A} \cap B\}) \ge P(B \setminus \{\widetilde{A} \cap B\})$. As a consequence, $P(\widetilde{A}) \ge P(B)$. ■

Since $P(\omega_k)$ increases with k if $p > 1/2$ and decreases if $p < 1/2$, to apply the lemma we need the monotonicity of the measure $\overline{Q}$. In fact we are interested in the monotonicity of $\overline{Q}$ only on the set where it is strictly positive. Set

$$\begin{aligned} a_k &:= \overline{Q}(\omega_k) = (S(1+u)^k(1+d)^{N-k} - \overline{K})^+ p^{*k}(1-p^*)^{N-k}, \\ b_k &:= \frac{(S(1+u)^{k+1}(1+d)^{N-k-1} - \overline{K})^+}{(S(1+u)^k(1+d)^{N-k} - \overline{K})^+}, \\ q &:= \frac{1+u}{1+d}, \end{aligned}$$

where the sequence b_k is well defined under the convention that $a/0 = \infty$ for $a \ge 0$. Then $\overline{Q}(\omega_k)$ is increasing if $a_{k+1}/a_k \ge 1$ for each $k = 0, 1, \ldots, N-1$. This condition is equivalent to $b_k \ge (1-p^*)/p^*$ for each $k = 0, 1, \ldots, N-1$. But the sequence b_k is decreasing, since one can calculate that

$$\frac{b_{k+1}}{b_k} \le 1 \;\Leftrightarrow\; (q-1)^2 \ge 0.$$

Thus $\overline{Q}(\omega_k)$ is increasing if

$$b_{N-1} = \frac{(S(1+u)^N - \overline{K})^+}{(S(1+u)^{N-1}(1+d) - \overline{K})^+} \ge \frac{1-p^*}{p^*}.$$

Note that this case includes the situations when $p^* \ge 1/2$.

By analogous arguments one can obtain a condition under which $\overline{Q}(\omega_k)$ is decreasing. This is the case when $b_{\overline{k}} \le (1-p^*)/p^*$, where $\overline{k}$ is the minimal k for which $b_k \neq \infty$. Indeed, then $b_k \le (1-p^*)/p^*$ for all $k \ge \overline{k}$, which implies that $a_{k+1} < a_k$ for $k \ge \overline{k}$.

Before summarizing the above considerations let us introduce the notation

$$A_k := \{\omega \in \Omega : \text{the number of jumps upwards is equal to } k\}$$

for the set containing all elements ω_k. The following lemma is a consequence of Lemma 5.1.

LEMMA 5.2. (1) (*P increasing, $\overline{Q}$ decreasing*) *Let $\overline{k} = \min\{k : b_k \neq \infty\}$. If $p \geq 1/2$ and $b_{\overline{k}} \leq (1-p^*)/p^*$ then $\widetilde{A} = A_N \cup A_{N-1} \cup \cdots \cup A_{N-k} \cup B_{N-k-1}$, where the number k is such that $\overline{Q}(A_N \cup A_{N-1} \cup \cdots \cup A_{N-k}) \leq x_0$ and $\overline{Q}(A_N \cup A_{N-1} \cup \cdots \cup A_{N-k-1}) > x_0$ and the set B_{N-k-1} contains a maximal number of elements from A_{N-k-1} such that $\overline{Q}(B_{N-k-1}) \leq x_0 - \overline{Q}(A_N \cup A_{N-1} \cup \cdots \cup A_{N-k})$.*

(2) (*P decreasing, $\overline{Q}$ increasing*) *If $p \leq 1/2$ and*

$$\frac{(S(1+u)^N - \overline{K})^+}{(S(1+u)^{N-1}(1+d) - \overline{K})^+} \geq \frac{1-p^*}{p^*}$$

(for example when $p^* \geq 1/2$) then $\widetilde{A} = A_0 \cup A_1 \cup \cdots \cup A_k \cup B_{k+1}$, where the number k is such that $\overline{Q}(A_0 \cup A_1 \cup \cdots \cup A_k) \leq x_0$ and $\overline{Q}(A_0 \cup A_1 \cup \cdots \cup A_{k+1}) > x_0$ and the set B_{k+1} contains a maximal number of elements from A_{k+1} such that $\overline{Q}(B_{k+1}) \leq x_0 - \overline{Q}(A_0 \cup A_1 \cup \cdots \cup A_k)$.

