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ON UNIFORM TAIL EXPANSIONS OF MULTIVARIATE COPULAS AND WIDE CONVERGENCE OF MEASURES

Abstract. The theory of copulas provides a useful tool for modeling dependence in risk management. In insurance and finance, as well as in other applications, dependence of extreme events is particularly important, hence there is a need for a detailed study of the tail behaviour of multivariate copulas. We investigate the class of copulas having *regular* tails with a uniform expansion. We present several equivalent characterizations of uniform tail expansions. Next, basing on them, we determine the class of all possible leading parts of such expansions; we compute the leading parts of copulas popular in the literature, and discuss the statistical aspects of tail expansions.

1. Introduction. Copulas have recently become a very useful tool to handle dependence in risk management both in finance and in actuarial sciences. They enable specifying the marginal distributions to be decoupled from the dependence structure of variables, which is vital when one abandons the normality assumption in multidimensional problems (see for example [3–5, 10, 11]). In this paper we go one step further and concentrate on the tail behaviour of multivariate copulas, which is crucial in problems such as determining the Value at Risk for a portfolio consisting of several risky assets (see [9]) or dealing with extreme events in insurance (see [4]).

We investigate the class of copulas with *regular* tails, having a uniform expansion. It so happens that copulas popular among researchers usually belong to this class. Thus our study may help in the choice of a proper copula, suitable for a given task. At the end we deal with the statistical aspects of the tail behaviour of multivariate random data.

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2. Notation

2.1. *Copulas.* We recall that a function $C : [0,1]^n \to [0,1]$ is called a *copula* (see [13, §2.10], [3, §4.1]) if for every $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ ($u_i, v_i \in [0,1]$) and every $j \in \{1, \dots, n\}$,

(i) $u_j = 0 \Rightarrow C(u) = 0$;
(ii) $(\forall i \neq j \ \ u_i = 1) \Rightarrow C(u) = u_j$;
(iii) $u \preceq v \Rightarrow V_C(u,v) \geq 0$,

where $u \preceq v$ denotes the partial ordering on $\mathbb{R}^n$,

$$u \preceq v \ \Leftrightarrow \ \forall i \ \ u_i \leq v_i,$$

and $V_C(u,v)$ is the C-volume of the rectangle $I(u,v)$ with lower vertex u and upper vertex v,

$$V_C(u,v) = \Delta^1_{v_1-u_1} \dots \Delta^n_{v_n-u_n} C(u_1, \dots, u_n),$$

where

$$\Delta^k_h C(t_1, \dots, t_n) \\ = C(t_1, \dots, t_{k-1}, t_k + h, t_{k+1}, \dots, t_n) - C(t_1, \dots, t_{k-1}, t_k, t_{k+1}, \dots, t_n).$$

The functions with property (iii) are called *n-nondecreasing* and those which satisfy (i) are called *grounded.*

Note that every copula is nondecreasing not only with respect to each variable but also with respect to the partial ordering $\preceq$. Moreover it is continuous and even Lipschitz ([13, Theorem 2.10.7], [3, Lemma 4.2]):

$$|C(v) - C(u)| \leq \sum_{i=1}^{n} |v_i - u_i|.$$

Remark 1 (cf. [1, Th. 12.5]). Every continuous, grounded, n-nondecreasing function $H : [0,a]^n \to \mathbb{R}$ is the distribution function of a Borel measure μ_H on $[0,a]^n$:

$$H(u) = \mu_H(I(0,u)), \qquad \mu_H(I(u,v)) = \mu_H(\mathrm{int}(I(u,v))) = V_H(u,v).$$

Due to condition (ii) every copula is the distribution function of a probability measure on the unit rectangle $[0,1]^n$ with uniform margins (compare [10, §1.6]). Furthermore, the above remark remains true if H is defined on the whole multioctant $[0,\infty)^n$.

Let $\mathcal{X}_i$, $i = 1, \dots, n$, be random variables defined on the same probability space $(\Omega, \mathcal{M}, \mathbb{P})$. Their joint cumulative distribution $F_{\mathcal{X}}$ can be described using an appropriate copula $C_{\mathcal{X}}$ (see [13, Theorem 2.10.11], [3, Theorem 4.5]):

$$F_{\mathcal{X}}(x) = C_{\mathcal{X}}(F_{\mathcal{X}_1}(x_1), \dots, F_{\mathcal{X}_n}(x_n)),$$

where $F_{\mathcal{X}_i}$ are the cumulative distributions of $\mathcal{X}_i$. Note that strictly increasing transformations of the random variables $\mathcal{X}_i$ do not affect the copula.

Indeed, if

$$\mathcal{X}_i' = f_i(\mathcal{X}_i), \qquad i = 1, \dots, n,$$

where f_i is strictly increasing (and so invertible), then

$$\begin{aligned} F_{\mathcal{X}'}(x) &= F_{\mathcal{X}}(f_1^{-1}(x_1), \dots, f_n^{-1}(x_n)) \\ &= C_{\mathcal{X}}(F_{\mathcal{X}_1}(f_1^{-1}(x)), \dots, F_{\mathcal{X}_n}(f_n^{-1}(x_n))) = C_{\mathcal{X}}(F_{\mathcal{X}_1'}(x_1), \dots, F_{\mathcal{X}_n'}(x_n)). \end{aligned}$$

Therefore if one is interested in tail dependence of random variables rather than in their *individual* distribution, then the proper choice is to study the copula, and the more so since the copula is uniquely determined at every point u such that the equations $F_{\mathcal{X}_i}(x_i) = u_i$ have solutions.

In certain cases the copula $C_{\mathcal{X}}$ is the joint cumulative distribution of random variables defined on the same probability space as $\mathcal{X}_i$. Indeed, let $\mathcal{P}_i$, $i = 1, \dots, n$, be the random variables defined by

$$\mathcal{P}_i = F_{\mathcal{X}_i}(\mathcal{X}_i).$$

Proposition 1. *If the cumulative distributions $F_{\mathcal{X}_i}$ are continuous then*:

(i) *The $\mathcal{P}_i$ have uniform distributions on $[0,1]$.*
(ii) *The copula $C_{\mathcal{X}}$ is uniquely determined.*
(iii) *The n-dimensional cumulative distribution $F_{\mathcal{P}}$ coincides with $C_{\mathcal{X}}$.*

Proof of (iii). (The first two items are obvious.) We choose $p = (p_1, \dots, p_n)$ such that $0 \le p_i \le 1$. Since the $F_{\mathcal{X}_i}$ are continuous, there exists $x = (x_1, \dots, x_n)$ such that for all i,

$$p_i = F_{\mathcal{X}_i}(x_i).$$

Hence

$$\begin{aligned} F_{\mathcal{P}}(p) &= \mathbb{P}\Big(\bigwedge_i \mathcal{P}_i \le p_i\Big) = \mathbb{P}\Big(\bigwedge_i F_{\mathcal{X}_i}(\mathcal{X}_i) \le F_{\mathcal{X}_i}(x_i)\Big) \\ &= \mathbb{P}\Big(\bigwedge_i \mathcal{X}_i \le x_i\Big) = C_{\mathcal{X}}(F_{\mathcal{X}_1}(x_1), \dots, F_{\mathcal{X}_n}(x_n)) = C_{\mathcal{X}}(p). \end{aligned}$$

In order to study the dependence of extreme events one has to deal with the tail behaviour of a copula. Therefore it is useful to introduce some regularity conditions.

