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COVARIANCE STRUCTURE OF WIDE-SENSE MARKOV PROCESSES OF ORDER $k \geq 1$

Abstract. A notion of a wide-sense Markov process $\{X_t\}$ of order $k \geq 1$, $\{X_t\} \sim \mathrm{WM}(k)$, is introduced as a direct generalization of Doob's notion of wide-sense Markov process (of order $k = 1$ in our terminology). A base for investigation of the covariance structure of $\{X_t\}$ is the k-dimensional process $\{x_t = (X_{t-k+1}, \ldots, X_t)\}$. The covariance structure of $\{X_t\} \sim \mathrm{WM}(k)$ is considered in the general case and in the periodic case. In the general case it is shown that $\{X_t\} \sim \mathrm{WM}(k)$ iff $\{x_t\}$ is a k-dimensional WM(1) process and iff the covariance function of $\{x_t\}$ has the triangular property. Moreover, an analogue of Borisov's theorem is proved for $\{x_t\}$. In the periodic case, with period $d > 1$, it is shown that Gladyshev's process $\{Y_t = (X_{(t-1)d+1}, \ldots, X_{td})\}$ is a d-dimensional AR(p) process with $p = \lceil k/d \rceil$.

1. Introduction. The paper deals with a characterization of the structure of the covariance function of the periodic and nonperiodic Markov processes in the wide sense of order $k \geq 1$. A real-valued process $\{X_t, t \in \mathbb{Z}\} \equiv \{X_t\}$, in a discrete time $t \in \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$ with $EX_t^2 < \infty$, is called a *Markov process in the wide sense of order* $k \geq 1$, briefly a WM(k) process, if the best linear prediction of the process at time $u > t$, based on the past up to time t, denoted here by $\widehat{E}(X_u \mid X_s,\ s \leq t)$, is, with probability one, equal to the best linear prediction of the value X_u based on the vector $(X_t, X_{t-1}, \ldots, X_{t-k+1})$, i.e. $\widehat{E}(X_u \mid X_s,\ s \leq t) \stackrel{P.1}{=} \widehat{E}(X_u \mid X_t, X_{t-1}, \ldots, X_{t-k+1})$. The best linear prediction is meant in the sense of the minimum-mean-square-error prediction. This is a generaliza-

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tion of the notion of a WM(1) process introduced by J. Doob in [4, p. 233], which he called a Markov process in the wide sense. Doob considered those processes in discrete time as well as in continuous time, both complex and real valued. Here, presenting the results on WM(1) processes as well as considering WM(k) processes, we restrict our attention to real WM(k) processes in discrete time. The covariance function for such a process is denoted by $\gamma(s,t) = \mathrm{cov}(X_s, X_t)$. It is worth mentioning that the class of stationary WM(1) processes equals the class of autoregression processes of order one, i.e. AR(1). Similarly, the class of stationary WM(k) processes equals AR(k). In the case of wide-sense Markov processes of order $k = 1$ this follows from [4] and for any k it follows from our Corollary 1. Below we consider those processes in the nonstationary case.

Doob showed (see [4, p. 234]) that $\{X_t\}$ is a WM(1) process iff the function $R(t,s) := \gamma(s,t)/\gamma(s,s),\ s \le t$, has the *triangular property*, i.e.

$$R(s,u) = R(s,t)R(t,u) \quad \text{for } s \le t \le u. \tag{1}$$

Relation (1) can be rewritten as $\gamma(s,u)\gamma(t,t) = \gamma(s,t)\gamma(t,u)$ for $s \le t \le u$, which can also be rewritten as $\gamma(s,u) = \gamma(s,t)\gamma^{-1}(t,t)\gamma(t,u)$, if $\gamma(t,t) > 0$. Iteration of the last equality gives

$$\gamma(s,u) = \gamma(s,s+1)\,\frac{\gamma(s+1,s+2)}{\gamma(s+1,s+1)} \cdots \frac{\gamma(u-1,u)}{\gamma(u-1,u-1)}, \tag{2}$$

which means that the covariance function $\gamma(s,u)$ is determined by the values $0 < \sigma_t^2 := \gamma(t,t)$ and $b_t := \gamma(t-1,t)$ which satisfy $b_t^2 \le \sigma_{t-1}^2\sigma_t^2$.

Another characterization of covariance functions for WM(1) processes was given by I. S. Borisov in [3]. He showed that a function $f(s,t)$ is the covariance function of a WM(1) process iff

$$f(s,t) = G(\min(s,t))H(\max(s,t)), \tag{3}$$

where the functions G and H are determined uniquely up to a multiplicative constant and the ratio G/H is a positive and nondecreasing function.

An interesting case of nonstationary processes is when the covariance function is periodic. A process $\{X_t\}$ is called *periodically correlated with period* $d \ge 1$, briefly PC(d), if $\gamma(s,t) = \gamma(s+d, t+d)$ for all s,t and d is the smallest number with that property. Gladyshev showed (see [5]) that, if $\{X_t\}$ is PC(d), then the d-dimensional process $\{Y_t, t \in \mathbb{Z}\} \equiv \{Y_t\}$ defined as $Y_t = (X_{dt-d+1}, X_{dt-d+2}, \ldots, X_{dt}),\ t \in \mathbb{Z}$, is stationary. In the case when $\{X_t\}$ is a WM(k) and PC(d) process we say that $\{X_t\}$ is a WM(k)PC(d) process. The structure of covariance functions for WM(1)PC(d) processes was characterized by A. R. Nematollahi and A. R. Soltani in [8]. In that case the covariance function $\gamma(s,t)$ is determined by $2d$ numbers σ_t^2 and $b_t,\ 1 \le t \le d$ (see [8, Theorem 3.2]).