EXAMPLE. As an application of Lemma 5.2 we study a risk minimizing problem for a call option with strike $K = 600$ in a 3-period model with parameters: $S_0 = 1000$, $u = 0,1$, $d = -0,2$, $p = 1/4$. The price of the option at time 0 is $u_0 = \mathbf{E}^Q[(S_3 - 600)^+] = 398\frac{7}{27}$. Assume that we have only $x_0 = 150$ and $\alpha = 5$ is a level of acceptable losses measured by $u(x) = \sqrt{x}$. We denote by ω^{abc}, where $a, b, c \in \{u, d\}$, elementary events with interpretation of a, b, c as a history of the price process. For example ω^{udu} means the event where the price process moved up in the first and third periods and moved down in the second one. Since we cannot hedge the original contingent claim $H = (S_3 - 600)^+$:

$$H(\omega^{uuu}) = 731, \quad H(\omega^{uud}) = H(\omega^{udu}) = H(\omega^{duu}) = 368,$$
$$H(\omega^{udd}) = H(\omega^{dud}) = H(\omega^{ddu}) = 104, \quad H(\omega^{ddd}) = 0,$$

we have to hedge $\widetilde{H} = \mathbf{1}_{\widetilde{A}}(S_3 - 625)^+$. Since $p = 1/4$ and $p^* = 2/3$, we can apply Lemma 5.2(2) to construct $\widetilde{A}$. Below we present three possible right candidates for $\widetilde{H}$:

$$\widetilde{H}(\omega^{uuu}) = 0, \quad \widetilde{H}(\omega^{ddd}) = 0, \quad \widetilde{H}(\omega^{ddu}) = \widetilde{H}(\omega^{dud}) = \widetilde{H}(\omega^{udd}) = 79$$

and either

$$\widetilde{H}(\omega^{uud}) = \widetilde{H}(\omega^{udu}) = 343, \quad \widetilde{H}(\omega^{ddu}) = 0,$$

or

$$\widetilde{H}(\omega^{uud}) = 0, \quad \widetilde{H}(\omega^{udu}) = \widetilde{H}(\omega^{ddu}) = 343,$$

or

$$\widetilde{H}(\omega^{uud}) = 343, \quad \widetilde{H}(\omega^{udu}) = 0, \quad \widetilde{H}(\omega^{ddu}) = 343.$$

Moreover, $P(\widetilde{A}) = \left(\frac{3}{4}\right)^3 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 \cdot 3 + \left(\frac{1}{4}\right)^2 \cdot \frac{3}{4} \cdot 2 = \frac{15}{16}$.

6. Incomplete markets. Now let us consider the case when the equivalent martingale measure is not unique. This means that the market is incomplete and not every contingent claim can be replicated. We preserve all assumptions from the previous section. Recall that the wealth process $X_t^{x,\pi}$ is a supermartingale with respect to each martingale measure $Q \in \mathcal{Q}$. In this case the optimal strategy is described by the following theorem:

THEOREM 6.1. *Assume that there exists a set $\widetilde{A}$ which is a solution of the problem*

$$P(A) \to \max, \qquad \sup_{Q\in\mathcal{Q}} E^Q[\mathbf{1}_A(H-u^{-1}(\alpha))^+] \le x_0.$$

Then the strategy which hedges the contingent claim $\mathbf{1}_{\widetilde{A}}(H-u^{-1}(\alpha))^+$ is optimal.

