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ON THE INTEGRABILITY OF THE GENERALIZED YANG–MILLS SYSTEM

Abstract. We consider a hamiltonian system which, in a special case and under the gauge group $SU(2)$, can be considered as a reduction of the Yang–Mills field equations. We prove explicitly, using the Lax spectral curve technique and the van Moerbeke–Mumford method, that the flows generated by the constants of motion are straight lines on the Jacobi variety of a genus two Riemann surface.

1. Statement of the problem. The problem of integrating hamiltonian systems on symplectic manifolds has attracted a considerable amount of attention in recent years. In hamiltonian mechanics, to integrate a dynamical system with $n$ degrees of freedom, it is sufficient in most cases to know only the first $n$ integrals. This situation is known as Liouville complete integrability of a hamiltonian system. It seems still hopeless to describe, or even to recognize with any facility, those hamiltonian systems which are completely integrable, though they are quite exceptional. Now we shall recall their exact definition. A hamiltonian system on a $2n$-dimensional symplectic manifold is called completely integrable if it has $n$ integrals $H_1, \ldots, H_n$ in involution (i.e., such that the associated Poisson brackets $\{H_i, H_j\}$ all vanish) with linearly independent gradients (i.e., $dH_1 \wedge \ldots \wedge dH_n \neq 0$). For appropriate constants $c_1, \ldots, c_n$, the invariant manifold $\{H_1 = c_1, \ldots, H_n = c_n\}$ is compact, connected and therefore diffeomorphic to an $n$-dimensional torus, by the Arnold–Liouville theorem [9]. Also, there is a transformation to so-called action-angle variables, mapping the flow into a straight line motion on that torus.

In this paper, we consider the Yang–Mills system for a field with gauge group $SU(2)$:

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\[ \frac{\partial F_{ik}}{\partial x_i} + [A_i, F_{ik}] = 0, \]

where \( F_{ik}, A_i \in T_c SU(2), 1 \leq i, k \leq 4, \) and

\[ F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} + [A_i, A_k]. \]

In the case of a homogeneous two-component field, \( \partial A_k / \partial x_i = 0 \) (\( i \neq 1 \)), \( A_1 = A_2 = 0, A_3 = n_1 U_1 \in su(2), A_4 = n_2 U_2 \in su(2) \), where \( n_i \) are \( su(2) \)-generators. The system becomes

\[ \frac{\partial^2 U_1}{\partial t^2} + U_1 U_2^2 = 0, \quad \frac{\partial^2 U_2}{\partial t^2} + U_2 U_1^2 = 0, \]

with \( t = x_1 \). By setting \( U_i = q_i, \partial U_i / \partial t = p_i, i = 1, 2, \) the Yang–Mills equations reduce to a hamiltonian system with the hamiltonian

\[ H = \frac{1}{2} (p_1^2 + p_2^2 + q_1^2 q_2^2). \]

The symplectic transformation

\[ p_1 = \alpha (x_1 + x_2), \quad q_1 = \beta (y_1 + iy_2), \]
\[ p_2 = \alpha (x_1 - x_2), \quad q_2 = \beta (y_1 - iy_2), \]

where \( \alpha = \sqrt{2}/2 \) and \( \beta = \frac{1}{2} (\sqrt{2})^3 \), takes the hamiltonian into

\[ H = \frac{1}{2} (x_1^2 + x_2^2) + \frac{1}{4} y_1^4 + \frac{1}{4} y_2^4 + \frac{1}{2} y_1^2 y_2^2. \]

We start with a general expression of the Yang–Mills hamiltonian

(1) \[ H = \frac{1}{2} (x_1^2 + x_2^2 + a_1 y_1^2 + a_2 y_2^2) + \frac{1}{4} y_1^4 + \frac{1}{4} y_2^4 + \frac{1}{2} a_3 y_1^2 y_2^2. \]