In this paper we study the structure of covariance functions of WM(k) processes and WM(k)PC(d) processes. For that it is natural to consider the multivariate process $\{x_t, t \in \mathbb{Z}\} \equiv \{x_t\}$, defined as

$$x_t := (x_t^{(1)}, \ldots, x_t^{(k)})^T \equiv (X_{t-k+1}, X_{t-k+2}, \ldots, X_t)^T, \quad t \in \mathbb{Z},$$

i.e. $x_t^{(i)} = X_{t-k+i}$, where $(\cdot)^T$ denotes the transposition of a vector. This suggests considering multivariate wide-sense Markov processes. A process $\{Z_t = (Z_t^{(1)}, \ldots, Z_t^{(k)})^T, t \in \mathbb{Z}\} \equiv \{Z_t\}$, where Z_t are random vectors in $\mathbb{R}^k$, is called a *multivariate wide-sense Markov process of order* m, briefly an MWM(m) *process*, if the best linear prediction $\widehat{E}(Z_u \,|\, Z_s,\ s \le t)$ is, with probability one, equal to the best linear prediction $\widehat{E}(Z_u \,|\, Z_t, Z_{t-1}, \ldots, Z_{t-m+1})$. Processes of that type were considered by F. J. Beutler [2] and V. Mandrekar [6] for $m = 1$. They obtained some results similar to the case of WM(1) processes. For example, an analogue of the triangular property (1) for the covariance function of those processes is given in [2].

In this paper we study the structure of covariance functions of WM(k) processes $\{X_t\}$ via studying the structure of covariance functions of the MWM(1) processes $\{x_t\}$. The main results are given in Section 3. First, we show that $\{X_t\}$ is a WM(k) process iff $\{x_t\}$ is a MWM(1) process. Next we show that the covariance function of the $\{x_t\}$ process, denoted here by $\Gamma(s, u)$, satisfies an analogue of the triangular property (1) (it can also be obtained from Beutler's results in [2]). Moreover it satisfies an analogue of the recursive relation (2) and an analogue of (3). Finally, we characterize the structure of covariance functions of those processes in the periodic case. Such a function is specified by d vectors in $\mathbb{R}^k$ and d covariance matrices satisfying some conditions. It turns out that the notion of WM(k)PC(d) processes is related to the notion of periodic autoregressive processes considered by M. Pagano in [7].

2. Preliminaries. In this section we give the main definitions, notation and auxiliary results. For a process $\{X_t\}$ we assume that $EX_t^2 < \infty$ and $EX_t = 0$, where the last assumption is only for simplicity of notation. For any such $\{X_t\}$ we consider the k-dimensional process $\{x_t = (x_t^{(1)}, \ldots, x_t^{(k)})^T\}$, where $x_t^{(i)} = X_{t-k+i}$, $1 \le i \le k$. The expectation of a random vector is meant here as the vector of the expectations of its coordinates. Analogously we understand the conditional expectation and the best linear prediction of a random vector based on some random vector. Namely, for the random vector x_t we define

$$\widehat{E}(x_u \,|\, x_s, s \le t) := (\widehat{E}(x_u^{(1)} \,|\, x_s,\ s \le t), \widehat{E}(x_u^{(2)} \,|\, x_s,\ s \le t), \ldots, \widehat{E}(x_u^{(k)} \,|\, x_s,\ s \le t))^T.$$

Here $\widehat{E}(x_u^{(i)} \mid x_s,\ s \le t)$ is the best linear prediction of $x_u^{(i)}$ based on $(x_s,\ s \le t)$. It is equivalent to the best linear prediction of $x_u^{(i)}$ on $X_t, X_{t-1}, \ldots$. In a similar way we mean define the best linear prediction of x_u on x_t, denoted by $\widehat{x}_{u,t} := \widehat{E}(x_u \mid x_t) := (\widehat{E}(x_u^{(1)}|x_t), \ldots, \widehat{E}(x_u^{(k)}|x_t))^T$.

In the set of nonsingular $k \times k$ matrices we use the operation * defined by $A^* := (A^{-1})^T$. Of course $A^* = (A^T)^{-1}$. Furthermore $(AB)^* = A^*B^*$ and $(B^{-1}A^T)^T = AB^*$. We use the notation $A \succeq 0$ to mean that A is a *covariance matrix*, i.e. a symmetric and nonnegative definite matrix, while $A \succeq B$ denotes that $A - B \succeq 0$. It is well known that if $A \succeq 0$, then

$$MAM^T \succeq 0 \tag{4}$$

for any $m \times k$ matrix M, $m \in \mathbb{N}$.

Furthermore for $t \le u$ we define

$$\begin{gathered}
\Gamma(t,u) := \operatorname{cov}(x_t, x_u) \equiv E x_t x_u^T \equiv (E x_u x_t^T)^T, \\
\Gamma(u,t) := \Gamma(t,u)^T, \qquad \Gamma_t := \Gamma(t,t), \\
\widetilde{R}(t,u) := \Gamma_t^{-1}\Gamma(t,u) \qquad \text{if } \det(\Gamma_t) \neq 0,
\end{gathered}$$

and

$$\begin{aligned}
&\Gamma(t,u) \\
&= \begin{pmatrix}
\gamma(t-k+1, u-k+1) & \gamma(t-k+1, u-k+2) & \ldots & \gamma(t-k+1, u) \\
\gamma(t-k+2, u-k+1) & \gamma(t-k+2, u-k+2) & \ldots & \gamma(t-k+2, u) \\
\ldots & \ldots & \ldots & \ldots \\
\gamma(t, u-k+1) & \gamma(t, u-k+2) & \ldots & \gamma(t,u)
\end{pmatrix} \\
&\equiv \begin{pmatrix}
\Gamma(t,u)_1 \\ \Gamma(t,u)_2 \\ \ldots \\ \Gamma(t,u)_k
\end{pmatrix}
\end{aligned}$$

where $\Gamma(t,u)_i$ denotes the ith row of the matrix $\Gamma(t,u)$, $1 \le i \le k$. Hence

$$\begin{aligned}
\Gamma(t,u)_i &= \operatorname{cov}(x_t^{(i)}, x_u) = E x_t^{(i)} x_u^T \\
&= [\gamma(t-k+i, u-k+1), \gamma(t-k+i, u-k+2), \ldots, \gamma(t-k+i, u)].
\end{aligned}$$

Adapting the well known results on prediction (see for example [1]) to our notation we give the form of the predictions

$$\widehat{E}(x_u^{(i)} \mid x_t) \equiv \widehat{E}(X_{u-k+i} \mid X_t, X_{t-1}, \ldots, X_{t-k+1}),$$

and $\widehat{E}(x_u \mid x_t) \equiv \widehat{x}_{u,t}$.