Proof. Consider an arbitrary admissible strategy (x,π), where $x \le x_0$. We will show that $P(u(H-X_T^{x,\pi})^+ \le \alpha) \le P(\widetilde{A})$.

Notice that for any $a,b,c \ge 0$ we have $(a-b)^+ \le c \Leftrightarrow b \ge (a-c)^+$ and thus $u((H-X_T^{x,\pi})^+) \le \alpha \Leftrightarrow X_T^{x,\pi} \ge (H-u^{-1}(\alpha))^+$. As a consequence, for any $Q \in \mathcal{Q}$ we obtain

$$\begin{aligned}E^Q[\mathbf{1}_{\{u((H-X_T^{x,\pi})^+)\le\alpha\}}(H-X_T^{x,\pi})^+] &\le E^Q[\mathbf{1}_{\{u((H-X_T^{x,\pi})^+)\le\alpha\}}X_T^{x,\pi}]\\ &\le E^Q[X_T^{x,\pi}] \le x \le x_0,\end{aligned}$$

where the last but one inequality follows from the fact that $X_t^{x,\pi}$ is a Q-supermartingale. Taking the supremum over all martingale measures gives

$$\sup_{Q\in\mathcal{Q}} E^Q[\mathbf{1}_{\{u((H-X_T^{x,\pi})^+)\le\alpha\}}(H-u^{-1}(\alpha))^+] \le x_0.$$

From the definition of the set $\widetilde{A}$ we have $P(u(H-X_T^{x,\pi})^+ \le \alpha) \le P(\widetilde{A})$.

Now let us consider the strategy $(\widetilde{x},\widetilde{\pi})$ which hedges $\mathbf{1}_{\widetilde{A}}(H-u^{-1}(\alpha))^+$. Then

$$\begin{aligned}\{u(H-X_T^{\widetilde{x},\widetilde{\pi}})^+ \le \alpha\} &= \{X_T^{\widetilde{x},\widetilde{\pi}} \ge (H-u^{-1}(\alpha))^+\}\\ &\supseteq \{X_T^{\widetilde{x},\widetilde{\pi}} \ge \mathbf{1}_{\widetilde{A}}(H-u^{-1}(\alpha))^+\} \supseteq \widetilde{A}\end{aligned}$$

and so $P(u(H-X_T^{\widetilde{x},\widetilde{\pi}})^+ \le \alpha) \ge P(\widetilde{A})$. It follows that $(\widetilde{x},\widetilde{\pi})$ is optimal and moreover $P(u(H-X_T^{\widetilde{x},\widetilde{\pi}})^+ \le \alpha) = P(\widetilde{A})$. ∎

The main problem which needs to be investigated is the existence of the set $\widetilde{A}$. We are not in a position to prove a general existence result for $\widetilde{A}$ but we will show an example of a trinomial model where it can be explicitly found.

EXAMPLE (Trinomial model). Let us consider a one-step model where the stock price is given by

$$S_1 = S(1+\xi), \quad \text{where} \quad P(\xi = a) = p_1,\ P(\xi = b) = p_2,\ P(\xi = c) = p_3,$$
$$a > b > c, \qquad p_1, p_2, p_3 > 0, \qquad p_1 + p_2 + p_3 = 1,$$

and where the interest rate is equal to 0. Here S is an initial price and S_1 is the price at time 1. To obtain an arbitrage-free model we assume that $a > 0$ and $c < 0$. The contingent claim is denoted by $H = (H_1, H_2, H_3) = (H(\omega_1), H(\omega_2), H(\omega_3))$.