We say that a copula has a uniform lower tail expansion if near the origin it can be *uniformly* approximated by a homogeneous function of degree 1. In more detail:

Definition 1. We say that a copula $C : [0,1]^n \to [0,1]$ *has a uniform lower tail expansion* if there exist a homogeneous function $L : [0,\infty)^n \to \mathbb{R}$ of degree 1, i.e.

$$\forall t \ge 0 \qquad L(tu) = tL(u),$$

and a bounded function $R : [0,1]^n \to \mathbb{R}$ with

$$\lim_{u\to 0} R(u) = 0$$

such that

$$\forall u \in [0,1]^n \quad C(u) = L(u) + R(u)(u_1 + \ldots + u_n).$$

The function L will be called the *leading part* of the expansion. When $L \equiv 0$ we shall say that the expansion is *trivial.*

Similarly we define the tail expansion at other vertices of the unit cube $[0,1]^n$. Namely for any vertex $e = (e_1, \ldots, e_n)$, $e_i = 0, 1$, there is an associated copula $\widehat{C}_e$, defined by

$$\widehat{C}_e : [0,1]^n \to [0,1], \qquad \widehat{C}_e(v) = V_C(u, u+v),$$

where $u_i = e_i(1 - v_i)$. We say that C has a *uniform tail expansion at the vertex* e if $\widehat{C}_e$ has a uniform lower tail expansion. If $e = (1, \ldots, 1)$ then $\widehat{C}_e$ is called a *survival copula* and the expansion is called an *upper tail expansion.*

Note that if C describes the joint distribution of the $\mathcal{X}_i$'s, then $\widehat{C}$ does the same for the $(-1)^{e_i}\mathcal{X}_i$'s.

There are several reasons why it is important to study copulas having uniform tail expansions.

- The commonly used copulas have this property.
- Any copula can be approximated by copulas with this property.
- Copulas having uniform tail expansions fit to experimental data.
- The presence of a uniform tail expansion simplifies modeling.

Note that our conditions are a little stronger than the ones introduced by P. Embrechts ([4]) and other authors, but they are still satisfied by nearly all copulas studied in the literature.

There are also other ways of introducing the leading part L. For example one may adopt the definition with a "strong" derivative. Indeed:

Lemma 1. *For a copula $C : [0,1]^n \to [0,1]$ the following conditions are equivalent*:

(i) *C has a uniform lower tail expansion.*

(ii) *There exists a homogeneous function $L : [0,\infty)^n \to \mathbb{R}$, of degree* 1, *such that*

$$\lim_{u\to 0^+} \frac{|C(u) - C(0) - L(u)|}{\|u\|} = 0.$$

2.2. *σ-finite measures.* In order to study the tail behaviour of copulas one has to deal with σ-finite measures on $[0,\infty)^n$—probability (i.e. finite) measures are not enough.

We recall the basic facts.

2.2.1. *Wide convergence of measures.* The notion of wide convergence of measures is based on the fact that every σ-finite Borel measure μ on $\mathbb{R}^n$ may be considered as a continuous linear functional on the space of continuous functions with compact support endowed with the topology of uniform convergence (see [15, §2.2]):

$$\mu(f) = \int_{\mathbb{R}^n} f\,d\mu.$$

Definition 2 ([2, §III.1.9], [16, §IV.7]). We say that a one-parameter family of measures μ_t, $t > 0$, *converges* to the measure μ *in the wide sense* as t tends to 0 if for every continuous function f with compact support,

$$\lim_{t\to 0} \mu_t(f) = \mu(f).$$

2.2.2. *Relatively invariant measures.* Let Ξ denote the action of the multiplicative group $\mathbb{R}_+$ on $[0,\infty)^n$,

$$\Xi : \mathbb{R}_+ \times [0,\infty)^n \to [0,\infty)^n, \qquad \Xi(t,u) = tu.$$

Definition 3. The Borel measure μ on $[0,\infty)^n$ is called *relatively invariant with respect to Ξ with multiplicator $\kappa(t) = t$* if for any Borel set $A \subset [0,\infty)^n$ and any $t > 0$,

$$\mu(tA) = t\mu(A).$$

Following the ideas of [2, Chapter VII], one can construct the factor measure on the space of orbits of the $\mathbb{R}_+$-action on $[0,\infty)^n \setminus \{0\}$. This orbit space is homeomorphic to the unit simplex in $[0,\infty)^n$,

$$\Delta = \{x \in [0,\infty)^n : x_1 + \cdots + x_n = 1\}.$$

Furthermore, note that Ξ restricted to $\mathbb{R}_+ \times \Delta$ is a diffeomorphism onto $[0,\infty)^n \setminus \{0\}$. Therefore every relatively invariant measure μ is a product of the factor measure on Δ and a relatively invariant measure on $\mathbb{R}_+$.

3. Main results. Let C be an n-dimensional copula and μ_C the associated measure on the unit rectangle,

$$C(u) = \mu_C(I(0,u)) \quad \text{and} \quad \mu_C(\partial[0,1]^n) = 0.$$

We extend both of them to $[0,\infty)^n$,

$$C_1(u) = C(\min(1,u_1),\ldots,\min(1,u_n)), \qquad \mu_1(A) = \mu_C(A \cap [0,1]^n).$$

Obviously $C_1(u) = \mu_1(I(0,u)) = \mu_1(\operatorname{cl} I(0,u))$.

We define a family of functions on $[0,\infty)^n$ by

$$C_t(u) = \frac{1}{t}\,C_1(tu), \qquad t > 0,$$

and a family of measures on $[0,\infty)^n$ by

$$\mu_t(A) = \frac{1}{t}\,\mu(tA), \qquad t > 0.$$

As above we have $C_t(u) = \mu_t(I(0,u)) = \mu_t(\operatorname{cl} I(0,u))$. Note that for a function f we have

$$\mu_t(f(u)) = \int f(u)\,d\mu_t(u) = \frac{1}{t}\int f\left(\frac{u}{t}\right) d\mu_1(u) = \frac{1}{t}\,\mu_1\left(f\left(\frac{\cdot}{t}\right)\right).$$

The main result of this paper is the following theorem.