Proposition 1. *For $t < u$,*

$$\widehat{x}_{u,t} \equiv \widehat{E}(x_u \mid x_t) = \Phi(u,t)x_t, \tag{5}$$

where

$$\widehat{E}(x_u^{(i)} \mid x_t) = \Phi(u,t)_i x_t, \qquad 1 \le i \le k, \tag{6}$$

and

$$\Phi(u,t)_i = [\phi(u,t)_{i,1}, \dots, \phi(u,t)_{i,k}]$$

is the i*th row of the matrix* $\Phi(u,t)$ *defined for* $t \le u$ *as the solution of the equation*

$$\Gamma_t \Phi^T(u,t) = \Gamma(t,u). \tag{7}$$

Equation (7) gives

$$\Phi^T(u,t) = \Gamma_t^{-1}\Gamma(t,u) \equiv \widetilde{R}(t,u). \tag{8}$$

Below we assume that $\det \Gamma(s,u) \neq 0$ for all s,u.

COROLLARY 1. *Every stationary* WM(k) *process is autoregressive of order* k, *i.e.* AR(k).

Proof. Putting $u = t+1, i = k$ and using (6) we obtain

$$X_{t+1} = \widehat{x}_{t+1,t} + \varepsilon_{t+1} = \sum_{j=1}^{k} \phi(t+1,t)_{k,j} X_{t+1-k+j} + \varepsilon_{t+1},$$

where $\{\varepsilon_t\}$ is a white noise with mean zero and ε_{t+1} is uncorrelated with X_s, $s \le t$. By stationarity and (7) it follows that $\phi(t+1,t)_{k,j} = \phi_{k-j+1}$, $j = 1,\dots,k$, are independent of t, $\{\varepsilon_t\}$ is stationary, so

$$X_{t+1} = \sum_{j=1}^{k} \phi_j X_{t+1-j} + \varepsilon_{t+1},$$

which finishes the proof of the corollary. ■

REMARK 1. Corollary 1 also follows from Theorem 5 below with $d = 1$.

3. Main results

3.1. *Autocovariance structure for* MWM(1) *processes* $\{x_t\}$

THEOREM 1. *The following statements are equivalent*:

(i) $\{X_t\}$ *is a* WM(k) *process*;
(ii) $\{x_t\}$ *is an* MWM(1) *process*;
(iii) *for any* $t \le u$ *the random vector* $x_u - \widehat{x}_{u,t}$ *is orthogonal to all random vectors* x_s, $s \le t$, *i.e.*

$$Ex_s(x_u - \widehat{x}_{u,t})^T = \mathbf{0} \quad \text{for } s \le t; \tag{9}$$

(iv) *for any* $s \le t \le u$,

$$\widetilde{R}(s,u) = \widetilde{R}(s,t)\widetilde{R}(t,u). \tag{10}$$

Proof. To prove the implication (i)⇒(ii) observe that

$$\begin{aligned}\widehat{E}(x_u^{(i)} \,|\, x_s,\, s \le t) &= \widehat{E}(X_{u-k+i} \,|\, X_s,\, s \le t)\\ &= \widehat{E}(X_{u-k+i} \,|\, X_t, \ldots, X_{t-k+1}) = \widehat{E}(x_u^{(i)} \,|\, x_t),\end{aligned}$$

where the last equality is a consequence of the Markov property in the case $u-k+i > t$, while for $u-k+i \le t$, $x_u^{(i)}$ is $\sigma(x_t, \ldots, x_{t-k+1})$-measurable. This proves that $\{x_t\}$ is an MWM(1) process.

To prove (ii)⇒(i) observe that

$$\begin{aligned}\widehat{E}(X_u \,|\, X_s,\, s \le t) &= \widehat{E}(x_u^{(k)} | x_s,\, s \le t) = \widehat{E}(x_u^{(k)} | x_t)\\ &= \widehat{E}(X_u \,|\, X_t, X_{t-1}, \ldots, X_{t-k+1}),\end{aligned}$$

which means that $\{X_t\}$ is a WM(k) process.

To prove (ii)⇒(iii) notice that (ii) implies

$$x_u - \widehat{x}_{u,t} \overset{P.1}{=} x_u - \widehat{E}(x_u \,|\, x_s,\, s \le t).$$

By the well known properties of prediction it follows that the random vectors on the right hand side of the above equality are orthogonal to all x_s, $s \le t$. Therefore the random vectors on the left hand side are orthogonal to all x_s, $s \le t$. This gives (iii).

To prove (iii)⇒(ii) observe that (iii) implies

$$\widehat{E}((x_u - \widehat{x}_{u,t}) \,|\, x_s,\, s \le t) = \mathbf{0} \quad \text{and} \quad \widehat{E}((x_u - \widehat{x}_{u,t}) \,|\, x_t) = \mathbf{0}. \tag{11}$$

Since $\widehat{E}(\widehat{x}_{u,t} \,|\, x_s,\, s \le t) = \widehat{E}(\widehat{x}_{u,t} \,|\, x_t) = \widehat{x}_{u,t}$, from (11) we get

$$\widehat{E}(x_u \,|\, x_s,\, s \le t) = \widehat{E}(\widehat{x}_{u,t} \,|\, x_t) = \widehat{x}_{u,t},$$

which proves (ii).

To prove (iii)⇒(iv) notice that from (iii), i.e. from the orthogonality of the random vectors $x_u - \widehat{x}_{u,t}$ to x_s for $s \le t$, we get

$$\begin{aligned}\Gamma(s,u) = Ex_s x_u^T = Ex_s \widehat{x}_{u,t}^T = Ex_s(\Phi(u,t)x_t)^T = Ex_s x_t^T \Phi^T(u,t)\\ = \Gamma(s,t)\Phi^T(u,t).\end{aligned}$$

Hence

$$\Gamma_s^{-1}\Gamma(s,u) = \Gamma_s^{-1}\Gamma(s,t)\Phi^T(u,t),$$

which by the definition of $\widetilde{R}(s,u)$ and by the equality $\Phi^T(u,t) = \widetilde{R}(t,u)$, given in (8), give equality (10) in (iv).

