APPLICATIONES MATHEMATICAE
31,3 (2004), pp. 345–351

A. Lesfari and A. Elachab (El-Jadida)

ON THE INTEGRABILITY OF THE GENERALIZED YANG–MILLS SYSTEM

Abstract. We consider a hamiltonian system which, in a special case and under the gauge group $SU(2)$, can be considered as a reduction of the Yang–Mills field equations. We prove explicitly, using the Lax spectral curve technique and the van Moerbeke–Mumford method, that the flows generated by the constants of motion are straight lines on the Jacobi variety of a genus two Riemann surface.

1. Statement of the problem. The problem of integrating hamiltonian systems on symplectic manifolds has attracted a considerable amount of attention in recent years. In hamiltonian mechanics, to integrate a dynamical system with n degrees of freedom, it is sufficient in most cases to know only the first n integrals. This situation is known as Liouville complete integrability of a hamiltonian system. It seems still hopeless to describe, or even to recognize with any facility, those hamiltonian systems which are completely integrable, though they are quite exceptional. Now we shall recall their exact definition. A hamiltonian system on a $2n$-dimensional symplectic manifold is called *completely integrable* if it has n integrals $H_1, \ldots, H_n$ in involution (i.e., such that the associated Poisson brackets $\{H_i, H_j\}$ all vanish) with linearly independent gradients (i.e., $dH_1 \wedge \ldots \wedge dH_n \neq 0$). For appropriate constants $c_1, \ldots, c_n$, the invariant manifold $\{H_1 = c_1, \ldots, H_n = c_n\}$ is compact, connected and therefore diffeomorphic to an n-dimensional torus, by the Arnold–Liouville theorem [9]. Also, there is a transformation to so-called action-angle variables, mapping the flow into a straight line motion on that torus.

In this paper, we consider the Yang–Mills system for a field with gauge group $SU(2)$:

2000 *Mathematics Subject Classification*: 70H05, 70H06, 14H70, 14H40.

Key words and phrases: hamiltonian, integrable systems, Lax representation, curves, Jacobian varieties.

$$\frac{\partial F_{ik}}{\partial x_i} + [A_i, F_{ik}] = 0,$$

where $F_{ik}, A_i \in T_e SU(2), 1 \leq i, k \leq 4$, and

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} + [A_i, A_k].$$

In the case of a homogeneous two-component field, $\partial A_k/\partial x_i = 0$ $(i \neq 1)$, $A_1 = A_2 = 0$, $A_3 = n_1 U_1 \in su(2)$, $A_4 = n_2 U_2 \in su(2)$, where n_i are $su(2)$-generators. The system becomes

$$\frac{\partial^2 U_1}{\partial t^2} + U_1 U_2^2 = 0, \qquad \frac{\partial^2 U_2}{\partial t^2} + U_2 U_1^2 = 0,$$

with $t = x_1$. By setting $U_i = q_i$, $\partial U_i/\partial t = p_i$, $i = 1, 2$, the Yang–Mills equations reduce to a hamiltonian system with the hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + q_1^2 q_2^2).$$

The symplectic transformation

$$\begin{aligned} p_1 &= \alpha(x_1 + x_2), \qquad & q_1 &= \beta(y_1 + i y_2), \\ p_2 &= \alpha(x_1 - x_2), \qquad & q_2 &= \beta(y_1 - i y_2), \end{aligned}$$

where $\alpha \equiv \sqrt{2}/2$ and $\beta \equiv \frac{1}{2}(\sqrt[4]{2})^3$, takes the hamiltonian into

$$H = \tfrac{1}{2}(x_1^2 + x_2^2) + \tfrac{1}{4}y_1^4 + \tfrac{1}{4}y_2^4 + \tfrac{1}{2}y_1^2 y_2^2.$$

We start with a general expression of the Yang–Mills hamiltonian

$$H = \tfrac{1}{2}(x_1^2 + x_2^2 + a_1 y_1^2 + a_2 y_2^2) + \tfrac{1}{4}y_1^4 + \tfrac{1}{4}a_3 y_2^4 + \tfrac{1}{2}a_4 y_1^2 y_2^2. \tag{1}$$

The corresponding system is given by

$$\begin{aligned} \dot{y}_1 &= x_1, \qquad & \dot{x}_1 &= -a_1 y_1 - y_1^3 - a_4 y_1 y_2^2, \\ \dot{y}_2 &= x_2, \qquad & \dot{x}_2 &= -a_2 y_2 - a_3 y_2^3 - a_4 y_1^2 y_2. \end{aligned} \tag{2}$$

