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BOUNDED SOLUTIONS FOR ARMA MODEL WITH VARYING COEFFICIENTS

Abstract. The paper deals with ARMA systems of equations with varying coefficients. A complete description of bounded solutions to ARMA$(1, q)$ systems is obtained and their uniqueness is studied. Some special cases are discussed, including the case of significant interest of systems with periodic coefficients. The paper generalizes results of [9] and opens a new direction of study.

1. Introduction. An ARMA(p, q) *system* is a system of linear equations

$$X_n - \sum_{k=1}^{p} b_k(n)X_{n-k} = \sum_{j=0}^{q-1} a_j(n)\xi_{n-j}, \qquad n \in \mathbb{Z}, \tag{1}$$

where $(b_k(n))$, $k = 1, \ldots, p$, and $(a_k(n))$, $k = 0, \ldots, q-1$, are sequences of complex numbers, $b_k(n) \neq 0$ for all $k = 1, \ldots, p$ and $n \in \mathbb{Z}$, and (ξ_n) is a sequence of uncorrelated complex random variables with mean zero and unit variances. In what follows, $\mathbb{Z}$ will stand for the set of all integers. All random variables X are complex of second order and mean zero. The correlation $E(X\overline{Y}')$ of X and Y is denoted by (X, Y) and the variance of X by $\|X\|^2$. The L^2-closed linear space generated by the sequence ξ_n, $n \in \mathbb{Z}$, is denoted by M_ξ. The space M_ξ with norm $\|\cdot\|$ is a Hilbert space; in particular the sequence (ξ_n) is orthonormal in M_ξ.

The notation $\lim_{n\to\infty} X_n = Y$, for $\{X_n\}$ a stochastic sequence and Y a random variable, will mean $\lim_{n\to\infty} \|X_n - Y\|^2 = 0$. Throughout the paper we use the convention that if $s < r$ then $\prod_{k=r}^{s} c_k = 1$ and $\sum_{k=r}^{s} c_k = 0$.

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Any sequence (X_n) in M_ξ that satisfies (1) is called a *solution*. A solution (X_n) is *bounded* if $\sup_{n\in\mathbb{Z}} \|X_n\|^2 < \infty$. We consider first the system ARMA$(1,q)$ defined as

$$X_n - b_n X_{n-1} = \sum_{p=0}^{q-1} a_p(n)\xi_{n-p}, \qquad n \in \mathbb{Z}, \tag{2}$$

where the coefficients and innovations have the same properties as in the general model. Let

$$Y_n = \sum_{p=0}^{q-1} a_p(n)\xi_{n-p}, \qquad B_k^n = \prod_{j=k}^{n} b_j.$$

It is obvious that an initial value $X_0 = X$ determines a solution (X_n) of (2). Indeed, iterating the equation (2) k times we find that for all $n \in \mathbb{Z}$ and $k \geq 1$,

$$X_n = B_{n-k+1}^n X_{n-k} + \sum_{s=0}^{k-1} B_{n+1-s}^n Y_{n-s}, \tag{3}$$

$$X_n = \frac{X_{n+k}}{B_{n+1}^{n+k}} - \sum_{j=1}^{k} \frac{1}{B_{n+1}^{n+j}} Y_{n+j}. \tag{4}$$

If in (3) we let $n \geq 1$ and $k = n$, and in (4) for each $n \leq -1$ we let $k = -n$, then we obtain

$$X_n = \begin{cases} B_1^n X_0 + \displaystyle\sum_{j=1-n}^{0} B_{n+1+j}^n Y_{n+j} & \text{if } n \geq 1, \\ \dfrac{X_0}{B_{n+1}^0} - \displaystyle\sum_{j=1}^{-n} \frac{1}{B_{n+1}^{n+j}} Y_{n+j} & \text{if } n \leq -1. \end{cases} \tag{5}$$