To prove (iv)⇒(iii) observe that (9) is equivalent to the equality $\Gamma(s,u) = \Gamma(s,t)\Phi^T(u,t)$, which can be rewritten as

$$\mathbf{0} = \Gamma(s,u) - \Gamma(s,t)\Phi^T(u,t) = Ex_s(x_u - \Phi(u,t)x_t)^T = Ex_s(x_u - \widehat{x}_{u,t})^T = \mathbf{0},$$

and that in turn implies (iii). This finishes the proof of the theorem. ■

THEOREM 2 (Structure of $\Gamma(s,t)$ and $\Phi(t,s)$). *Let $\{X_t\}$ be a* WM(k) *process. Then the matrices $\Gamma(s,t)$, $\Phi(t,s)$ and $\Phi_t := \Phi(t,t-1)$ satisfy the recurrent relation*

$$\Gamma(s,t+1) = \Gamma(s,t)\cdot\Phi_{t+1}^T \quad \text{for } s \le t, \tag{12}$$

which has the solution

$$\Gamma(s,t) = \Gamma_s\Phi_{s+1}^T\cdots\Phi_t^T \quad \text{for any } s \le t. \tag{13}$$

Furthermore the matrices $\Phi(t,s)$ are as follows: $\Phi(t,t)$ is the identity matrix,

$$\begin{aligned} (14)\quad \Phi(t+1,t) &\equiv \Phi_{t+1} \\ &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \phi_{t+1,1} & \phi_{t+1,2} & \phi_{t+1,3} & \dots & \phi_{t+1,k-1} & \phi_{t+1,k} \end{pmatrix} \end{aligned}$$

where

$$\phi_{t+1,j} = \phi(t+1,t)_{k,j}, \qquad 1 \le j \le k,$$

and the ith row of $\Phi(t+j,t)$, $1 \le i \le k-j$, is

$$\Phi(t+j,t)_i = [\underbrace{0,\dots,0}_{j+i-1},1,\underbrace{0,\dots,0}_{k-j-i}].$$

Proof. Since $\{X_t\}$ is WM(k), by Theorem 1 the random vector $x_{t+1} - \widehat{x}_{t+1,t}$ is orthogonal to all x_s, $s \le t$. Hence and by equality (5) for $u = t+1$ we get

$$\begin{aligned} \mathbf{0} &= Ex_s(x_{t+1} - \widehat{x}_{t+1,t})^T = Ex_s x_{t+1}^T - Ex_s\widehat{x}_{t+1,t}^T \\ &= \Gamma(s,t+1) - Ex_s(\Phi(t+1,t)x_t)^T \\ &= \Gamma(s,t+1) - Ex_s x_t^T\Phi^T(t+1,t) = \Gamma(s,t+1) - \Gamma(s,t)\Phi^T(t+1,t), \end{aligned}$$

which proves (12).

Equality (13) can be obtained by iteration of (12).

The form of the matrix $\Phi(t+1,t)$, given in (14), follows from (5) and Proposition 1, i.e. from the equality

$$\widehat{x}_{t+1,t} = \Phi(t+1,t)x_t. \tag{15}$$

Indeed, the right hand side of (15) equals

$$\begin{aligned} (16)\quad & \Phi(t+1,t)x_t = \Phi(t+1,t)\cdot[X_{t-k+1},X_{t-k+2},\dots,X_t]^T \\ &= [X_{t-k+2},X_{t-k+3},\dots,X_t,\phi_{t+1,1}X_{t-k+1}+\phi_{t+1,2}X_{t-k+2}+\dots+\phi_{t+1,k}X_t]^T \\ &= [X_{t-k+2},X_{t-k+3},\dots,X_t,\Phi(t+1,t)_k x_t]^T. \end{aligned}$$

To find the left hand side of (15), notice that for $1 \le i \le k-1$,

$$\widehat{x}^{(i)}_{t+1} = \widehat{E}(x^{(i)}_{t+1} \,|\, x_t) = \widehat{E}(X_{t+1-k+i} \,|\, X_t, \ldots, X_{t-k+1}) = X_{t+1-k+i}, \tag{17}$$

while for $i = k$,

$$\widehat{x}^{(k)}_{t+1} = \widehat{E}(x^{(k)}_{t+1} \,|\, x_t) = \widehat{E}(X_{t+1} \,|\, X_t, \ldots, X_{t-k+1}) = \Phi(t+1,t)_k x_t. \tag{18}$$

Altogether this proves that $\Phi(t+1,t)$ is of the form (14). This finishes the proof of Theorem 2. ∎

From the relation $\Phi^T(u,t) = \widetilde{R}(t,u)$ for $t \le u$ and from the triangular property for $\widetilde{R}(t,u)$, given in (10), we get the following relation:

$$\Phi(t+h+1,t) = \Phi(t+h+1,t+h)\Phi(t+h,t), \qquad h \ge 0. \tag{19}$$

This in turn and the form of $\Phi(t+1,t)$, given in (14), imply the following corollary.

Corollary 2. *The following relations hold*:

$$\Phi(t+h+1,t)_i = \Phi(t+h,t)_{i+1} \quad \text{for } 1 \le i \le k-1, \tag{20}$$

$$\Phi(t+h+1,t)_k = \Phi(t+h+1,t+h)_k \Phi(t+h,t), \tag{21}$$

$$\Phi(t+h+1,t)_i = \Phi(t+h+1-j,t)_{i+j} \quad \text{for } 1 \le j \le k-i. \tag{22}$$

Theorem 3. *A function $\{f(s,u),\ s,u \in \mathbb{Z}\}$ is the autocovariance function of an* MWM(1) *process $\{x_t\}$ iff there exist $k \times k$ matrices $G_t,\ H_t,\ t \in \mathbb{Z}$, such that*

$$f(s,u) = G_s H_u \quad \text{and} \quad f(u,s) = (f(s,u))^T \quad \text{for } s \le u, \tag{23}$$

$$H^*_t G_t \succeq 0 \quad \text{and} \quad H^*_{t+1} G_{t+1} - H^*_t G_t \succeq 0 \quad \text{for all } t \in \mathbb{Z}. \tag{24}$$