This hamiltonian system also arises in connection with some problems in scalar field theory and in the semi-classical method in quantum field theory. The integrability of the system (2) has been studied by several authors (e.g. [2], [13], [14]). It has been shown [2] that the hamiltonian (1) has the Painlevé property (i.e., the general solutions have no movable singularities other than poles) only if

$$\begin{aligned} &\text{(i)} \quad a_1 = a_2, \qquad a_3 = a_4 = 1, \\ &\text{(ii)} \quad a_1 = a_2, \qquad a_3 = 1, \qquad a_4 = 3. \end{aligned}$$

In case (i), the second integral has the form

$$H_2 = x_2 y_1 - x_1 y_2,$$

whereas in case (ii) the second integral is

$$H_2 = x_1 x_2 + y_1 y_2 (a_1 + y_1^2 + y_2^2).$$

Recently, it was shown [5] that if

$$a_1 = a_2/4, \qquad a_3 = 16, \qquad a_4 = 6,$$

then the system (2) is integrable and the second integral is

$$H_2 = y_1^2 y_2(a_1 + y_1^2 + 2y_2^2) + x_1(x_2 y_1 - x_1 y_2).$$

This paper deals with the problem of integrability of the system (2) corresponding to the choice: a_1, a_2 arbitrary and $a_3 = a_4 = 1$. We study this case using the Lax spectral curve technique and the van Moerbeke–Mumford linearization method. We show that in this case the system (2) is linearized in the jacobian variety of a genus two hyperelliptic Riemann surface Γ.

2. Lax representation and complete integrability. New examples of completely integrable hamiltonian systems, which have recently been discovered, are based on the Lax representation of the equations of motion. For more information, see [1], [3], [4], [6] and [12]. Using the results given in [3] and [4], we consider the Lax representation in the form

$$\dot{A}_h = [B_h, A_h] = B_h A_h - A_h B_h, \qquad \dot{} \equiv \frac{\partial}{\partial t}, \tag{3}$$

with the following ansatz for the Lax operator:

$$A_h = \begin{pmatrix} U(h) & V(h) \\ W(h) & -U(h) \end{pmatrix}, \qquad B_h = \begin{pmatrix} 0 & 1 \\ R(h) & 0 \end{pmatrix},$$

where

$$\begin{aligned} V(h) &= -(a_1 + h)(a_2 + h)\left(1 + \frac{1}{2}\left(\frac{y_1^2}{a_1 + h} + \frac{y_2^2}{a_2 + h}\right)\right), \\ U(h) &= \frac{1}{2}(a_1 + h)(a_2 + h)\left(\frac{x_1 y_1}{a_1 + h} + \frac{x_2 y_2}{a_2 + h}\right), \\ W(h) &= (a_1 + h)(a_2 + h)\left(\frac{1}{2}\left(\frac{x_1^2}{a_1 + h} + \frac{x_2^2}{a_2 + h}\right) - h + \frac{1}{2}(y_1^2 + y_2^2)\right), \\ R(h) &= h - y_1^2 - y_2^2. \end{aligned} \tag{4}$$

Equation (3) is equivalent to (2) with $a_3 = a_4 = 1$. The proof is straightforward and based on direct computation: we have

$$\dot{A}_h = \begin{pmatrix} \dot{U}(h) & \dot{V}(h) \\ \dot{W}(h) & -\dot{U}(h) \end{pmatrix},$$

$$[B_h, A_h] = \begin{pmatrix} W(h) - V(h)R(h) & -2U(h) \\ 2U(h)R(h) & V(h)R(h) - W(h) \end{pmatrix},$$

and it follows from (4) and (2) with $a_3 = a_4 = 1$ that

$$\begin{aligned} \dot{U}(h) &= W(h) - V(h)R(h), \\ \dot{V}(h) &= -2U(h), \\ \dot{W}(h) &= 2U(h)R(h). \end{aligned}$$

Equation (3) means that for $h \in \mathbb{C}$ and under the time evolution of the system, $A_h(t)$ remain similar to $A_h(0)$. So the spectrum of A_h is conserved, i.e. it undergoes an isospectral deformation. The eigenvalues of A_h, viewed as functionals, represent the integrals (constants of motion) of the system. To be precise, a hamiltonian flow of the type (3) preserves the spectrum of A_h and therefore its characteristic polynomial $\det(A_h - zI)$. We form the Riemann surface in (z, h) space

$$\Gamma : \quad \det(A_h - zI) = 0, \tag{5}$$

whose coefficients are functions of the phase space. Explicitly, equation (5) looks as follows:

$$\begin{aligned} \Gamma : \quad z^2 &= U^2(h) + V(h)W(h), \\ &= (a_1 + h)(a_2 + h)(h^3 + (a_1 + a_2)h^2 + (a_1a_2 - H_1)h - H_2) \\ &\equiv P_5(h), \end{aligned} \tag{6}$$

where $H_1 = H$ is defined by (1) with a_1, a_2 arbitrary, $a_3 = a_4 = 1$ and a second quartic integral H_2 of the form

$$\begin{aligned} H_2 = \tfrac{1}{4}(a_2y_1^4 + a_1y_2^4 + (a_1 + a_2)y_1^2y_2^2 + (x_1y_2 - x_2y_1)^2) \\ + \tfrac{1}{2}(a_2x_1^2 + a_1x_2^2 + a_1a_2(y_1^2 + y_2^2)). \end{aligned} \tag{7}$$

The Riemann surface Γ determined by the fifth-order equation (6) is smooth, hyperelliptic and its genus is two. Obviously, Γ is invariant under the hyperelliptic involution $(h, z) \curvearrowright (h, -z)$. The second hamiltonian vector field is written as

$$\begin{aligned} \dot{y}_1 &= \tfrac{1}{2}(x_1y_2 - x_2y_1)y_2 + a_2x_1, \\ \dot{y}_2 &= -\tfrac{1}{2}(x_1y_2 - x_2y_1)y_1 + a_1x_2, \\ \dot{x}_1 &= -a_2y_1^3 - \tfrac{1}{2}(a_1 + a_2)y_1y_2^2 + \tfrac{1}{2}(x_1y_2 - x_2y_1)x_2 - a_1a_2y_1, \\ \dot{x}_2 &= -a_1y_2^3 - \tfrac{1}{2}(a_1 + a_2)y_1^2y_2 - \tfrac{1}{2}(x_1y_2 - x_2y_1)x_1 - a_1a_2y_2. \end{aligned}$$

These vector fields are in involution with respect to the associated Poisson bracket. For generic $c = (c_1, c_2) \in \mathbb{C}^2$ the affine variety defined by

$$M_c = \{H_1 = c_1,\ H_2 = c_2\}, \tag{8}$$

is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and thus M_c is a smooth affine surface. Using the van Moerbeke–Mumford linearization method [11], we show that the linearized flow can be realized on the jacobian variety $\mathrm{Jac}(\Gamma)$

of the Riemann surface (6) associated to (3). Recall that the jacobian of Γ is

$$\begin{aligned}\mathrm{Jac}(\Gamma) &\equiv \mathrm{Pic}^0(\Gamma) = \text{line bundles of degree zero},\\ &= H^1(\mathcal{O}_\Gamma)/H^1(\Gamma,\mathbb{Z}) \quad \text{via the exponential sheaf sequence},\\ &\simeq H^0(\Omega_\Gamma)^*/H_1(\Gamma,\mathbb{Z}) \quad \text{via the duality given by Abel's theorem},\end{aligned}$$

where Ω_Γ is the sheaf of holomorphic 1-forms on Γ. We can construct an algebraic map from M_c to the Jacobi variety $\mathrm{Jac}(\Gamma)$:

$$M_c \to \mathrm{Jac}(\Gamma), \qquad p \in M_c \curvearrowright (s_1 + s_2) \in \mathrm{Jac}(\Gamma),$$

and the flows generated by the constants of motion are straight lines on $\mathrm{Jac}(\Gamma)$, i.e., the linearizing equations are given by

$$\sum_{i=1}^{2} \int_{s_i(0)}^{s_i(t)} \omega_k = c_k t, \quad 0 \le k \le 2,$$

where ω_1, ω_2 span the two-dimensional space of holomorphic differentials on the Riemann surface Γ and s_1, s_2 are two appropriate variables, algebraically related to the originally given ones, for which the Hamilton–Jacobi equation can be solved by separation of variables. Consequently, we have

THEOREM 1. *Suppose that $a_3 = a_4 = 1$. Then the system* (2) *is completely integrable for all a_1, a_2 and admits a Lax representation given by* (3). *The invariants of A_h are integrals of motion in involution. The first integral is given by the hamiltonian $H_1 = H$ (see* (1)), *whereas the second integral H_2 is also quartic and has the form* (7). *The flows generated by H_1 and H_2 are straight line motions on the jacobian variety* $\mathrm{Jac}(\Gamma)$ *of a smooth genus two hyperelliptic Riemann surface Γ given by* (6) *associated to Lax equation* (3).