The corresponding system is given by

(2) \[ \dot{y}_1 = x_1, \quad \dot{x}_1 = -a_1 y_1 - y_1^3 + a_3 y_1 y_2, \]
\[ \dot{y}_2 = x_2, \quad \dot{x}_2 = -a_2 y_2 - a_3 y_2^3 - a_4 y_1^2 y_2. \]

This hamiltonian system also arises in connection with some problems in scalar field theory and in the semi-classical method in quantum field theory. The integrability of the system (2) has been studied by several authors (e.g. [2], [13], [14]). It has been shown [2] that the hamiltonian (1) has the Painlevé property (i.e., the general solutions have no movable singularities other than poles) only if

(i) \[ a_1 = a_2, \quad a_3 = a_4 = 1, \]

(ii) \[ a_1 = a_2, \quad a_3 = 1, \quad a_4 = 3. \]

In case (i), the second integral has the form

\[ H_2 = x_2 y_1 - x_1 y_2, \]

whereas in case (ii) the second integral is

\[ H_2 = x_1 x_2 + y_1 y_2 (a_1 + y_1^2 + y_2^2). \]
Recently, it was shown [5] that if
\[ a_1 = a_2/4, \quad a_3 = 16, \quad a_4 = 6, \]
then the system (2) is integrable and the second integral is
\[ H_2 = y_1^2 y_2 (a_1 + y_1^2 + 2y_2^2) + x_1(x_2y_1 - x_1y_2). \]

This paper deals with the problem of integrability of the system (2) corresponding to the choice: \( a_1, a_2 \) arbitrary and \( a_3 = a_4 = 1 \). We study this case using the Lax spectral curve technique and the van Moerbeke–Mumford linearization method. We show that in this case the system (2) is linearized in the jacobian variety of a genus two hyperelliptic Riemann surface \( \Gamma \).

2. Lax representation and complete integrability. New examples of completely integrable hamiltonian systems, which have recently been discovered, are based on the Lax representation of the equations of motion. For more information, see [1], [3], [4], [6] and [12]. Using the results given in [3] and [4], we consider the Lax representation in the form
\[ \dot{A}_h = [B_h, A_h] = B_h A_h - A_h B_h, \quad \equiv \frac{\partial}{\partial t}, \]
with the following ansatz for the Lax operator:
\[ A_h = \begin{pmatrix} U(h) & V(h) \\ W(h) & -U(h) \end{pmatrix}, \quad B_h = \begin{pmatrix} 0 & 1 \\ R(h) & 0 \end{pmatrix}, \]
where
\[ V(h) = -(a_1 + h)(a_2 + h)
\left( 1 + \frac{1}{2} \left( \frac{y_1^2}{a_1 + h} + \frac{y_2^2}{a_2 + h} \right) \right), \]
\[ U(h) = \frac{1}{2}(a_1 + h)(a_2 + h)
\left( \frac{x_1 y_1}{a_1 + h} + \frac{x_2 y_2}{a_2 + h} \right), \]
\[ W(h) = (a_1 + h)(a_2 + h)
\left( \frac{x_1^2}{a_1 + h} + \frac{x_2^2}{a_2 + h} \right) - h + \frac{1}{2}(y_1^2 + y_2^2), \]
\[ R(h) = h - y_1^2 - y_2^2. \]