Proof. To prove necessity we show that if $\{x_t\}$ is an MWM(1) process with autocovariance function $\Gamma(t,u) = E x_t x_u^T$ for $t \le u$, then there exist $k \times k$ matrices $G_t,\ H_t,\ t \in \mathbb{Z}$, which satisfy conditions (23)–(24). To this end, fix t_0 and define

$$G_t = \Gamma(t,t_0)\Gamma_{t_0}^{-1/2}\mathbf{1}(t \le t_0) + \Gamma_t \Gamma^{-1}(t_0,t)\Gamma_{t_0}^{1/2}\mathbf{1}(t > t_0),$$
$$H_t = \Gamma_{t_0}^{1/2}\Gamma^{-1}(t,t_0)\Gamma_t \mathbf{1}(t \le t_0) + \Gamma_{t_0}^{-1/2}\Gamma(t_0,t)\mathbf{1}(t > t_0),$$

where $\mathbf{1}(A)$ denotes the indicator of A. Notice that equality (10) in Theorem 1 is equivalent to $\Gamma(s,u) = \Gamma(s,t)\Gamma_t^{-1}\Gamma(t,u)$ for $s \le t \le u$. We use that equality to show (23), which we verify below separately in three cases. For $t_0 \le s \le u$ we have

$$\Gamma(s,u) = \Gamma_s \Gamma^{-1}(t_0,s)\Gamma(t_0,u) = (\Gamma_s \Gamma^{-1}(t_0,s)\Gamma_{t_0}^{1/2})(\Gamma_{t_0}^{-1/2}\Gamma(t_0,u)) = G_s H_u.$$

For $s \le t_0 \le u$ we have

$$\Gamma(s,u) = \Gamma(s,t_0)\Gamma_{t_0}^{-1}\Gamma(t_0,u) = (\Gamma(s,t_0)\Gamma_{t_0}^{-1/2})(\Gamma_{t_0}^{-1/2}\Gamma(t_0,u)) = G_s H_u.$$

For $s \le u \le t_0$ we have

$$\begin{aligned}\Gamma(s,u) &= \Gamma(s,t_0)\Gamma^{-1}(u,t_0)\Gamma_u = (\Gamma(s,t_0)\Gamma_{t_0}^{-1/2})(\Gamma_{t_0}^{1/2}\Gamma^{-1}(u,t_0)\Gamma_u)\\ &= G_s H_u.\end{aligned}$$

Hence the matrices G_t and H_t satisfy (23).

Since $\Gamma_t \succeq 0$ implies $\Gamma_t^{-1} \succeq 0$, we have

$$H_t^* G_t = G_t^T (G_t^T)^{-1}(H_t^T)^{-1} G_t = G_t^T (H_t^T G_t^T)^{-1} G_t = G_t^T \Gamma_t^{-1} G_t \succeq 0,$$

which proves the first relation in (24), i.e.

$$H_t^* G_t = G_t^T \Gamma_t^{-1} G_t \succeq 0. \tag{25}$$

To show the second relation in (24) notice that

$$\begin{aligned}&(x_{t+1} - \Phi_{t+1}x_t)(x_{t+1} - \Phi_{t+1}x_t)^T\\ &\qquad = x_{t+1}x_{t+1}^T - x_{t+1}x_t^T\Phi_{t+1}^T - \Phi_{t+1}x_t x_{t+1}^T + \Phi_{t+1}x_t x_t^T \Phi_{t+1}^T.\end{aligned}$$

Hence the covariance matrix of the random vector $x_{t+1} - \Phi_{t+1}x_t$ equals

$$\Gamma_{t+1} - \Gamma^T(t,t+1)\Phi_{t+1}^T - \Phi_{t+1}\Gamma(t,t+1) + \Phi_{t+1}\Gamma_t\Phi_{t+1}^T,$$

which by putting $\Phi_{t+1}^T = \Gamma_t^{-1}\Gamma(t,t+1)$ has the form

$$\begin{aligned}\Gamma_{t+1} - \Gamma^T(t,t+1)\Gamma_t^{-1}\Gamma(t,t+1) - \Gamma^T(t,t+1)(\Gamma_t^{-1})^T\Gamma(t,t+1)&\\ + \Gamma^T(t,t+1)(\Gamma_t^{-1})^T\Gamma_t\Gamma_t^{-1}\Gamma(t,t+1)&\\ = \Gamma_{t+1} - \Gamma^T(t,t+1)\Gamma_t^{-1}\Gamma(t,t+1)&.\end{aligned}$$

Using (23) and the fact that the above is a covariance matrix we get

$$A := G_{t+1}H_{t+1} - (G_t H_{t+1})^T (G_t H_t)^{-1} G_t H_{t+1} \succeq 0.$$

Hence using (25) we get

$$0 \preceq H_{t+1}^* A H_{t+1}^{-1} = H_{t+1}^* G_{t+1} - H_t^* G_t,$$

which gives the second relation in (24). This finishes the proof of the necessity.

Now assume that the function $f(s,u)$ is defined by (23) and the matrices $G_t, H_t,\ t \in \mathbb{Z}$, satisfy (24). We show that $f(s,u)$ has the triangular property and is nonnegative definite. Indeed, for $s \le t \le u$ we have

$$f(s,u) = G_s H_u = G_s H_t H_t^{-1} G_t^{-1} G_t H_u = f(s,t)(f(t,t))^{-1} f(t,u),$$

which means that $f(s,u)$ has the triangular property, i.e. satisfies (10).

To prove the nonnegative definiteness of f we use the following notation: $A_t := G_{kt}$, $B_t := H_{kt}$, $\mathbf{s}_N := (\mathbf{z}_1^T, \ldots, \mathbf{z}_n^T)^T$, where $\mathbf{z}_t := (z_{(t-1)k+1}, \ldots, z_{tk})^T$, while $\widetilde{f}_{i,j} := f(ki,kj)$ and $S_N := (\widetilde{f}_{i,j},\ 1 \le i,j \le N)$. We will show that

$S_N \succeq 0$ for all positive integers N. Notice that

$$\begin{aligned}
s_N^T S_N s_N &= \sum_{i,j=1}^{N} \mathbf{z}_i^T \widetilde{f}_{i,j} \mathbf{z}_j = \sum_{1\le j\le i\le N} \mathbf{z}_i^T (\widetilde{f}_{j,i})^T \mathbf{z}_j + \sum_{1\le i<j\le N} \mathbf{z}_i^T \widetilde{f}_{i,j} \mathbf{z}_j \\
&= \sum_{1\le j\le i\le N} \mathbf{z}_i^T B_i^T A_j^T \mathbf{z}_j + \sum_{1\le i<j\le N} \mathbf{z}_i^T A_i B_j \mathbf{z}_j \\
&= \sum_{j=1}^{N} \sum_{i=j}^{N} \mathbf{z}_i^T B_i^T A_j^T \mathbf{z}_j + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \mathbf{z}_i^T A_i B_j \mathbf{z}_j \\
&= \sum_{j=1}^{N-1} \sum_{i=j}^{N} \mathbf{z}_i^T B_i^T A_j^T \mathbf{z}_j + \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \mathbf{z}_j^T A_j B_i \mathbf{z}_i + \mathbf{z}_N^T B_N^T A_N^T \mathbf{z}_N .
\end{aligned}$$