This formula describes all solutions. In this note we deal with the problem whether there is a bounded solution among them and when it is unique. Systems like (2) arise in the time series framework and the uniqueness of solution and its form are important in the analysis of such models. If (b_n) and $(a_k(n))$ do not depend on n, then (2) takes the form $X_n - bX_{n-1} = \sum_{p=0}^{q-1} a_p \xi_{n-p}$, and is a special case of a stationary ARMA system ([4]). It is well known that such a system has a unique bounded solution iff $|b| \neq 1$, the solution is stationary and has a one-sided moving average representation. Recently ARMA systems with periodic coefficients (PARMA) became of significant interest (for example see [1], [2], [3], [8], [11], [14], [16]). Such systems arise in climatology, economics, hydrology, electrical engineering and other disciplines. They are usually treated by converting them to vector ARMA systems [14], [16], which yields some conditions for existence of a unique bounded solution. An alternative method of treatment of such systems was proposed in [3].

To our knowledge the only system of the form (2) with nonperiodic coefficients was studied in [9], where (b_n) was assumed to be almost periodic, $q = 1$ and $a_0(n) = 1$. Under these assumptions the authors found some sufficient conditions for existence of a unique bounded solution, which turned out to be almost periodically correlated. They also derived conditions for existence and uniqueness of a bounded solution to the PAR(1) system without referring to vector ARMAs.

In this note we give necessary and sufficient conditions for the system (2) to have a unique bounded solution and obtain its form, without any additional assumptions on the coefficients (b_n) and $(a_k(n))$. Some simple cases are also discussed, e.g. the case of constant and periodic coefficients. In the last section we discuss necessary conditions in general ARMA(p,q) model to have a bounded solution.

2. ARMA$(1,q)$ **system.** We will follow the approach from [9] and split the analysis into three cases:

(C1) $\quad \sup_{n\ge 1} |B_1^n| = \infty,$

(C2) $\quad \sup_{n\le 0} |B_n^0|^{-1} = \infty,$

(C3) $\quad \sup_{n\ge 1} |B_1^n| < \infty$ and $\sup_{n\le 0} |B_n^0|^{-1} < \infty.$

If $b_n = b = \text{const}$, they correspond to the cases $|b| > 1$, $|b| < 1$, $|b| = 1$.

The following lemma will be used repeatedly.

Lemma 2.1. *Let c_j, $j \in \mathbb{Z}$, be a sequence of scalars,*

$$Y_n = \sum_{p=0}^{q-1} a_p(n)\xi_{n-p},$$

and let $M < N$, $M, N \in \mathbb{Z}$. Then

$$\sum_{j=N}^{M} c_j Y_{n+j} = \sum_{k=N+n-q+1}^{M+n} \Big(\sum_{j=\max(N,k-n)}^{\min(M,q-1+k-n)} c_j a_{n+j-k}(n+j) \Big) \xi_k, \tag{6}$$

$$\Big\| \sum_{j=N}^{M} c_j Y_{n+j} \Big\|^2 = \sum_{k=N+n-q+1}^{M+n} \Big| \sum_{j=\max(N,k-n)}^{\min(M,q-1+k-n)} c_j a_{n+j-k}(n+j) \Big|^2. \tag{7}$$

Proof. Since $Y_n = \sum_{p=0}^{q-1} a_p(n)\xi_{n-p}$, we have

$$(Y_{n+j}, \xi_k) = \begin{cases} a_{n+j-k}(n+j) & \text{if } n+j-q+1 \le k \le n+j, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

It follows that for a fixed k the inner product (Y_{n+j},ξ_k) is possibly nonzero only if $k-n\le j\le q-1+k-n$. Hence

$$\Big(\sum_{j=N}^{M} c_jY_{n+j},\xi_k\Big)=\sum_{j=\max(N,k-n)}^{\min(M,q-1+k-n)} c_ja_{n+j-k}(n+j). \tag{9}$$

The sum above is 0 if $k-n>M$ or $q-1+k-n<N$. Therefore

$$\sum_{j=N}^{M} c_jY_{n+j}=\sum_{k=N+n-q+1}^{M+n}\Big(\sum_{j=\max(N,k-n)}^{\min(M,q-1+k-n)} c_ja_{n+j-k}(n+j)\Big)\xi_k,$$

which implies (7) since (ξ_n) is orthonormal. ∎

PROPOSITION 2.1. *Suppose that* $\sup_{n\ge1}|B_1^n|=\infty$. *Then the system* (2) *has a bounded solution in* M_ξ *iff*

$$\sup_{n\in\mathbb{Z}}\sum_{s=2-q}^{\infty}\Big|\sum_{j=\max(1,s)}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\,a_{j-s}(n+j)\Big|^2<\infty. \tag{10}$$