THEOREM 1. *Let C be an n-dimensional copula and L a homogeneous function of degree* 1. *Then the following conditions are equivalent*:

(i) *L is the leading part of a uniform lower tail expansion of C.*
(ii) *$C_t \to L$ almost uniformly as $t \to 0$.*
(iii) *L is continuous*, *grounded*, *n-nondecreasing and $\mu_t \to \mu_L$ in the wide sense as $t \to 0$.*
(iv) *For every $u \succeq 0$ the ray-like limit* $\lim_{t\to 0^+} C(tu)/t$ *exists and equals $L(u)$.*

We recall that two identically distributed random variables $\mathcal{X}_1$ and $\mathcal{X}_2$ are called *lower tail asymptotically independent* (cf. [12, p. 170] for the upper tail case and [3, §§1.8.5, 3.1.5]) if

$$\lim_{x\to x^*} \mathbb{P}(\mathcal{X}_1 < x \,|\, \mathcal{X}_2 < x) = 0,$$

where x^* is the lower bound of the support,

$$x^* = \inf\{x \in \mathbb{R} : \mathbb{P}(\mathcal{X}_j \le x) > 0\}.$$

Now let C be a copula describing the dependence of identically distributed random variables $\mathcal{X}_1,\ldots,\mathcal{X}_n$ such that

$$\lim_{x\to x^*} \mathbb{P}(\mathcal{X}_j < x) = 0.$$

THEOREM 2. *If any two $\mathcal{X}_i$'s are asymptotically independent then the copula C has a trivial uniform lower tail expansion.*

REMARK 2. For the bivariate case ($n = 2$) there is an equivalence: $\mathcal{X}_1$ and $\mathcal{X}_2$ are lower tail asymptotically independent if and only if the expansion is trivial.

Next we deal with the leading part L. We show that the corresponding measure μ_L is the product of the Lebesgue measure m on the real half-line and a measure μ_Δ on the unit simplex Δ. Basing on this we prove that L is superadditive and concave.

Our second main result is the characterization of all possible leading parts.

THEOREM 3. *For a homogeneous function* $L : [0,\infty)^n \to \mathbb{R}$ *of degree* 1, *the following conditions are equivalent*:

(i) *L is the leading part of the lower tail of some copula C.*
(ii) *L is n-nondecreasing and*

$$0 \le L(u) \le \min(u_1,\dots,u_n) \quad \textit{for } u \succeq 0.$$

(iii) *L is continuous, grounded, n-nondecreasing and $\mu_L = m \times \mu_\Delta$, where*

$$\int_{\Delta} \frac{1}{q_i}\, d\mu_\Delta(q) \le 1 \quad \textit{for } i = 1,\dots,n.$$

In the next section we find the leading parts of the uniform tail expansions for some copulas which are popular in the literature, like Gaussian, Archimedean or MEV.

In the last part of the paper we deal with the statistical aspects of tail expansions. We consider the following questions:

- How to detect the occurrence of a nontrivial uniform tail expansion?
- How to select the proper benchmark γ, past which the tail is close enough to its uniform approximation?
- How to construct the estimator of the leading part L?

4. Proof of Theorem 1

(i)⇒(ii). We have

$$C_t(u) = \frac{C_1(tu)}{t} = \frac{L(tu) + \|tu\| R_1(tu)}{t} = L(u) + \|u\| R_1(tu),$$

where R_1 is an extension of R from the definition of the uniform expansion ($R_1(u) = R(u)$ for $u \in I(0,1)$) and

$$\|u\| = u_1 + \dots + u_n.$$

For u bounded, $\|u\| \le \delta$, we get

$$\sup_{\|u\|\le\delta} |C_t(u) - L(u)| = \sup_{\|u\|\le\delta} \|u\| R_1(tu) \le \delta \sup_{\|u\|\le\delta} |R_1(tu)|.$$

Since R has limit 0 at the origin, $R_1(tu)$ tends uniformly to 0 as $t \to 0$, which finishes the proof of the implication.

(ii)⇒(iii). The C_t are continuous, grounded and n-nondecreasing, hence their limit L has the same properties.

Next, we adapt the *probabilistic* approach from [1, Theorem 29.1]. For every rectangle $I(u,v)$, $0 \preceq u \preceq v$, we get, as $t \to 0$,

$$\mu_t(I(u,v)) = V_{C_t}(u,v) \to V_L(u,v) = \mu_L(I(u,v)).$$

Now let G be an open bounded nonempty subset of $[0,\infty)^n$. We shall show that

$$\liminf_{t\to 0}\mu_t(G)\ge \mu_L(G).$$

Indeed, for every $\varepsilon>0$ there exists a finite sequence of rectangles $I_k = I(u_k,v_k)$ with disjoint interiors such that

$$G\supset \bigcup_{k=1}^{K} I_k \quad\text{and}\quad \mu_L(G)\le \sum_{k=0}^{K}\mu_L(I_k)+\varepsilon.$$

Therefore

$$\begin{aligned}\liminf_{t\to 0}\mu_t(G) &\ge \liminf_{t\to 0}\mu_t\Big(\bigcup_{k=1}^{K} I_k\Big)=\sum_{k=1}^{K}\lim_{t\to 0}\mu_t(I_k)=\sum_{k=1}^{K}\mu_L(I_k)\\ &\ge \mu_L(G)-\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$ we obtain the required inequality.

Next, let f be a continuous function on $[0,\infty)^n$ with a compact support S. Let I be any closed bounded rectangle containing S. Then

$$\begin{aligned}\mu_t(f) &= \int_I f(u)\,d\mu_t(u)=\int_I (f(u)-\min f)\,d\mu_t(u)+(\min f)\mu_t(I)\\ &=\int_I\Big(\int_{\min f}^{f(u)} 1\,dx\Big)\,d\mu_t(u)+(\min f)\mu_t(I)\\ &=\int_I\Big(\int_{\min f}^{\max f}\mathbb{I}_{f(u)>x}\,dx\Big)\,d\mu_t(u)+(\min f)\mu_t(I).\end{aligned}$$

Hence by the Fubini theorem,

$$\begin{aligned}\mu_t(f) &= \int_{\min f}^{\max f}\Big(\int_I \mathbb{I}_{f(u)>x}\,d\mu_t(u)\Big)\,dx+(\min f)\mu_t(I)\\ &=\int_{\min f}^{\max f}\mu_t(\{u\in I: f(u)>x\})\,dx+(\min f)\mu_t(I).\end{aligned}$$

The sets $\{f(u)>x\}\cap \operatorname{int} I$ are open, hence applying the above and the Fatou lemma ([1, Theorem 16.3]) we obtain

$$\begin{aligned}\liminf_{t\to 0}\mu_t(f) &\ge \int_{\min f}^{\max f}\mu_L(\{u\in \operatorname{int} I: f(u)>x\})\,dx+(\min f)\mu_L(I)\\ &=\int_I f(u)\,d\mu_L(u)=\mu_L(f).\end{aligned}$$

Since $-f$ is also a continuous function with compact support, we have

$$\mu_L(f) = -\mu_L(-f) \geq -\liminf_{t\to 0} \mu_t(-f) = \limsup_{t\to 0} \mu_t(f).$$

Thus the upper and lower limits of $\mu_t(f)$ are equal and so

$$\lim_{t\to 0} \mu_t(f) = \mu_L(f).$$

This finishes the proof of the wide convergence.