Equation (3) is equivalent to (2) with \( a_3 = a_4 = 1 \). The proof is straightforward and based on direct computation: we have
\[ \dot{A}_h = \begin{pmatrix} \dot{U}(h) & \dot{V}(h) \\ \dot{W}(h) & -\dot{U}(h) \end{pmatrix}, \]
\[ [B_h, A_h] = \begin{pmatrix} W(h) - V(h)R(h) & -2U(h) \\ 2U(h)R(h) & V(h)R(h) - W(h) \end{pmatrix}. \]
and it follows from (4) and (2) with $a_3 = a_4 = 1$ that
\[
\begin{align*}
\dot{U}(h) &= W(h) - V(h)R(h), \\
\dot{V}(h) &= -2U(h), \\
\dot{W}(h) &= 2U(h)R(h).
\end{align*}
\]
Equation (3) means that for $h \in \mathbb{C}$ and under the time evolution of the system, $A_h(t)$ remain similar to $A_h(0)$. So the spectrum of $A_h$ is conserved, i.e. it undergoes an isospectral deformation. The eigenvalues of $A_h$, viewed as functionals, represent the integrals (constants of motion) of the system. To be precise, a hamiltonian flow of the type (3) preserves the spectrum of $A_h$ and therefore its characteristic polynomial $\det(A_h - zI)$. We form the Riemann surface in $(z, h)$ space
\[
\Gamma : \quad \det(A_h - zI) = 0,
\]
whose coefficients are functions of the phase space. Explicitly, equation (5) looks as follows:
\[
\begin{align*}
\Gamma : \quad z^2 &= U^2(h) + V(h)W(h), \\
&= (a_1 + h)(a_2 + h)(h^3 + (a_1 + a_2)h^2 + (a_1a_2 - H_1)h - H_2) \\
&\equiv P_5(h),
\end{align*}
\]
where $H_1 = H$ is defined by (1) with $a_1, a_2$ arbitrary, $a_3 = a_4 = 1$ and a second quartic integral $H_2$ of the form
\[
\begin{align*}
H_2 &= \frac{1}{4}(a_2y_1^4 + a_1y_2^4 + (a_1 + a_2)y_1^2y_2^2 + (x_1y_2 - x_2y_1)^2) \\
&\quad + \frac{1}{2}(a_2x_1^2 + a_1x_2^2 + a_1a_2(y_1^2 + y_2^2)).
\end{align*}
\]
The Riemann surface $\Gamma$ determined by the fifth-order equation (6) is smooth, hyperelliptic and its genus is two. Obviously, $\Gamma$ is invariant under the hyperelliptic involution $(h, z) \sim (h, -z)$. The second hamiltonian vector field is written as
\[
\begin{align*}
\dot{y}_1 &= \frac{1}{2}(x_1y_2 - x_2y_1)y_2 + a_2x_1, \\
\dot{y}_2 &= -\frac{1}{2}(x_1y_2 - x_2y_1)y_1 + a_1x_2, \\
\dot{x}_1 &= -a_2y_1^3 - \frac{1}{2}(a_1 + a_2)y_1y_2^2 + \frac{1}{2}(x_1y_2 - x_2y_1)x_2 - a_1a_2y_1, \\
\dot{x}_2 &= -a_1y_2^3 - \frac{1}{2}(a_1 + a_2)y_2^2y_1 - \frac{1}{2}(x_1y_2 - x_2y_1)x_1 - a_1a_2y_2.
\end{align*}
\]
These vector fields are in involution with respect to the associated Poisson bracket. For generic $c = (c_1, c_2) \in \mathbb{C}^2$ the affine variety defined by
\[
\begin{align*}
M_c &= \{H_1 = c_1, \ H_2 = c_2\},
\end{align*}
\]
is the fibre of a morphism from $\mathbb{C}^4$ to $\mathbb{C}^2$ and thus $M_c$ is a smooth affine surface. Using the van Moerbeke–Mumford linearization method [11], we show that the linearized flow can be realized on the jacobian variety $\text{Jac}(\Gamma)$
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of the Riemann surface (6) associated to (3). Recall that the jacobian of \( \Gamma \) is

\[
\text{Jac}(\Gamma) = \text{Pic}^0(\Gamma) = \text{line bundles of degree zero},
\]

\[
= H^1(\mathcal{O}_\Gamma)/H^1(\Gamma, \mathbb{Z}) \quad \text{via the exponential sheaf sequence},
\]

\[
\simeq H^0(\mathcal{O}_\Gamma)^*/H_1(\Gamma, \mathbb{Z}) \quad \text{via the duality given by Abel’s theorem},
\]