Moreover, *if this is the case*, *then there is a unique bounded solution of* (2) *in* M_ξ *and it is given by*

$$X_n=-\sum_{s=2-q}^{\infty}\Big[\sum_{j=\max(1,s)}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\,a_{j-s}(n+j)\Big]\xi_{n+s}. \tag{11}$$

Proof. Suppose first that (X_n) is a bounded solution of (2). Since $\sup_r|B_1^r|=\infty$ and all b_k's are nonzero, for every n there is a sequence k_r such that $\lim_r|B_{n+1}^{n+k_r}|=\infty$. Hence, from (4), we conclude that for every n,

$$X_n=-\lim_r\sum_{j=1}^{k_r}\frac{1}{B_{n+1}^{n+j}}\,Y_{n+j}.$$

From Lemma 2.1 we have

$$\sum_{j=1}^{k_r}\frac{1}{B_{n+1}^{n+j}}\,(Y_{n+j},\xi_k)=\sum_{j=\max(1,k-n)}^{\min(k_r,q-1+k-n)}\frac{1}{B_{n+1}^{n+j}}\,a_{n+j-k}(n+j).$$

Letting $r\to\infty$, we obtain

$$(X_n,\xi_k)=-\lim_r\sum_{j=1}^{k_r}\frac{1}{B_{n+1}^{n+j}}\,(Y_{n+j},\xi_k)=\sum_{j=\max(1,k-n)}^{q-1+k-n}\frac{1}{B_{n+1}^{n+j}}\,a_{n+j-k}(n+j)$$

if $k>1-q+n$, and $(X_n,\xi_k)=0$ if $k\le 1-q+n$. Therefore

$$X_n=-\sum_{k=2+n-q}^{\infty}\Big[\sum_{j=\max(1,k-n)}^{q-1+k-n}\frac{1}{B_{n+1}^{n+j}}\,a_{n+j-k}(n+j)\Big]\xi_k,$$

which after substituting $s = k - n$ gives (11). This also shows that the solution is unique. The variance of X_n is

$$\|X_n\|^2 = \sum_{s=2-q}^{\infty} \Big| \sum_{j=\max(1,s)}^{q-1+s} \frac{1}{B_{n+1}^{n+j}} a_{j-s}(n+j) \Big|^2,$$

hence (X_n) is bounded iff (10) holds.

Conversely, suppose that (10) is satisfied. Define

$$S_n^M = -\sum_{j=1}^{M} \frac{1}{B_{n+1}^{n+j}} Y_{n+j}, \qquad n \in \mathbb{Z},\ M \geq 1.$$

From Lemma 2.1 it follows that

$$S_n^M = -\sum_{k=2+n-q}^{M+n} \Big(\sum_{j=\max(1,k-n)}^{\min(M,q-1+k-n)} \frac{1}{B_{n+1}^{n+j}} a_{n+j-k}(n+j) \Big) \xi_k.$$

Therefore (10) implies that for each fixed n, the limit $X_n = \lim_{M\to\infty} S_n^M$ exists and is given by (11). From the latter and (10) it follows that $X_n \in M_\xi$ and $\sup_{n\in\mathbb{Z}} \|X_n\| < \infty$. Note that

$$S_n^M - b_n S_{n-1}^M = Y_n - \frac{1}{B_{n+1}^{n+M}} Y_{n+M}. \tag{12}$$

The last term in (12) equals $S_n^M - S_n^{M-1}$ and hence it converges to zero. Therefore, letting $M \to \infty$, we conclude that

$$X_n - b_n X_{n-1} = Y_n - \lim_M \Big(\frac{1}{B_{n+1}^{n+M}} Y_{n+M} \Big) = Y_n,$$

that is, (X_n) satisfies (2). ■

Note that in fact we have proved that if the series (11) converges, then (X_n) is a solution (possibly unbounded) of the system, even if $\sup_{n\geq 1} |B_1^n| < \infty$.