(iii)⇒(iv). Basing on the fact that $\mu_L(\partial I(0,u)) = 0$ we get

$$\frac{C(tu)}{t} = C_t(u) = \mu_t(I(0,u)) \to \mu_L(I(0,u)) = L(u)$$

(cf. [16, §IV.7 Corollary 2] and [2, Theorem 29.1.iv]).

(iv)⇒(i) (this implication is based on the proof of the two-dimensional case by M. Wawruszczak [17]).

Step 1. *The quotient $C(tv)/t$ satisfies the uniform Cauchy condition*

$$\forall \varepsilon > 0\ \exists r > 0\ \forall s,t \in (0,r) \qquad \sup_{v\in[0,1]^n} \left| \frac{C(tv)}{t} - \frac{C(sv)}{s} \right| < \varepsilon.$$

First for every $\varepsilon > 0$ we construct an $\varepsilon/4$-net N_ε on the unit cube $[0,1]^n$. Let $m = [\varepsilon/4] + 1$. We put

$$N_\varepsilon = \{(k_1/m, \ldots, k_n/m) : k_i \in \{0,1,\ldots,m\}\}.$$

Obviously for every $v \in [0,1]^n$ there exists $v^* \in N_\varepsilon$ such that

$$\|v - v^*\| = \sum_{i=1}^{n} |v_i - v_i^*| < n\,\frac{1}{m} < n\,\frac{\varepsilon}{4n} = \frac{\varepsilon}{4}.$$

Next we choose r so small that for every $v^* \in N_\varepsilon$,

$$\forall s,t \in (0,r) \qquad \left| \frac{C(tv^*)}{t} - \frac{C(sv^*)}{s} \right| < \frac{\varepsilon}{2}.$$

To finish the proof of the Cauchy condition we have to apply the Lipschitz property of C (see [13, Theorem 2.10.7]). For $s,t \in (0,r)$, $v \in [0,1]^n$ and $v^* \in N_\varepsilon$ such that $\|v - v^*\| < \varepsilon/4$ we have

$$\begin{aligned}
&\left| \frac{C(tv)}{t} - \frac{C(sv)}{s} \right| \\
&\qquad \leq \left| \frac{C(tv)}{t} - \frac{C(tv^*)}{t} \right| + \left| \frac{C(tv^*)}{t} - \frac{C(sv^*)}{s} \right| + \left| \frac{C(sv^*)}{s} - \frac{C(sv)}{s} \right| \\
&\qquad \leq \frac{\|tv - tv^*\|}{t} + \frac{\varepsilon}{2} + \frac{\|sv - sv^*\|}{s} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}$$

Step 2. *$R(u)$ tends to 0 as $u \to 0$.* Indeed, from the Cauchy condition we find that for sufficiently small t,

$$\sup_{v\in[0,1]^n}\left|\frac{C(tv)}{t} - L(v)\right| = \sup_{v\in[0,1]^n}\left|\frac{C(tv)}{t} - \lim_{s\to 0^+}\frac{C(sv)}{s}\right|$$
$$= \sup_{v\in[0,1]^n}\lim_{s\to 0^+}\left|\frac{C(tv)}{t} - \frac{C(sv)}{s}\right| \le \varepsilon.$$

Therefore setting $v = u/\|u\|$ and $t = \|u\|$ we obtain

$$|R(u)| = \frac{|C(u) - L(u)|}{\|u\|} = \left|\frac{C(tv)}{t} - L(v)\right| \le \varepsilon,$$

which finishes the proof of the theorem.

5. The asymptotic independence. First we prove the following proposition.

Proposition 2. *For any copula C the following conditions are equivalent*:

(i) $\exists v \succ 0 \ \ \lim_{t\to 0^+} C(tv)/t = 0$.
(ii) $\forall u \succeq 0 \ \ \lim_{t\to 0^+} C(tu)/t = 0$.
(iii) *C has a trivial lower tail expansion.*

Proof. The equivalence (ii)⇔(iii) follows from Theorem 1. The implication (ii)⇒(i) is evident. Thus it is enough to show (i)⇒(ii).

Let u be any point from the positive multioctant, $u \succeq 0$. Since $v \succ 0$, there exists $s \in \mathbb{R}_+$ such that $sv \succeq u$. Therefore for every sufficiently small, positive t,

$$C(tsv) \ge C(tu),$$

and we have

$$0 \le \frac{C(tu)}{t} \le \frac{C(tsv)}{t} = \frac{C(tsv)}{ts}\, s \overset{t\to 0^+}{\longrightarrow} 0.$$

Hence $\lim_{t\to 0^+} C(tu)/t = 0$.

Now we can prove Theorem 2.

Proof of Theorem 2. Let $i \ne j$. Then

$$\mathbb{P}(\mathcal{X}_i < x \mid \mathcal{X}_j < x) = \frac{\mathbb{P}(\mathcal{X}_i < x, \mathcal{X}_j < x)}{\mathbb{P}(\mathcal{X}_j < x)}$$
$$\ge \frac{\mathbb{P}(\mathcal{X}_1 < x, \ldots, \mathcal{X}_n < x)}{\mathbb{P}(\mathcal{X}_j < x)} = \frac{C(F(x), \ldots, F(x))}{F(x)},$$

where F is the cumulative distribution of the $\mathcal{X}_i$'s. Since

$$\lim_{x\to x^*} F(x) = 0,$$

passing to the limit we get

$$\lim_{x\to x^*} \mathbb{P}(\mathcal{X}_i < x \,|\, \mathcal{X}_j < x) \geq \lim_{x\to x^*} \frac{C(F(x),\ldots,F(x))}{F(x)} = \lim_{t\to 0^+} \frac{C(t,\ldots,t)}{t}.$$

As C is nonnegative, $\lim_{x\to x^*} \mathbb{P}(\mathcal{X}_i < x \,|\, \mathcal{X}_j < x) = 0$ yields

$$\lim_{t\to 0^+} \frac{C(t,\ldots,t)}{t} = 0.$$

Note that in the two-dimensional case

$$\mathbb{P}(\mathcal{X}_1 < x \,|\, \mathcal{X}_2 < x) = \frac{\mathbb{P}(\mathcal{X}_1 < x, \mathcal{X}_2 < x)}{\mathbb{P}(\mathcal{X}_2 < x)} = \frac{C(F(x),F(x))}{F(x)},$$

hence we get the equivalence from Remark 2.

6. Properties of the leading part L

6.1. *Basics.* Let L be the leading part of a uniform lower tail expansion of a given copula C. From Theorem 1 we know that L is continuous, grounded and n-nondecreasing. Moreover since all copulas satisfy the estimate

$$0 \leq C(u) \leq \min(u_1,\ldots,u_n)$$

([13, Th. 2.10.12]), we have:

PROPOSITION 3. *L is nonnegative and bounded by the smallest of its arguments*,

$$0 \leq L(u) \leq \min(u_1,\ldots,u_n).$$

From the above and the homogeneity of L we obtain its further properties (cf. [7, Cor. 4]).