Setting

$$M_j = \sum_{i=j}^{N} \mathbf{z}_i^T B_i \quad \text{for } 1 \le j \le N$$

and using the relations

$$(\mathbf{z}_j^T A_j B_i \mathbf{z}_i)^T = \mathbf{z}_i^T B_i^T A_j^T \mathbf{z}_j,$$

$$(A_j B_j)^T = A_j B_j, \qquad A_j^T = B_j^* A_j B_j, \qquad (B_j^* A_j)^T = B_j^* A_j,$$

we get

$$\begin{aligned}
s_N^T S_N s_N &= \sum_{j=1}^{N-1} M_j A_j^T \mathbf{z}_j + \Big(\sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \mathbf{z}_j^T A_j B_i \mathbf{z}_i \Big)^T + M_N A_N^T \mathbf{z}_N \\
&= \sum_{j=1}^{N-1} M_j A_j^T \mathbf{z}_j + \sum_{j=1}^{N-1} \Big(\sum_{i=j+1}^{N} \mathbf{z}_i^T B_i^T \Big) A_j^T \mathbf{z}_j + M_N A_N^T \mathbf{z}_N \\
&= \sum_{j=1}^{N-1} M_j A_j^T \mathbf{z}_j + \sum_{j=1}^{N-1} M_{j+1} A_j^T \mathbf{z}_j + M_N A_N^T \mathbf{z}_N \\
&= \sum_{j=1}^{N-1} M_j B_j^* A_j B_j \mathbf{z}_j + \sum_{j=1}^{N-1} M_{j+1} B_j^* A_j B_j \mathbf{z}_j + M_N B_N^* A_N B_N \mathbf{z}_N .
\end{aligned}$$

But

$$\begin{aligned}
M_j B_j^* A_j B_j \mathbf{z}_j &= M_j B_j^* A_j \Big(\sum_{i=j}^{N} B_i \mathbf{z}_i - \sum_{i=j+1}^{N} B_i \mathbf{z}_i \Big) \\
&= M_j B_j^* A_j M_j^T - M_j B_j^* A_j M_{j+1}^T \\
&= M_j B_j^* A_j M_j^T - M_{j+1} A_j^T (B_j^*)^T M_j^T \\
&= M_j B_j^* A_j M_j^T - M_{j+1} B_j^* A_j M_j^T .
\end{aligned}$$

Analogously we show that

$$M_{j+1}B_j^*A_jB_j\mathbf{z}_j = M_{j+1}B_j^*A_j\Big(\sum_{i=j}^{N} B_i\mathbf{z}_i - \sum_{i=j+1}^{N} B_i\mathbf{z}_i\Big)$$
$$= M_{j+1}B_j^*A_jM_j^T - M_{j+1}B_j^*A_jM_{j+1}^T.$$

Hence

$$\begin{aligned} s_N^T S_N s_N &= M_1B_1^*A_1M_1^T + \sum_{j=2}^{N-1} M_jB_j^*A_jM_j^T \\ &\quad - \sum_{j=1}^{N-1} M_{j+1}B_j^*A_jM_{j+1}^T + M_NB_N^*A_NM_N^T \\ &= M_1B_1^*A_1M_1^T + \sum_{j=1}^{N-2} M_{j+1}B_{j+1}^*A_{j+1}M_{j+1}^T \\ &\quad + M_NB_N^*A_NM_N^T - \sum_{j=1}^{N-1} M_{j+1}B_j^*A_jM_{j+1}^T \\ &= M_1B_1^*A_1M_1^T + \sum_{j=1}^{N-1} M_{j+1}(B_{j+1}^*A_{j+1} - B_j^*A_j)M_{j+1}^T \geq 0. \end{aligned}$$

Thus $f(s,n)$ is nonnegative definite, which finishes the proof of the theorem. ∎

Corollary 3 (Construction of autocovariance functions for WM(k) processes). *Let $\{\varphi_{t+1} = (\varphi_{t+1,1},\ldots,\varphi_{t+1,k}), t\in\mathbb{Z}\}$ be a sequence of vectors in $\mathbb{R}^k$ such that $\varphi_{t+1,1}\neq 0$, let $\{\Phi_{t+1},\ t\in\mathbb{Z}\}$ be the sequence of $k\times k$ matrices defined by*

$$(26)\qquad \Phi_{t+1} := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ \varphi_{t+1,1} & \varphi_{t+1,2} & \varphi_{t+1,3} & \cdots & \varphi_{t+1,k-1} & \varphi_{t+1,k} \end{pmatrix}$$

and let $\{\Gamma_t, t\in\mathbb{Z}\}$ be a sequence of $k\times k$ matrices such that

$$(27)\qquad \Phi_{t+1}^{-1}\Gamma_{t+1}(\Phi_{t+1}^{-1})^T - \Gamma_t \succeq 0 \quad \textit{for all integers } t.$$

Then the sequence of matrices $\{\Gamma(s,t)\}$ defined by

$$\Gamma(s,t) := \Gamma_s\Phi_{s+1}^T\Phi_{s+2}^T\cdots\Phi_t^T \quad \textit{for } s\leq t$$

and $\Gamma(t,s) := (\Gamma(s,t))^T$ *is the autocovariance function for some* MWM(1) *process* $\{x_t\}$.

Proof. For $t \geq 0$ define $H_t := \Phi_0^T \Phi_1^T \cdots \Phi_t^T$ and $H_{-t} := \Phi_0^T \Phi_{-1}^T \cdots \Phi_{-t}^T$, while $G_t := \Gamma_t H_t^{-1}$ for all t. We show that H_t and G_t satisfy the conditions in (24) of Theorem 3. In the proof we use (4). First notice that for $t \geq 0$,

$$H_t^* G_t = ((\Phi_0^T \Phi_1^T \cdots \Phi_t^T)^{-1})^T \Gamma_t (\Phi_0^T \Phi_1^T \cdots \Phi_t^T)^{-1}.$$

Hence and by (4) we get $H_t^* G_t \succeq 0$. In a similar way we show that $H_t^* G_t \succeq 0$ for $t \leq 0$.