PROPOSITION 2.2. *Suppose that* $\sup_{n\leq -1} |B_n^0|^{-1} = \infty$. *Then the system* (2) *has a bounded solution in* M_ξ *iff*

$$\sup_{n\in\mathbb{Z}} \sum_{s=0}^{\infty} \Big| \sum_{j=-s}^{\min(0,q-1-s)} B_{n+j+1}^n a_{s+j}(n+j) \Big|^2 < \infty. \tag{13}$$

Moreover, if this is the case, then (2) *has a unique bounded solution in* M_ξ *which is given by*

$$X_n = \sum_{s=0}^{\infty} \Big[\sum_{j=-s}^{\min(0,q-1-s)} B_{n+j+1}^n a_{s+j}(n+j) \Big] \xi_{n-s}. \tag{14}$$

Proof. The proof is similar to the proof of Proposition 2.1, so we only sketch it. If $\sup_r |B_r^1|^{-1} = \infty$, then for every n there is a sequence $k_r \to \infty$ such that $\lim_r |B_{n-k_r+1}^{n+1}| = 0$. Suppose first that (X_n) is a bounded solution of (2). Then from (3) it follows that

$$X_n = \lim_r \sum_{j=1-k_r}^{0} B_{n+j+1}^{n+1} Y_{n+j}.$$

Using Lemma 2.1 we conclude that

$$X_n = \sum_{s=0}^{\infty} \Big[\sum_{j=-s}^{\min(0,q-1-s)} B_{n+j+1}^{n} a_{s+j}(n+j) \Big] \xi_{n-s},$$

which also yields (13).

Conversely, suppose that (13) is satisfied; let $S_n^M = \sum_{j=1-M}^{0} B_{n+j+1}^{n+1} Y_{n+j}$. From Lemma 2.1 it follows that S_n^M converges, as $M \to \infty$, to

$$X_n = \sum_{s=0}^{\infty} \Big[\sum_{j=-s}^{\min(0,q-1-s)} B_{n+j+1}^{n} a_{s+j}(n+j) \Big] \xi_{n-s}.$$

By (13) the sequence (X_n) is bounded. As $S_n^M - b_n S_{n-1}^M = Y_n - B_{n+1-M}^n Y_{n-M}$ and $B_{n+1-M}^n Y_{n-M} = S_n^{M+1} - S_n^M$, the sequence (X_n) satisfies (2). ∎

Note again that in fact we have proved that for (X_n) to be a solution (possibly unbounded) of the system it is enough that the series (14) converges.

PROPOSITION 2.3. *Suppose that* $\sup_r |B_1^r| < \infty$ *and* $\sup_r |B_r^0|^{-1} < \infty$. *Then* (2) *has a bounded solution iff*

$$\sup_{n\ge 1} \sum_{s=0}^{n+q-2} \Big| \sum_{j=\max(1-n,-s)}^{\min(0,q-1-s)} B_{n+1+j}^{n} a_{j+s}(n+j) \Big|^2 < \infty, \tag{15}$$

$$\sup_{n\le -1} \sum_{s=2-q}^{-n} \Big| \sum_{j=\max(1,s)}^{\min(0-n,q-1+s)} \frac{1}{B_{n+1}^{n+j}} a_{j-s}(n+j) \Big|^2 < \infty. \tag{16}$$

In this case (2) *has infinitely many bounded solutions given by* (5).