COROLLARY 1. *Let L be grounded*, *n-nondecreasing*, *bounded by its arguments and homogeneous of degree* 1. *Then for all $u, v \succeq 0$, L satisfies the following*:

(i) *L is nondecreasing with respect to the partial ordering $\preceq$*,

$$u \preceq v \;\Rightarrow\; L(u) \leq L(v).$$

(ii) *L is Lipschitz with Lipschitz constant* 1,

$$|L(v) - L(u)| \leq \sum_i |v_i - u_i|.$$

(iii) *L is continuous.*

6.2. *Measures with homogeneous distribution functions.* For the action of the multiplicative group $\mathbb{R}_+$ on $[0,\infty)^n$,

$$\Xi : \mathbb{R}_+ \times [0,\infty)^n \to [0,\infty)^n, \qquad \Xi(t,u) = tu,$$

we denote by Ξ_u the parametrization of the orbit of the point u,

$$\Xi_u : \mathbb{R}_+ \to [0,\infty)^n, \qquad \Xi_u(t) = \Xi(t,u) = tu,$$

and by Ξ_Δ the diffeomorphism

$$\Xi_\Delta : \mathbb{R}_+ \times \Delta \to [0,\infty)^n \setminus \{0\}, \qquad \Xi_\Delta(t,x) = tx.$$

Lemma 2. *If L is grounded, n-nondecreasing and homogeneous of degree 1 then the corresponding measure μ_L is relatively invariant with respect to Ξ, i.e. for any Borel set $A \subset [0,\infty)^n$ and any $t > 0$,*

$$\mu_L(tA) = t\mu_L(A).$$

Proof. Since L is homogeneous of degree 1, for every segment $I(u,v)$ with vertices u, v ($u \preceq v$) and any $t > 0$ we have

$$\mu_L(tI(u,v)) = V_L(tu,tv) = tV_L(u,v) = t\mu_L(I(u,v)).$$

The above equality remains true for Borel sets (see [1, Theorem 10.3]).

Let μ_Δ be the factor measure on Δ defined by

$$\mu_\Delta(A) = \mu_L\Big(\bigcup_{t\le 1} tA\Big)$$

for any Borel subset A of Δ. Since μ_L vanishes on the boundary of $[0,\infty)^n$, μ_Δ vanishes on all faces of the simplex Δ,

$$\forall i \quad \mu_\Delta(\{x \in \Delta : x_i = 0\}) = 0.$$

Since Ξ_Δ is a diffeomorphism, the measure μ_L is the product of μ_Δ and a relatively invariant measure on $\mathbb{R}_+$. It turns out that the latter is the Lebesgue measure m. Indeed,

$$m((ta,tb]) = tm((a,b]).$$

Two relatively invariant measures with the same multiplicator are proportional (see [2, Chapter VII]), but our choice of μ_Δ determines only one of them. Indeed, only m satisfies the product rule

$$\mu_L\Big(\bigcup_{t\le 1} t\Delta\Big) = \mu_\Delta(\Delta)\cdot m((0,1]).$$

Thus, by the Fubini theorem ([15, Th.7.8]) we get (cf. also [2, §VII.2]):

Corollary 2. *For any bounded Borel set $A \subset [0,\infty)^n$,*

$$\mu_L(A) = \int_\Delta m(\Xi_\xi^{-1}(A))\, d\mu_\Delta(\xi).$$

Furthermore setting $A = I(0,u)$ we obtain

Corollary 3.

$$L(u) = \mu_L(I(0,u)) = \int_\Delta \min(u_1/q_1,\ldots,u_n/q_n)\, d\mu_\Delta(q).$$

Now we are able to continue the characterization of the leading parts.

PROPOSITION 4. *Any grounded, n-nondecreasing, homogeneous function $L : [0,\infty)^n \to \mathbb{R}$ of degree 1 is*

(i) *superadditive*:
$$L(u+v) \ge L(u) + L(v).$$

(ii) *concave*:
$$\forall \lambda_1, \lambda_2 \ge 0, \lambda_1 + \lambda_2 = 1 \qquad L(\lambda_1 u + \lambda_2 v) \ge \lambda_1 L(u) + \lambda_2 L(v).$$

Proof. (i) We have
$$L(u+v) - (L(u) + L(v)) = \mu_L(I(0, u+v)) - \mu_L(I(0,u)) - \mu_L(I(0,v)) = *.$$
Next we apply Corollary 3 and the inequality $\min(a_i + b_i) \ge \min(a_i) + \min(b_i)$ to get
$$\begin{aligned} * &= \int_{\Delta} m(\Xi_\xi^{-1}(I(0,u+v)))\, d\mu_\Delta(\xi) - \int_{\Delta} m(\Xi_\xi^{-1}(I(0,u)))\, d\mu_\Delta(\xi) \\ &\quad - \int_{\Delta} m(\Xi_\xi^{-1}(I(0,v)))\, d\mu_\Delta(\xi) \\ &= \int_{\Delta} (\min\{(u_i+v_i)/\xi_i : i = 1,\dots,n\} - \min\{u_i/\xi_i : i = 1,\dots,n\} \\ &\quad - \min\{v_i/\xi_i : i = 1,\dots,n\})\, d\mu_\Delta(\xi) \ge 0. \end{aligned}$$

(ii) is a direct consequence of (i). Indeed, let $\lambda_1, \lambda_2 \ge 0$ and $\lambda_1 + \lambda_2 = 1$. Then
$$L(\lambda_1 u + \lambda_2 v) \ge L(\lambda_1 u) + L(\lambda_2 v) = \lambda_1 L(u) + \lambda_2 L(v).$$

REMARK 3. Since any n-nondecreasing, homogeneous function H of degree 1 is a sum of a linear function H_1 and a grounded, n-nondecreasing, homogeneous function H_2 of degree 1, Proposition 4 remains valid without the "grounded" assumption.

One can also characterize the factor measures corresponding to leading parts.

PROPOSITION 5. *For a grounded, n-nondecreasing, homogeneous function $L : [0,\infty)^n \to \mathbb{R}$ of degree 1 the following conditions are equivalent*:

(i) $L(u) \le \min(u_1, \dots, u_n)$ *for* $u \succeq 0$.
(ii) $\int_\Delta (1/q_i)\, d\mu_\Delta(q) \le 1$ *for* $i = 1, \dots, n$.