First let us study the structure of the set $\mathcal{Q}$ of all martingale measures. Each $Q \in \mathcal{Q}$ is a triplet $Q = (q_1, q_2, q_3)$ which is a solution of the system

$$\begin{cases} q_1 S_0(1+a) + q_2 S_0(1+b) + q_3 S_0(1+c) = S_0, \\ q_1 + q_2 + q_3 = 1, \\ q_1, q_2, q_3 > 0. \end{cases}$$

By direct computation we find that such triplet can be parametrized by q_1. Precisely, each martingale measure is of the form

$$Q = \left(q_1,\ \frac{c-a}{b-c} q_1 + \frac{c}{c-b},\ \frac{a-b}{b-c} q_1 + \frac{b}{b-c} \right),$$
$$\text{where } q_1 \in (\underline{q}, \overline{q}) := \left(0 \vee \frac{b}{b-a},\ \frac{c}{c-a} \right).$$

That means that each $Q \in \mathcal{Q}$ can be represented by

$$Q = \alpha Q_1 + (1-\alpha) Q_2, \qquad \text{where } \alpha \in (0,1) \text{ and}$$
$$Q_1 = \left(\underline{q},\ \frac{c-a}{b-c} \underline{q} + \frac{c}{c-b},\ \frac{a-b}{b-c} \underline{q} + \frac{b}{b-c} \right),$$
$$Q_2 = \left(\overline{q},\ \frac{c-a}{b-c} \overline{q} + \frac{c}{c-b},\ \frac{a-b}{b-c} \overline{q} + \frac{b}{b-c} \right).$$

Thus $\mathcal{Q}$ is a convex set with two vertices Q_1, Q_2. Now notice that for any $A \in \mathcal{F}$ we have

$$\sup_{Q \in \mathcal{Q}} E^Q[\mathbf{1}_A (H - u^{-1}(\alpha))^+] \le x_0 \quad \text{if and only if}$$
$$E^{Q_1}[\mathbf{1}_A (H - u^{-1}(\alpha))^+] \le x_0 \text{ and } E^{Q_2}[\mathbf{1}_A (H - u^{-1}(\alpha))^+] \le x_0,$$

so the constraints for $\widetilde{A}$ reduce to two vertex measures. As a consequence, we are looking for a set $\widetilde{A}$ which is a solution of the problem

$$P(A) \to \max, \qquad \begin{cases} E^{Q_1}[\mathbf{1}_A (H - u^{-1}(\alpha))^+] \overline{Q}_1(A) \le x_0, \\ E^{Q_2}[\mathbf{1}_A (H - u^{-1}(\alpha))^+] \overline{Q}_2(A) \le x_0. \end{cases}$$

Now let us make concrete calculations for the case when $b > 0$. Then $Q_1 = \left(0, \frac{c}{c-b}, \frac{b}{b-c}\right)$, $Q_2 = \left(\frac{c}{c-a}, 0, \frac{a}{a-c}\right)$. Let $\overline{H} := (H - u^{-1}(\alpha))^+$, $\overline{H}_i := (H_i - u^{-1}(\alpha))^+$. Our problem is of the form

$$\mathbf{1}_{\omega_1}(A)p_1 + \mathbf{1}_{\omega_2}(A)p_2 + \mathbf{1}_{\omega_3}(A)p_3 \to \max,$$
$$\mathbf{1}_{\omega_2}(A)\,\frac{c}{c-b}\,\overline{H}_2 + \mathbf{1}_{\omega_3}(A)\,\frac{b}{b-c}\,\overline{H}_3 \le x_0,$$
$$\mathbf{1}_{\omega_1}(A)\,\frac{c}{c-a}\,\overline{H}_1 + \mathbf{1}_{\omega_3}(A)\,\frac{a}{a-c}\,\overline{H}_3 \le x_0.$$

Since we do not have a general method of solution, we will check all possibilities depending on S, a, b, c, H, u, α. Set $L_1 := \frac{c}{c-b}\overline{H}_2 + \frac{b}{b-c}\overline{H}_3$ and $L_2 := \frac{c}{c-a}\overline{H}_1 + \frac{a}{a-c}\overline{H}_3$. We have the following description of the set $\widetilde{A}$.