Proof. Since the sequences $|B_1^n|$, $n \ge 1$, and $|1/B_{n+1}^0|$, $n \le -1$, are bounded, the sequence (X_n) in (5) is bounded iff

$$\sup_{n\ge 1} \Big\| \sum_{j=1-n}^{0} B_{n+1+j}^{n} Y_{n+j} \Big\|^2 < \infty, \qquad \sup_{n\le -1} \Big\| \sum_{j=1}^{-n} \frac{1}{B_{n+1}^{n+j}} Y_{n+j} \Big\|^2 < \infty. \tag{17}$$

Using Lemma 2.1 we obtain

$$\Big\| \sum_{j=1-n}^{0} B^n_{n+1+j} Y_{n+j} \Big\|^2 = \sum_{k=2-q}^{n} \Big| \sum_{j=\max(1-n,k-n)}^{\min(0,q-1+k-n)} B^n_{n+1+j} a_{n+j-k}(n+j) \Big|^2,$$

which implies (15). A similar computation shows that the second condition in (17) is equivalent to (16). ■

Since cases (C1)–(C3) cover all possible situations, we obtain the following theorem.

THEOREM 2.1. *The system* (2) *has a bounded solution iff at least one of the following three conditions is satisfied*:

$$\text{I. } \sup_{n\in\mathbb{Z}} \sum_{s=2-q}^{\infty} \Big| \sum_{j=\max(1,s)}^{q-1+s} \frac{1}{B^{n+j}_{n+1}} a_{j-s}(n+j) \Big|^2 < \infty,$$

$$\text{II. } \sup_{n\in\mathbb{Z}} \sum_{s=0}^{\infty} \Big| \sum_{j=-s}^{\min(0,q-1-s)} B^n_{n+j+1} a_{s+j}(n+j) \Big|^2 < \infty,$$

$$\text{III. (i) } \sup_{n\ge 1} \sum_{s=0}^{n+q-2} \Big| \sum_{j=\max(1-n,-s)}^{\min(0,q-1-s)} B^n_{n+1+j} a_{j+s}(n+j) \Big|^2 < \infty,$$

$$\text{(ii) } \sup_{n\le -1} \sum_{s=2-q}^{-n} \Big| \sum_{j=\max(1,s)}^{\min(0-n,q-1+s)} \frac{1}{B^{n+j}_{n+1}} a_{j-s}(n+j) \Big|^2 < \infty.$$

Conditions I–III are not disjoint and none of them implies uniqueness of a bounded solution. To see this, it is enough to take $a_k(n) \equiv 0$, $k = 0, \ldots, q-1$.

The following theorem gives sufficient and necessary conditions for existence of a unique bounded solution of (2). The theorem follows immediately from Propositions 2.1–2.3.

THEOREM 2.2. *The system* (2) *has a unique bounded solution iff either*

$$\text{I. (i) } \sup_{n\ge 1} |B^n_1| = \infty,$$

$$\text{(ii) } \sup_{n\in\mathbb{Z}} \sum_{s=2-q}^{\infty} \Big| \sum_{j=\max(1,s)}^{q-1+s} \frac{1}{B^{n+j}_{n+1}} a_{j-s}(n+j) \Big|^2 < \infty, \ \textit{or}$$

$$\text{II. (i) } \sup_{n\le -1} |B^0_n|^{-1} = \infty,$$

$$\text{(ii) } \sup_{n\in\mathbb{Z}} \sum_{s=0}^{\infty} \Big| \sum_{j=-s}^{\min(0,q-1-s)} B^n_{n+j+1} a_{s+j}(n+j) \Big|^2 < \infty.$$

If I *is satisfied then the solution has the form*

$$X_n = -\sum_{s=2-q}^{\infty}\Big[\sum_{j=\max(1,s)}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\,a_{j-s}(n+j)\Big]\xi_{n+s}.$$

If II *is satisfied then the solution is given by*

$$X_n = \sum_{s=0}^{\infty}\Big[\sum_{j=-s}^{\min(0,q-1-s)} B_{n+j+1}^{n}a_{s+j}(n+j)\Big]\xi_{n-s}.$$

Conditions I and II are not disjoint. It is possible that both $\sup_{n\ge1}|B_1^n|$ and $\sup_{n\le-1}|B_n^0|^{-1}$ are infinite. If this is the case and the system has a bounded solution, then one can show that the coefficients (b_n) and $(a_k(n))$ must satisfy

$$\sum_{j=k-n}^{q-1+k-n} B_{n+j+1}^{k+q-1}a_{j+k-n}(n+j)=0 \qquad \text{for all } n,k.$$

In particular, if $q=1$ then conditions I and II are satisfied simultaneously only if the system is homogeneous, that is, if $a_0(n)\equiv 0$.