Proof. (ii)⇒(i). By Corollary 3 we have
$$\begin{aligned} L(u) &= \mu_L(I(0,u)) = \int_{\Delta} \min(u_1/q_1, \dots, u_n/q_n)\, d\mu_\Delta(q) \\ &\le \min\Big(u_1 \int_{\Delta} \frac{1}{q_1}\, d\mu_\Delta(q), \dots, u_n \int_{\Delta} \frac{1}{q_n}\, d\mu_\Delta(q)\Big) \le \min(u_1, \dots, u_n). \end{aligned}$$

$\neg$(ii)$\Rightarrow\neg$(i). Assume that

$$\int_{\Delta} \frac{1}{q_1}\, d\mu_{\Delta}(q) > 1.$$

Consider the half-line $u = (1, t, \ldots, t)$, $t > 1$. From the Fatou lemma we get

$$\begin{aligned}\liminf_{t\to\infty} L(u) &= \liminf_{t\to\infty} \int_{\Delta} \min\left(\frac{1}{q_1}, \frac{t}{q_2}, \ldots, \frac{t}{q_n}\right) d\mu_{\Delta}(q)\\ &\ge \int_{\Delta} \frac{1}{q_1}\, d\mu_{\Delta}(q) > 1.\end{aligned}$$

So for large t,

$$L(1, t, \ldots, t) > 1 = \min(1, t).$$

6.3. *Copulas with prescribed leading part L.* In this section we show that Proposition 3 completely describes all possible leading parts.

PROPOSITION 6. *Let $L : [0,\infty)^n \to \mathbb{R}$ be an n-nondecreasing function, homogeneous of degree* 1, *such that*

$$\forall u \quad 0 \le L(u) \le \min(u_1, \ldots, u_n).$$

Then there exists a copula C having L as the leading part of a uniform expansion of its lower tail.

Proof. If $L(1, \ldots, 1) = 1$ then, by concavity, $L(u) = \min(u_1, \ldots, u_n)$ and L restricted to the unit rectangle is a copula. In this case we put $C(u) = L_{|\mathrm{cl}\, I(0,(1,\ldots,1))}(u)$.

Now assume that $L(1, \ldots, 1) < 1$. Let w_i, $i = 1, \ldots, n$, be the parametrizations of the edges of the unit rectangle which end at the point $(1, \ldots, 1)$,

$$w_i : [0, 1] \to [0, 1]^n, \qquad w_i(t)_j = \begin{cases} 1 & \text{if } j \ne i,\\ t & \text{if } j = i.\end{cases}$$

We consider the function

$$G(t) = \sum_{i=1}^{n} G_i(t), \qquad G_i(t) = t - L(w_i(t)), \qquad t \in [0, 1].$$

For all t and i we have $L(w_i(t)) \le \min(1, t) = t$, hence each G_i is nonnegative. Furthermore, if $s > t$ then $L(w_i(s)) - L(w_i(t)) \le s - t$, hence each G_i is nondecreasing. Moreover $G(0) = 0 < 1 - L(1, \ldots, 1)$ and $G(1) = n(1 - L(1, \ldots, 1)) \ge 1 - L(1, \ldots, 1)$. Since G is continuous, there is t_0, $0 < t_0 < 1$, such that

$$G(t_0) = 1 - L(1, \ldots, 1), \qquad \forall t < t_0 \quad G(t) < 1 - L(1, \ldots, 1).$$

We define a new singular Borel measure μ_+ on $[0,\infty)^n$. The support of μ_+ consists of n segments joining the point $(t_0, \ldots, t_0)$ and the vertices $w_i(0)$ of the unit rectangle, $i = 1, \ldots, n$. The mass is distributed according to the rule

$$\mu_+(I(0, w_i(t))) = G_i(t) \quad \text{for } 0 \le t \le t_0.$$

Note that for such t the rectangle $I(0, w_i(t))$ intersects only one segment, the one ending at $w_i(0)$.

Let μ be the restriction of the sum of the measures μ_L and μ_+ to the unit rectangle I. Then

$$\mu(I) = \mu_L(I) + \mu_+(I) = L(1, \ldots, 1) + \sum_{i=1}^{n} G_i(t_0) = 1.$$

Thus μ is a probability measure on I. Furthermore, for each i and $0 \le t \le t_0$ we have

$$\mu(I(0, w_i(t))) = \mu_L(I(0, w_i(t))) + \mu_+(I(0, w_i(t))) = L(w_i(t)) + G_i(t) = t,$$

and for $t_0 < t \le 1$,

$$\begin{aligned}\mu(I(0, w_i(t))) &= \mu_L(I(0, w_i(t))) + \mu_+(I(0, w_i(t))) \\ &= L(w_i(t)) + G_i(t_0) + \sum_{j \ne i}\Big(G_j(t_0) - G_j\Big(t_0 \frac{1-t}{1-t_0}\Big)\Big) \\ &= 1 - (L(1, \ldots, 1) - L(w_i(t))) - \sum_{j \ne i} G_j\Big(t_0 \frac{1-t}{1-t_0}\Big).\end{aligned}$$

Let $\mathcal{X}_i$ be the projection onto the ith axis restricted to the unit rectangle I,

$$\mathcal{X}_i(u_1, \ldots, u_n) = u_i.$$

Since $\mathcal{X}_i$ is μ-measurable, it is a random variable. Moreover its distribution function F_i is continuous and $F_i(t) = t$ for $0 \le t \le t_0$.

The copula C we are looking for is the joint distribution function of the random variables $F_i(\mathcal{X}_i)$ (see Proposition 1). For $u \preceq (t_0, \ldots, t_0)$ we have

$$C(u) = \mu\Big(\bigwedge_i F_i(\mathcal{X}_i) \le u_i\Big) = *,$$

but $F_i(t) = t$ for $t < t_0$, thus

$$* = \mu\Big(\bigwedge_i \mathcal{X}_i \le u_i\Big) = \mu(I(0, u)) = \mu_L(I(0, u)) + \mu_+(I(0, u)) = L(u) + 0.$$

This finishes the proof: the lower tail of C equals L.

The above results remain valid for pairs of tails and pairs of functions.

Proposition 7. *Any two n-nondecreasing functions $L_i : [0, \infty)^n \to \mathbb{R}$, $i = 1, 2$, homogeneous of degree 1, and such that*

$$0 \le L_i(u) \le \min(u_i),$$

are the leading parts of the lower and upper tail expansion of the same copula.

Proof. We apply the patchwork technique (cf. [13, §3.2.2]). For $i = 1, 2$ let C_i be copulas having L_i as lower tails (as in the proof of the previous proposition). Then the function $C : [0,1]^n \to [0,1]$, defined by

$$C(u) = \begin{cases} C_1(2u)/2 & \text{for } 0 \preceq u \preceq (1/2,\dots,1/2), \\ (1+\widehat{C}_2(2u_1-1,\dots,2u_n-1))/2 & \text{for } (1/2,\dots,1/2) \preceq u \preceq (1,\dots,1), \\ C_1(\min(1,2u_1),\dots,\min(1,2u_n))/2 & \text{otherwise,} \end{cases}$$

is a copula.

Indeed, for the lower subrectangle we apply the measure induced by C_1, for the upper one the measure induced by $\widehat{C}_2$ and for the rest the null measure.