Now notice that for $t \geq 0$ we have

$$\begin{aligned} &H_{t+1}^* G_{t+1} - H_t^* G_t \\ &\quad = ((\Phi_0^T \Phi_1^T \cdots \Phi_t^T)^{-1})^T (\Phi_{t+1}^{-1} \Gamma_{t+1} (\Phi_{t+1}^{-1})^T - \Gamma_t)(\Phi_0^T \Phi_1^T \cdots \Phi_t^T)^{-1}. \end{aligned}$$

By (27) and (4) it follows that $H_{t+1}^* G_{t+1} - H_t^* G_t \succeq 0$ for $t \geq 0$. In a similar way we show that the last inequality holds for $t \leq 0$. Therefore using Theorem 3 we get the assertion of the corollary. ∎

3.2. *Covariance structure for* MWM(1)PC(d) *processes.* A process $\{X_t\}$ is said to be *periodically correlated* with period $d \geq 1$ if its autocovariance function $\gamma(t,s)$ is periodic with period d, i.e. $\gamma(t+d, s+d) = \gamma(t,s)$ for all t, s, and d is the smallest such value. Then

$$\Gamma(t+d, u+d) = \Gamma(t,u) \quad \text{and} \quad \Phi(t+d, u+d) = \Phi(t,u). \tag{28}$$

Hence we get the following lemma.

Lemma 1. *If* $\{X_t\}$ *is a* WM(k) *process with period* d, *then*

$$\Gamma_{t+d} = \Gamma_t, \qquad \Gamma(t, t+nd) = \Gamma_t (\Phi_{t+1}^T \Phi_{t+2}^T \cdots \Phi_{t+d}^T)^n, \tag{29}$$

and for $1 \leq i \leq d$,

$$\Gamma(t, t+nd+i) = \Gamma_t (\Phi_{t+1}^T \Phi_{t+2}^T \cdots \Phi_{t+d}^T)^n (\Phi_{t+1}^T \Phi_{t+2}^T \cdots \Phi_{t+i}^T). \tag{30}$$

Theorem 4. *The autocovariance function of a* WM(k)PC(d) *process* $\{x_t\}$ *is determined by* d *covariance matrices* Γ_t *and* d *vectors* $\varphi_{t+1} = (\varphi_{t+1,1}, \ldots, \varphi_{t+1,k})$ *such that* $\varphi_{t+1,1} \neq 0$ *with* $0 \leq t \leq d-1$ *and satisfying* (27).

Immediately from the definition of the process $\{x_t\}$ and the Gladyshev process $\{Y_t\}$, associated with $\{X_t\}$, where $Y_t = (Y_{t,1}, \ldots, Y_{t,d})^T = (X_{dt-d+1}, X_{dt-d+2}, \ldots, X_{dt})^T$, we get some relations between these processes. To state them, we define an $\ell \times r$ matrix $A_{\ell,r} = (a_{i,j})$, for $r > \ell$, as follows: $a_{i,i+r-\ell} = 1$ and $a_{i,j} = 0$ for other i, j. The first $r - \ell$ columns of $A_{\ell,r}$ are zero vectors.

REMARK 2. If $\{X_t\}$ is a WM(k)PC(d) process, then

$$\begin{aligned} Y_t &= x_{dt} && \text{for } k = d, \\ x_{dt} &= A_{k,d} Y_t && \text{for } k < d, \\ Y_t &= A_{d,k} x_{dt} && \text{for } k > d. \end{aligned}$$

To give the structure of the Gladyshev process $\{Y_t\}$ associated with a WM(k)PC(d) process $\{X_t\}$ we define some matrices formed from the matrices $\Phi(u,t)$. Recall that for $u > t$, $\Phi(u,t)$ is a $k \times k$ matrix such that $\widehat{x}_{u,t} = \Phi(u,t)x_t$, i.e. it satisfies the equation $\Gamma_t \Phi^T(u,t) = \Gamma(t,u)$ (see Proposition 1, formula (7)). Furthermore $\Phi(u,t)_i$ denotes the ith row of $\Phi(u,t)$, and $\phi(u,t)_{k,k}$ is the last entry in the last row of $\Phi(u,t)$. For simplicity we set $\theta_{u,t} = \phi(u,t)_{k,k}$.

Let $\mathbf{F}$ be the $d \times d$ matrix

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 & & & . \\ \theta_{2,1} & 1 & 0 & . & . & . \\ \theta_{3,1} & \theta_{3,2} & 1 & 0 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ \theta_{d,1} & \theta_{d,2} & . & . & \theta_{d,d-1} & 1 \end{pmatrix}.$$

In case $k < d$, we define a $d \times d$ matrix $\widetilde{\Phi}(d,0)$ whose ith row is

$$\widetilde{\Phi}(d,0)_i = [\underbrace{0,\ldots,0}_{d-k}, \phi(d,0)_{i,1}, \ldots, \phi(d,0)_{i,k}], \qquad 1 \le i \le d.$$

In case $k > d$, with $k = (p-1)d + r$, $0 < r < d$, we define $d \times d$ matrices $\Psi_1, \ldots, \Psi_p$ formed from the $k \times k$ matrix $\Phi(d,0)$ in the following way. The ith row of Ψ_j, denoted by $\Psi_{j,i}$, equals

$$\Psi_{j,i} = [\phi(d,0)_{i,k-jd+1}, \phi(d,0)_{i,k-jd+2}, \ldots, \phi(d,0)_{i,k-(j-1)d}] \qquad \text{for } j < p,$$

and

$$\Psi_{p,i} = [\underbrace{0,\ldots,0}_{k-r}, \phi(d,0)_{i,1}, \ldots, \phi(d,0)_{i,k-(p-1)}].$$

If $\{X_t\}$ is a WM(k)PC(d) process then $\widehat{E}(X_t \mid x_{t-1}) = \Phi(t,t-1)_k x_{t-1}$. Hence $X_t = \Phi(t,t-1)_k x_{t-1} + \varepsilon_t$, where $\{\varepsilon_t\}$ is a white noise such that ε_t is uncorrelated with X_s, $s < t$, and $E\varepsilon_t = 0$, $E\varepsilon_i^2 = \sigma_i^2$ for $1 \le i \le d$.