3. Examples. In this section we examine some special systems. Most of the results derived here are known or may be obtained in a more straightforward way. Our purpose is to illustrate Theorems 2.1 and 2.2 and their consistency with known results.

3.1. *Homogeneous system.* Let $q=1$ and $a_0(n)\equiv 0$, so the system (2) takes the form

$$X_n - b_nX_{n-1}=0,\qquad n\in\mathbb{Z}. \tag{18}$$

Then conditions I–III of Theorem 2.1 and conditions I(ii) and II(ii) of Theorem 2.2 are trivially satisfied. Since $X_n=0$ satisfies (18), from Theorem 2.2 we conclude that the system (18) has a nonzero bounded solution iff $\sup_{n\ge1}|B_1^n|<\infty$ and $\sup_{n\le-1}|B_n^0|^{-1}<\infty$.

3.2. *Constant coefficients.* Suppose that $b_n=b$ and $a_j(n)=a_j$, $j=0,\dots,q-1$, do not depend on n, that is,

$$X_n - bX_{n-1}=\sum_{p=0}^{q-1}a_p\xi_{n-p},\qquad n\in\mathbb{Z}. \tag{19}$$

Then $B_s^r=b^{s-r+1}$ for $s\ge r$, $\sup_{n\ge1}|B_1^n|=\infty$ iff $|b|>1$, and $\sup_{n\le-1}|B_n^0|^{-1}=\infty$ iff $|b|<1$. If $|b|>1$ then the sum in (10) equals

$$\sum_{s=2-q}^{0}\Big|\sum_{j=1}^{q-1+s}\frac{a_{j-s}}{b^j}\Big|^2+\sum_{s=1}^{\infty}\frac{1}{|b|^s}\Big|\sum_{k=0}^{q-1}\frac{a_k}{b^k}\Big|^2,$$

which is finite and does not depend on n. Hence if $|b| > 1$ then I(ii) of Theorem 2.2 holds true. Similarly, if $|b| < 1$, then II(ii) is satisfied. Therefore from Theorem 2.2 we obtain

If $|b| \neq 1$ then the system (19) *has a unique bounded solution given by*

$$X_n = \begin{cases} -\sum_{s=2-q}^{\infty}\Big[\sum_{j=\max(1,s)}^{q-1+s} \frac{a_{j-s}}{b^j}\Big]\xi_{n+s} & \text{if } |b| > 1, \\ \sum_{s=0}^{\infty}\Big[\sum_{j=\max(0,s-q+1)}^{s} b^j a_{s-j}\Big]\xi_{n-s} & \text{if } |b| < 1. \end{cases}$$

Clearly (X_n) is stationary, that is, (X_n, X_m) depends only on $n - m$.

If $|b| = 1$ then the assumptions of Proposition 2.3 are satisfied, and so the system (19) has a bounded solution iff $z = 1/b$ is a zero of a polynomial $\sum_{k=0}^{q-1} a_k z^k$ (see [4]).

3.3. ARMA$(1,1)$. Suppose now that $q = 1$. Then the system (2) takes the form

$$X_n - b_n X_{n-1} = a_0(n)\xi_n, \qquad n \in \mathbb{Z}. \tag{20}$$

By Theorem 2.2, the system (20) has a unique bounded solution iff either

$$\text{I. (i)} \sup_{n\geq 1}|B_1^n| = \infty \quad \text{and} \quad \text{(ii)} \sup_{n\in\mathbb{Z}}\sum_{s=1}^{\infty}|a_0(n+s)/B_{n+1}^{n+s}|^2 < \infty,$$

or

$$\text{II. (i)} \sup_{n\leq -1}|B_n^0|^{-1} = \infty \quad \text{and} \quad \text{(ii)} \sup_{n\in\mathbb{Z}}\sum_{s=0}^{\infty}|B_{n-s+1}^n a_0(n-s)|^2 < \infty.$$

If $a_0(n) = \text{const} \neq 0$ then condition I(ii) above implies I(i), and II(ii) implies II(i). In particular, if $a_0(n) \equiv 1$ our conditions are consistent with those obtained in [9].