Therefore for u enough small

$$C(u) = \frac{C_1(2u)}{2} = L_1(u), \qquad \widehat{C}(u) = \frac{C_2(2u)}{2} = L_2(u).$$

This finishes the proof: the lower tail of C equals L_1 and the upper one equals L_2.

6.4. *Proof of Theorem 3.* Theorem 3 follows from Proposition 3 (i)⇒(ii), Proposition 5 (ii)⇔(iii) and Proposition 6 (ii)⇒(i).

7. Examples

7.1. *Examples of trivial expansions*

• Assume that the $\mathcal{X}_i$ are independent. Then

$$C(u) = \prod_i u_i.$$

In this case $L(u) = 0$ and $R(u) = u_1 \dots u_n/(u_1 + \dots + u_n)$. Hence the expansion of the lower tail is trivial. The same is valid for the survival copula. Hence the expansion of the upper tail is also trivial.

• Assume that $\mathcal{X}_2 = -\mathcal{X}_1$. Then

$$C(u) \le \max(0, u_1 + u_2 - 1) \quad \text{and} \quad \widehat{C}(u) \le \max(0, u_1 + u_2 - 1).$$

Once more the expansions of both tails are trivial.

• *The Gaussian copula.* Let $\mathcal{X}_i$ have the same standard normal distribution $N(0,1)$ and normal joint distribution. Then their copula

$$C_N(u) = F_{\mathcal{X}}(F^{-1}(u_1), \dots, F^{-1}(u_n)),$$

where F is the distribution function of the standard normal distribution $(N(0,1))$, is called *Gaussian.*

PROPOSITION 8. *The Gaussian copula C_N has trivial expansions at all vertices of the unit cube.*

Proof. Note that the random variable $\mathcal{Z} = \mathcal{X}_1 + \cdots + \mathcal{X}_n$ has the normal distribution,

$$E(\mathcal{Z}) = 0, \qquad D^2(\mathcal{Z}) = c^2 = \sum_{i,j} c_{i,j} < n^2, \qquad 0 < c < n,$$

where $c_{i,j} = \operatorname{cov}(\mathcal{X}_i, \mathcal{X}_j)$ (if $i \neq j$ then $-1 < c_{i,j} < 1$).

Let $u_i = F(x_i)$ and $x_i \le x_1 < 0$ for every i (due to symmetry we may renumber the variables). Then

$$\begin{aligned} 0 &\le \frac{C_N(u)}{u_1 + \cdots + u_n} = \frac{F_{\mathcal{X}}(x)}{F(x_1) + \cdots + F(x_n)} = \frac{\mathbb{P}(\bigwedge_i \mathcal{X}_i \le x_i)}{F(x_1) + \cdots + F(x_n)} \\ &\le \frac{\mathbb{P}(\mathcal{X}_1 + \cdots + \mathcal{X}_n \le x_1 + \cdots + x_n)}{F(x_1) + \cdots + F(x_n)} = \frac{F((x_1 + \cdots + x_n)/c)}{F(x_1) + \cdots + F(x_n)} \\ &\le \frac{F(nx_1/c)}{F(x_1)}. \end{aligned}$$

Next we apply the de l'Hospital rule:

$$\lim_{x_1 \to -\infty} \frac{F(nx_1/c)}{F(x_1)} = \lim_{x_1 \to -\infty} \frac{n}{c} \frac{\exp\left(-\frac{n^2 x_1^2}{2c^2}\right)}{\exp\left(-\frac{x_1^2}{2}\right)} = \lim_{x_1 \to -\infty} \frac{n}{c} \exp\left(\left(\frac{1}{2} - \frac{n^2}{2c^2}\right) x_1^2\right).$$

Since $c < n$, the argument of the exponential function is negative and

$$\lim_{x_1 \to -\infty} \exp\left(\left(\frac{1}{2} - \frac{n^2}{2c^2}\right) x_1^2\right) = 0.$$

By the theorem of three limits we get $L(u) = 0$. This finishes the proof for the lower tail. The other cases are quite similar.

Remark 4. The above result shows that the phenomenon of the nontrivial tail expansions does not exist in the "world ruled by the paradigm of *normality of all distributions*".

7.2. *Simple examples of nontrivial expansions.* Assume that $\mathcal{X}_1 = \cdots = \mathcal{X}_n$. Then

$$C(u) = \min(u_1, \ldots, u_n).$$

In this case $L(u) = \min(u_i)$ and $R(u) = 0$. The factor measure is concentrated at one point $q = (1/n, \ldots, 1/n)$:

$$\mu_\Delta(\{q\}) = \mu_L(I(0,q)) = L(q) = 1/n.$$

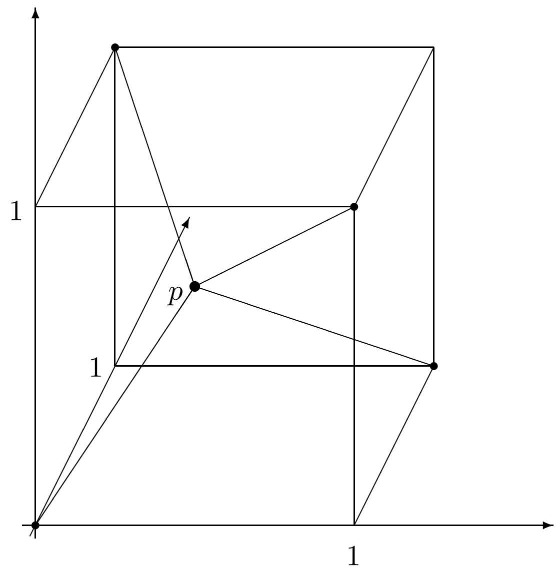
The same is true for the survival copula.

Note that the above remains true if we only assume that the $\mathcal{X}_i$ are comonotonic.

The above copula has a singular support (the diagonal $\{(t, \ldots, t) : t \in [0,1]\}$). Consider the singular copulas C_p, where $p = (p_1, \ldots, p_n)$, $p_1 + \cdots + p_n = 1 + (n-1)a$ and $0 < a < p_i < 1$, defined by

$$C_p(u) = a\min(1, u_1/p_1, \ldots, u_n/p_n) + \sum_{i=1}^{n}(p_i - a)\left(\min\left\{\frac{(u_j - p_j)^+}{1 - p_j} : j \neq i\right\} - \frac{(p_i - u_i)^+}{p_i}\right)^+.$$

The support consists of $n+1$ segments, one joining p and the origin, and n segments joining p and the vertices of the unit rectangle having $n-1$ coefficients equal to 1. On each of them the distribution of mass is uniform.