where \( \mathcal{O}_\Gamma \) is the sheaf of holomorphic 1-forms on \( \Gamma \). We can construct an algebraic map from \( M_c \) to the Jacobi variety \( \text{Jac}(\Gamma) \):

\[
M_c \to \text{Jac}(\Gamma), \quad p \in M_c \cap (s_1 + s_2) \in \text{Jac}(\Gamma),
\]

and the flows generated by the constants of motion are straight lines on \( \text{Jac}(\Gamma) \), i.e., the linearizing equations are given by

\[
\sum_{i=1}^{2} \int_{s_i(0)}^{s_i(t)} \omega_k = c_k t, \quad 0 \leq k \leq 2,
\]

where \( \omega_1, \omega_2 \) span the two-dimensional space of holomorphic differentials on the Riemann surface \( \Gamma \) and \( s_1, s_2 \) are two appropriate variables, algebraically related to the originally given ones, for which the Hamilton–Jacobi equation can be solved by separation of variables. Consequently, we have

**Theorem 1.** Suppose that \( a_3 = a_4 = 1 \). Then the system (2) is completely integrable for all \( a_1, a_2 \) and admits a Lax representation given by (3). The invariants of \( A_h \) are integrals of motion in involution. The first integral is given by the Hamiltonian \( H_1 = H \) (see (1)), whereas the second integral \( H_2 \) is also quartic and has the form (7). The flows generated by \( H_1 \) and \( H_2 \) are straight line motions on the jacobian variety \( \text{Jac}(\Gamma) \) of a smooth genus two hyperelliptic Riemann surface \( \Gamma \) given by (6) associated to Lax equation (3).

According to the schema of \([3]\) and \([4]\), we introduce coordinates \( s_1 \) and \( s_2 \) on the surface \( M_c \) given by (8) such that \( V(s_1) = V(s_2) = 0, a_1 \neq a_2, \)

\[
y_1^2 = 2 \frac{(a_1 + s_1)(a_1 + s_2)}{a_1 - a_2}, \quad y_2^2 = 2 \frac{(a_2 + s_1)(a_2 + s_2)}{a_2 - a_1},
\]

i.e.,

\[
s_1 + s_2 = \frac{1}{2}(y_1^2 + y_2^2) - a_1 - a_2, \quad s_1 s_2 = -\frac{1}{2}(a_2 y_1^2 + a_1 y_2^2) + a_1 a_2.
\]

After some algebraic manipulations, we obtain the following equations for \( s_1 \) and \( s_2 \):

\[
\dot{s}_1 = 2 \frac{\sqrt{P_5(s_1)}}{s_1 - s_2}, \quad \dot{s}_2 = 2 \frac{\sqrt{P_5(s_2)}}{s_2 - s_1},
\]

where \( P_5(s) \) is defined by (6). These equations can be integrated by the
abelian mapping

\[ \Gamma \to \text{Jac}(\Gamma) = \mathbb{C}^2 / L, \quad p \sim \left( \int_{p_0}^{p} \omega_1, \int_{p_0}^{p} \omega_2 \right), \]

where the hyperelliptic Riemann surface \( \Gamma \) of genus two is given by the equation (6), \( L \) is the lattice generated by the vectors \( n_1 + \Omega n_2, (n_1, n_2) \in \mathbb{Z}^2 \), \( \Omega \) is the period matrix of the Riemann surface \( \Gamma \), \( (\omega_1, \omega_2) \) is the canonical basis of holomorphic differentials on \( \Gamma \), i.e.,

\[ \omega_1 = \frac{ds}{\sqrt{P_5(s)}}, \quad \omega_2 = \frac{sds}{\sqrt{P_5(s)}}, \]

and \( p_0 \) is a fixed point. Consequently, we have

**Theorem 2.** The system of differential equations (2) with \( a_3 = a_4 = 1 \) can be integrated in terms of genus two hyperelliptic functions of time.

For \( a_1 = a_2 \), it is easy to show that the problem can be integrated in terms of elliptic functions.

**References**

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