3.4. *Homogeneous variance system.* Consider a system

$$X_n - b_n X_{n-1} = \eta_n + \eta_{n-1} + \cdots + \eta_{n-q+1}, \qquad n \in \mathbb{Z}, \tag{21}$$

where $b_n \neq 0$, $n \in \mathbb{Z}$, and (η_n) is a sequence of uncorrelated complex random variables with mean zero and possibly varying variances $\|\eta_n\|^2 = |a_n|^2$. The system above was introduced and studied in [17]. Writing $\eta_n = a_n\xi_n$, $n \in \mathbb{Z}$, we see that (21) is of the form (2) with $a_k(n) = a_{n-k}$, $k = 0, \ldots, q-1$, $n \in \mathbb{Z}$. From Theorem 2.2 and the discussion thereafter we conclude that the system (21) has a unique bounded solution iff at least one of the following two sets of conditions is satisfied:

I. (i) $\sup_{n\geq 1}|B_1^n| = \infty$,

(ii) $\sup_{n\in\mathbb{Z}}\left[\sum_{s=2-q}^{0}\left|\sum_{j=1}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2+\sum_{s=1}^{\infty}\left|\sum_{j=s}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\right|^2|a_{n+s}|^2\right]<\infty$;

II. (i) $\sup_{n\leq -1}|B_n^0|^{-1} = \infty$,

(ii) $\sup_{n\in\mathbb{Z}}\left[\sum_{s=0}^{q-2}\left|\sum_{j=-s}^{0}B_{n+j+1}^n\right|^2|a_{n-s}|^2+\sum_{s=q-1}^{\infty}\left|\sum_{j=-s}^{q-1-s}B_{n+j+1}^n\right|^2|a_{n-s}|^2\right]<\infty$.

3.5. *Periodic coefficients.* Suppose all the sequences (b_n) and $(a_k(n))$, $k = 0,\dots,q-1$, are periodic in n with the same period $T > 1$. Then $\sup_{k,n}|a_k(n)| < \infty$. Since b_n's are nonzero, also $\inf_n |b_n| < \infty$. Define $P = b_1\cdots b_T$. Then $\sup_{n\geq 1}|B_1^n| = \infty$ iff $|P| > 1$ and $\sup_{n\leq -1}|B_n^0|^{-1} = \infty$ iff $|P| < 1$.

Suppose first $|P| > 1$. Then

$$\begin{aligned}\sum_{s=2-q}^{\infty}\left|\sum_{j=\max(1,s)}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\,a_{j-s}(n+j)\right|^2 &\leq C+\sum_{s=1}^{\infty}\left|\sum_{j=s}^{q-1+s}\frac{1}{B_{n+1}^{n+j}}\,a_{j-s}(n+j)\right|^2\\ &\leq C+\sum_{s=1}^{\infty}\left|\sum_{w=0}^{q-1}\frac{1}{B_{n+1}^{n+w+s}}\,a_w(n+w+s)\right|^2\\ &\leq C+\sum_{N=0}^{\infty}\sum_{k=1}^{T}\left|\sum_{w=0}^{q-1}\frac{1}{B_{n+1}^{n+w+NT+k}}\,a_w(n+w+NT+k)\right|^2\\ &\leq C+\sum_{N=0}^{\infty}\sum_{k=1}^{T}\left|\sum_{w=0}^{q-1}\frac{1}{P^NB_{n+1}^{n+w+k}}\,a_w(n+w+k)\right|^2\\ &\leq C+\sum_{N=0}^{\infty}|P|^{-2N}\sum_{k=1}^{T}\left|\sum_{w=0}^{q-1}\frac{1}{B_{n+1}^{n+w+k}}\,a_w(n+w+k)\right|^2<\infty,\end{aligned}$$

and hence condition I(ii) of Theorem 2.2 is satisfied. A similar computation shows that if $|P| < 1$, then condition II(ii) of Theorem 2.2 is satisfied. Thus we have proved the following theorem.