According to the schema of [3] and [4], we introduce coordinates s_1 and s_2 on the surface M_c given by (8) such that $V(s_1) = V(s_2) = 0$, $a_1 \neq a_2$,

$$y_1^2 = 2\,\frac{(a_1+s_1)(a_1+s_2)}{a_1-a_2}, \qquad y_2^2 = 2\,\frac{(a_2+s_1)(a_2+s_2)}{a_2-a_1},$$

i.e.,

$$s_1 + s_2 = \tfrac{1}{2}(y_1^2 + y_2^2) - a_1 - a_2, \qquad s_1 s_2 = -\tfrac{1}{2}(a_2 y_1^2 + a_1 y_2^2) + a_1 a_2.$$

After some algebraic manipulations, we obtain the following equations for s_1 and s_2:

$$\dot{s}_1 = 2\,\frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \qquad \dot{s}_2 = 2\,\frac{\sqrt{P_5(s_2)}}{s_2 - s_1},$$

where $P_5(s)$ is defined by (6). These equations can be integrated by the abelian mapping

$$\Gamma \to \mathrm{Jac}(\Gamma) = \mathbb{C}^2/L, \qquad p \curvearrowright \Big(\int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2 \Big),$$

where the hyperelliptic Riemann surface Γ of genus two is given by the equation (6), L is the lattice generated by the vectors $n_1 + \Omega n_2$, $(n_1, n_2) \in \mathbb{Z}^2$, Ω is the period matrix of the Riemann surface Γ, (ω_1, ω_2) is the canonical basis of holomorphic differentials on Γ, i.e.,

$$\omega_1 = \frac{ds}{\sqrt{P_5(s)}}, \qquad \omega_2 = \frac{sds}{\sqrt{P_5(s)}},$$

and p_0 is a fixed point. Consequently, we have

THEOREM 2. *The system of differential equations* (2) *with* $a_3 = a_4 = 1$ *can be integrated in terms of genus two hyperelliptic functions of time.*

For $a_1 = a_2$, it is easy to show that the problem can be integrated in terms of elliptic functions.

References

[1] M. Adler and P. van Moerbeke, *Linearization of Hamiltonian systems, Jacobi varieties and representation theory*, Adv. Math. 38 (1980), 318–379.

[2] T. Bountis, H. Segur and F. Vivaldi, *Integrable Hamiltonian systems and the Painlevé property*, Phys. Rev. A 25 (1982), 1257–1264.

[3] J. C. Eilbeck, V. Z. Enolskii, V. B. Kuznetsov, and D. V. Leykin, *Linear r-matrix algebra for systems separable in parabolic coordinates*, Phys. Lett. A 180 (1993), 208–214.

[4] J. C. Eilbeck, V. Z. Enolskii, V. B. Kuznetsov and A. V. Tsiganov, *Linear r-matrix algebra for classical separable systems*, J. Phys. A: Math. Gen. 27 (1994), 567–578.

[5] S. Kasperczuk, *Integrability of the Yang–Mills Hamiltonian system*, Celes. Mech. Dyn. Astr. 58 (1994), 387–391.

[6] A. Lesfari, *Completely integrable systems*: *Jacobi's heritage*, J. Geom. Phys. 567 (1999), 1–22.

[7] —, *The problem of the motion of a solid in an ideal fluid. Integration of the Clebsch's case*, NoDEA Nonlinear Differ. Equ. Appl. 8 (2001), 1–13.

[8] —, *A new class of integrable systems*, Arch. Math. (Basel) 77 (2001), 347–353.

[9] —, *Le théorème d'Arnold–Liouville et ses conséquences*, Elem. Math. 58 (2003), 6–20.

[10] —, *Le système différentiel de Hénon–Heiles et les variétés Prym*, Pacific J. Math. 212 (2003), 125–132.

[11] P. van Moerbeke and D. Mumford, *The spectrum of difference operators and algebraic curves*, Acta Math. 143 (1979), 93–154.

[12] J. Moser, *Three integrable Hamiltonian systems connected with isospectral deformations*, Adv. Math. 16 (1975), 197–219.

[13] H. Yoshida, *Existence of exponentially unstable solutions and the non-integrability of homogeneous Hamiltonian*, Phys. D 21 (1986), 163–170.

[14] S. L. Ziglin, *Branching of solutions and the nonexistence of first integrals in Hamiltonian mechanics II*, Funct. Anal. Appl. 17 (1983), 6–17.

Department of Mathematics
Faculty of Sciences
University of Chouaïb Doukkali
B.P. 20, El-Jadida, Morocco
E-mail: Lesfari@ucd.ac.ma
Lesfariahmed@yahoo.fr

Received on 26.1.2004;
revised version on 8.4.2004 (1726)