THEOREM 5. *Let $\{X_t\}$ be a* WM(k)PC(d) *process. Then the stationary process $\{Y_t\}$ is a d-dimensional* AR(1) *process in the cases* (a) $k = d$ *and* (b) $k < d$, *while it is a d-dimensional* AR(p) *process in the case* (c) $k > d$, *where* $p = \lceil k/d \rceil$ (*i.e.* $k = (p-1)d + r,\ 0 \le r < d$). *Moreover*

$$Y_t = \Phi(d,0)Y_{t-1} + e_t \quad \textit{for } k = d,$$
$$Y_t = \widetilde{\Phi}(d,0)Y_{t-1} + e_t \quad \textit{for } k < d,$$

and

$$Y_t = \Psi_1 Y_{t-1} + \Psi_2 Y_{t-2} + \cdots + \Psi_p Y_{t-p} + e_t \quad \textit{for } k > d,$$

with $k = (p-1)d + r$, $0 < r < d$ *or* $k = pd$. *Here* $\{e_t = (e_{t,1}, \ldots, e_{t,d})^T,\ t \in \mathbb{Z}\}$, *in all cases, is a d-dimensional white noise with mean vector* $Ee_t = \mathbf{0}$, *covariance matrix* $Ee_te_t^T = C_e$ *and such that e_t is uncorrelated with Y_s for $s < t$, and* $C_e = \mathbf{F}\operatorname{diag}(\sigma_1^2, \ldots, \sigma_d^2)\mathbf{F}^T$, *where σ_i^2 is the variance of ε_i,* $1 \le i \le d$.

Proof. Case $k = d$. Since $\{Y_t\}$ is stationary and $x_{dt} = Y_t$, it follows that $\{x_{dt}, t \in \mathbb{Z}\}$ is stationary. Since $\{X_t\}$ is a WM(d) process, $\{x_t\}$ is a WM(1) process, which in turn implies that

$$\widehat{E}(x_{d(t+1)} \mid x_{ds}, s \le t) = \widehat{E}(x_{d(t+1)} \mid x_{dt}) = \Phi(d(t+1), dt)x_{dt} = \Phi(d,0)x_{dt},$$

where the last equality follows by periodicity of $\Phi(u,t)$. Hence $\{x_{dt}, t \in \mathbb{Z}\}$ is a d-dimensional AR(1) process, i.e. $Y_t = \Phi(d,0)Y_{t-1} + e_t$, where $\{e_t\}$ is a white noise such that e_t is uncorrelated with Y_s, $s < t$. The form of the covariance matrix of e_t will be given later. The proof in the case $k < d$ is similar.

Case $k > d$. Notice that

$$\widehat{E}(Y_{t,i} \mid Y_s, s < t) = \widehat{E}(Y_{t,i} \mid x_{d(t-1)}, x_{d(t-2)}, \ldots) = \widehat{E}(Y_{t,i} \mid x_{d(t-1)}),$$

where the last equality follows from the fact that $\{x_t\}$ is a (k-dimensional) MWM(1) process. But from Proposition 1 and periodicity of $\Phi(u,t)$ we get

$$\widehat{E}(Y_{t,i} \mid x_{d(t-1)}) = \Phi(dt, d(t-1))_s x_{d(t-1)} = \Phi(d,0)_s x_{d(t-1)},$$

where $\Phi(d,0)_s$ is the sth row of the $k \times k$ matrix $\Phi(d,0)$ and $s = k - d + i$.

Now notice that

$$\begin{aligned}\Phi(d,0)_s x_{d(t-1)} &= \sum_{j=1}^{k} \phi(d,0)_{s,j} X_{d(t-1)-k+j} \\ &= \Psi_{1,i}Y_{t-1} + \Psi_{2,i}Y_{t-2} + \ldots + \Psi_{p-1,i}Y_{t-(p-1)} + \Psi_{p,i}Y_{t-p} + e_{t,i}.\end{aligned}$$

This proves the asserted autoregressive structure of $\{Y_t\}$ in case $k > d$.

To find the covariance matrix for e_t define $(\underbrace{0, \ldots, 0}_{k-1}, 1)^T = a$ and notice that

$$\begin{aligned}X_1 &= \Phi(1,0)_k x_0 + \varepsilon_1, \\ X_2 &= \Phi(2,1)_k x_1 + \varepsilon_2 = \Phi(2,1)_k(\Phi(1,0)x_0 + a\varepsilon_1) + \varepsilon_2 \\ &= \Phi(2,0)_k x_0 + \Phi(2,1)_k a\varepsilon_1 + \varepsilon_2 = \Phi(2,0)_k x_0 + \theta_{2,1}\varepsilon_1 + \varepsilon_2,\end{aligned}$$

where in the last equalities we used the triangular property $\Phi(2,1)\Phi(1,0) = \Phi(2,0)$. Using the above idea we can write the following relations:

$$\begin{aligned} X_n &= \Phi(n,n-1)_k x_{n-1} + \varepsilon_n \\ &= \Phi(n,n-1)_k(\Phi(n-1,n-2)x_{n-2} + a\varepsilon_{n-1}) + \varepsilon_n \\ &= \Phi(n,n-1)_k\Phi(n-1,n-2)x_{n-2} + \Phi(n,n-1)_k a\varepsilon_{n-1} + \varepsilon_n \\ &= \Phi(n,n-2)_k x_{n-2} + \theta_{n,n-1}\varepsilon_{n-1} + \varepsilon_n. \end{aligned}$$

Finally, we get

$$X_n = \Phi(n,0)_k x_0 + \sum_{j=1}^{n} \theta_{n,j}\varepsilon_j, \qquad n = 1,\ldots,d,$$

where $\{\varepsilon_t\}$ is a white noise such that $E\varepsilon_h = 0$, $E\varepsilon_h^2 = \sigma_h^2$, $1 \le h \le d$, and ε_t is uncorrelated with X_s, $s < t$. Hence $e_{0,n} = \sum_{j=1}^{n} \theta_{n,j}\varepsilon_j$ for $n = 1,\ldots,d$. Since $e_0 = (e_{0,1},\ldots,e_{0,d})^T$, it follows that $Ee_0e_0^T = \mathbf{F}\,\mathrm{diag}(\sigma_1^2,\ldots,\sigma_d^2)\mathbf{F}$, which finishes the proof of the theorem. ■

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