THEOREM 3.1. *If* (b_n) *and* $(a_k(n))$, $k = 0,1,\dots,q-1$, *are periodic with the same period* T *and* $P = b_1\cdots b_T$, *then the system* (2) *has a unique bounded solution iff* $|P| \neq 1$. *Moreover, the solution is given by* (11) *if* $|P| > 1$, *and by* (14) *if* $|P| < 1$.

The form of the solution shows that it is periodically correlated, that is, the correlation (X_n, X_{n+k}) is T-periodic in n for every k (see [6], [7],

[12], [17]). We point out that having a one-sided series representation of the solution is important in the problem of prediction of such processes ([13]).

4. ARMA(p, q) **system.** In the last section we consider the general case of ARMA(p, q) system. Following [3] define

$$B_n = \begin{bmatrix} b_1(n) & b_2(n) & \dots & b_{p-1}(n) & b_p(n) \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \tag{22}$$

$$\mathbf{B}_k^n = B_n B_{n-1} \cdots B_k, \qquad n \geq k. \tag{23}$$

Let $\mathbf{Y}_n = [Y_n, 0, \dots, 0]'$ and $\mathbf{X}_n = [X_n, X_{n-1}, \dots, X_{n-p+1}]'$, where $Y_n = \sum_{p=0}^{q-1} a_p(n)\xi_{n-p}$ and $[\dots]'$ denotes the column vector. Then (see [3])

$$\mathbf{X}_n = B_n \mathbf{X}_{n-1} + \mathbf{Y}_n. \tag{24}$$

Consequently,

$$\mathbf{X}_n = \mathbf{B}_{n-k+1}^n \mathbf{X}_{n-k} + \sum_{s=0}^{k-1} \mathbf{B}_{n+1-s}^n \mathbf{Y}_{n-s}, \qquad k > 0, \tag{25}$$

where $\mathbf{B}_j^n = I$ if $j > n$. Since all matrices $\mathbf{B}_j^n$ are invertible, also

$$\mathbf{X}_n = (\mathbf{B}_{n+1}^{n+k})^{-1} \mathbf{X}_{n+k} - \sum_{j=1}^{k} (\mathbf{B}_{n+1}^{n+j})^{-1} \mathbf{Y}_{n+j}, \qquad k > 0. \tag{26}$$

The formulas above are almost exact copies of (3) and (4) respectively, which were the starting points in our analysis, and some partial results can be derived from them.

Let $\mathbf{a}_j(n)$ be the $p \times p$ matrix defined as

$$\mathbf{a}_j(n) = \begin{bmatrix} a_j(n) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The following proposition which gives a condition for bounded and unique solution of the ARMA(p, q) system is a generalization of Proposition 2.1:

PROPOSITION 4.1. *Suppose that* $\sup_{n\geq 1} \|(\mathbf{B}_1^n)^{-1}\| = 0$. *Then the system* (24) *has a bounded solution in* M_ξ *iff*

$$\sup_{n\in\mathbb{Z}} \sum_{s=2-q}^{\infty} \Big\| \sum_{j=\max(1,s)}^{q-1+s} (\mathbf{B}_{n+1}^{n+j})^{-1}\mathbf{a}_{j-s}(n+j)\Big\|^2 < \infty. \tag{27}$$

Moreover, *if this is the case*, *then there is a unique bounded solution of* (24) *in* M_ξ *and it is given by*

$$\mathbf{X}_n = -\sum_{s=2-q}^{\infty} \Big[\sum_{j=\max(1,s)}^{q-1+s} (\mathbf{B}_{n+1}^{n+j})^{-1}\mathbf{a}_{j-s}(n+j)\Big] \varXi_{n+s}, \tag{28}$$

where $\varXi_n$ *is the column* p-*vector* $[\xi_n, 0, \ldots, 0]'$.

Unfortunately, in the general case conditions for bounded solution of an ARMA(p, q) system are more complex. Notice that the matrix formula (27) is difficult to verify. We hope that simpler conditions can be derived by other methods. For example a promising reference is [5, Th. 2.3], where the sufficient conditions are considered in the framework of locally stationary processes. However, the resulting conditions are also difficult to verify in a practical situation.

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