1. If $L_1 \le x_0$ and $L_2 \le x_0$ then $\widetilde{A} = \{\omega_1, \omega_3, \omega_3\}$.
2. If $\min\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} > x_0$ or $\min\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} > x_0$ then $\widetilde{A} = \emptyset$.
3. If $L_1 \le x_0$ and $L_2 > x_0$ and $\min\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} \le x_0$ then if
   (a) $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} > x_0$ and if
      (i) $\frac{c}{c-a}\overline{H}_1 \ge \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_2, \omega_3\}$,
      (ii) $\frac{c}{c-a}\overline{H}_1 < \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$;
   (b) $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} \le x_0$ and if
      (i) $p_1 \ge p_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,
      (ii) $p_1 < p_3$ then $\widetilde{A} = \{\omega_2, \omega_3\}$.
4. If $L_1 > x_0$ and $L_2 \le x_0$ and $\min\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} \le x_0$ then if
   (a) $\max\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} > x_0$ and if
      (i) $\frac{c}{c-b}\overline{H}_2 \ge \frac{b}{b-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_3\}$,
      (ii) $\frac{c}{c-b}\overline{H}_2 < \frac{b}{b-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$;
   (b) $\max\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} \le x_0$ and if
      (i) $p_2 \ge p_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,
      (ii) $p_2 < p_3$ then $\widetilde{A} = \{\omega_1, \omega_3\}$.
5. If $L_1 > x_0$ and $L_2 > x_0$ and $\min\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} \le x_0$ and $\min\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\} \le x_0$ then if
   (a) $\max\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\} > x_0$ and $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} > x_0$ and if
      (i) $\frac{c}{c-b}\overline{H}_2 \le \frac{b}{b-c}\overline{H}_3$ and $\frac{c}{c-a}\overline{H}_1 \le \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,
      (ii) $\frac{c}{c-b}\overline{H}_2 \le \frac{b}{b-c}\overline{H}_3$ and $\frac{c}{c-a}\overline{H}_1 > \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_2\}$,
      (iii) $\frac{c}{c-b}\overline{H}_2 > \frac{b}{b-c}\overline{H}_3$ and $\frac{c}{c-a}\overline{H}_1 \le \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1\}$,
      (iv) $\frac{c}{c-b}\overline{H}_2 > \frac{b}{b-c}\overline{H}_3$ and $\frac{c}{c-a}\overline{H}_1 > \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_3\}$;

(b) $\max\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} > x_0$ and $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} \le x_0$ and if

(i) $\frac{c}{c-b}\overline{H}_2 \le \frac{b}{b-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,

(ii) $\frac{c}{c-b}\overline{H}_2 > \frac{b}{b-c}\overline{H}_3$ and if

(A) $p_3 \ge p_1$ then $\widetilde{A} = \{\omega_3\}$,

(B) $p_3 < p_1$ then $\widetilde{A} = \{\omega_1\}$;

(c) $\max\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} \le x_0$ and $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} > x_0$ and if

(i) $\frac{c}{c-a}\overline{H}_1 \le \frac{a}{a-c}\overline{H}_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,

(ii) $\frac{c}{c-a}\overline{H}_1 > \frac{a}{a-c}\overline{H}_3$ and if

(A) $p_2 \ge p_3$ then $\widetilde{A} = \{\omega_2\}$,

(B) $p_2 < p_3$ then $\widetilde{A} = \{\omega_3\}$;

(d) $\max\left\{\frac{c}{c-b}\overline{H}_2, \frac{b}{b-c}\overline{H}_3\right\} \le x_0$ and $\max\left\{\frac{c}{c-a}\overline{H}_1, \frac{a}{a-c}\overline{H}_3\right\} \le x_0$ and if

(i) $p_1 + p_2 \ge p_3$ then $\widetilde{A} = \{\omega_1, \omega_2\}$,

(ii) $p_1 + p_2 < p_3$ then $\widetilde{A} = \{\omega_3\}$.

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