The upper tail has a trivial expansion. But the lower tail always has a nonzero leading part:

$$L(u) = \frac{\varrho - 1}{n - 1}\min\left(\frac{u_1}{p_1}, \ldots, \frac{u_n}{p_n}\right), \qquad \varrho = p_1 + \cdots + p_n.$$

The factor measure is concentrated at one point $q = (1/\varrho)p$:

$$\mu_\Delta(\{q\}) = \mu_L(I(0, q)) = L(q) = \frac{\varrho - 1}{\varrho(n - 1)} < \frac{1}{n}.$$

7.3. *Archimedean copulas*

7.3.1. *Basics*

Definition 4. A copula $C : [0,1]^n \to [0,1]$ is called *Archimedean* if there exists a strictly decreasing, convex and continuous function $\varphi : [0,1] \to [0,\infty]$ with $\varphi(1) = 0$ and $\varphi(0) = \varphi_0 \leq \infty$ such that

$$C(u) = \varphi^{-1}(\min(\varphi_0, \varphi(u_1) + \cdots + \varphi(u_n)))$$

(see [13, §4.6], [3, §4.8.4] or [10, §4.2]).

Lemma 3. *Let $\varphi : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function such that $\varphi(1) = 0$ and $\varphi(0) = \varphi_0 \le \infty$ and the function*

$$\psi : [0,\infty] \to [0,1], \qquad \psi(x) = \varphi^{-1}(\min(\varphi_0, x)),$$

is $n-2$-times differentiable on $(0,\infty)$ and $(-1)^n \psi^{(n-2)}(x)$ is convex. Then the function

$$C_\varphi : [0,1]^n \to [0,1], \qquad C_\varphi(u) = \psi\Big(\sum_{i=1}^n \varphi(u_i)\Big),$$

is a copula.

Proof. We check the axioms.

(i) *C_φ is grounded.* Indeed, if $u_j = 0$ then

$$\sum \varphi(u_i) \ge \varphi(u_j) = \varphi_0.$$

Hence

$$\psi\Big(\sum \varphi(u_i)\Big) = \varphi^{-1}(\varphi_0) = 0.$$

(ii) *The marginal distributions are uniform.* Let $u_i = 1$ for $i \ne j$. Then $\varphi(u_i) = 0$ and

$$\sum \varphi(u_i) = \varphi(u_j).$$

Hence

$$\psi\Big(\sum \varphi(u_i)\Big) = \varphi^{-1}(\varphi(u_j)) = u_j.$$

(iii) *C_φ is n-nondecreasing.* If ψ is n-times differentiable and $\psi^{(n)}$ is continuous then the nth mixed derivative of C exists and

$$\frac{\partial^n C_\varphi}{\partial u_1 \ldots \partial u_n}(u) = \psi^{(n)}\Big(\sum \varphi(u_i)\Big) \cdot \prod_{i=1}^n \varphi'(u_i) \ge 0,$$

and for $v \succeq u$,

$$V_{C_\varphi}(u,v) = \int\limits_{I(u,v)} \frac{\partial^n C_\varphi}{\partial u_1 \ldots \partial u_n}(q)\, dq_1 \ldots dq_n \ge 0.$$

The general case is a bit more complicated. For fixed $u = (u_1, \ldots, u_n)$ we put

$$G(s) = (-1)^n \psi^{(n-2)}\Big(s + \sum_{i=3}^n \varphi(u_i)\Big).$$

Let $0 \le s_1 \le s_2$ and $0 \le s_3 \le s_4$. Then

$$s_1 + s_3 \le \min(s_1 + s_4, s_2 + s_3) \le \max(s_1 + s_4, s_2 + s_3) \le s_2 + s_4.$$

Since G is convex, we have

$$G(s_1 + s_3) + G(s_2 + s_4) \ge G(s_1 + s_4) + G(s_2 + s_3).$$

Therefore since $\psi^{(n-2)}$ is continuous, we get

$$V_{C_\varphi}(u,v) = \int_{u_3}^{v_3} \dots \int_{u_n}^{v_n} \Big(\psi^{(n-2)}\Big(\varphi(u_1) + \varphi(u_2) + \sum_{i=3}^{n} \varphi(q_i)\Big)$$

$$+ \psi^{(n-2)}\Big(\varphi(v_1) + \varphi(v_2) + \sum_{i=3}^{n} \varphi(q_i)\Big) - \psi^{(n-2)}\Big(\varphi(v_1) + \varphi(u_2) + \sum_{i=3}^{n} \varphi(q_i)\Big)$$

$$- \psi^{(n-2)}\Big(\varphi(u_1) + \varphi(v_2) + \sum_{i=3}^{n} \varphi(q_i)\Big)\Big) \prod_{i=3}^{n} \varphi'(q_i)\, dq_3 \dots dq_n \geq 0.$$

7.3.2. *Tails.* We shall show that the lower (resp. upper) tail expansion of the Archimedean copula depends on the limit elasticity of $\varphi(x)$ (resp. $\varphi(1-x)$) at 0, defined by

$$\mathcal{E}_x(0) = \lim_{x\to 0^+} \frac{x\varphi'(x)}{\varphi(x)}, \qquad \mathcal{E}_x(1) = \lim_{x\to 0^+} \frac{-x\varphi'(1-x)}{\varphi(1-x)}.$$

The only exception concerns the lower tail expansion. Namely if φ_0 is finite than $C_\varphi(u)$ vanishes in some neighbourhood of the origin, so $L(u) = 0$.

PROPOSITION 9. *If the limit $\mathcal{E}_x(1)$ exists then the Archimedean copula C_φ has a uniform upper tail expansion. Moreover if $\mathcal{E}_x(1) = d$, $1 < d < \infty$, then*

$$L(u) = (-1)^{n-1} V_{L^*}(0,u), \qquad L^*(u) = \sqrt[d]{u_1^d + \dots + u_n^d};$$

if $\mathcal{E}_x(1) = \infty$, then $L(u) = \min(u_i)$; and if $\mathcal{E}_x(0) = 1$, then $L(u) = 0$.

The proof is the same as the proof of Theorem 5 in [8]. Roughly speaking, it is based on the fact that if $\mathcal{E}_x(1) = d$, then for x close to 1,

$$\varphi(x) \approx c(1-x)^d, \qquad c > 0.$$

PROPOSITION 10. *If $\varphi_0 = \infty$ and the limit $\mathcal{E}_x(0)$ exists then the Archimedean copula C_φ has a uniform lower tail expansion. Moreover if $\mathcal{E}_x(0) = -d$, $0 < d < \infty$, then*

$$L(u) = \frac{1}{\sqrt[d]{u_1^{-d} + \dots + u_n^{-d}}};$$

if $\mathcal{E}_x(0) = -\infty$, then $L(u) = \min(u_i)$; and if $\mathcal{E}_x(0) = 0$, then $L(u) = 0$.

The proof is the same as the proof of Theorem 6 in [8]. Roughly speaking it is based on the fact that if $\mathcal{E}_x(0) = -d$, then for x close to 0,

$$\varphi(x) \approx cx^{-d}, \qquad c > 0.$$

PROPOSITION 11. *If $\varphi_0 = \infty$ then at any intermediate vertex e the Archimedean copula C has trivial tail expansion.*

Proof. Assume that $e_1 = 1$ and $e_2 = 0$. Since φ is convex, we have
$$\lim_{t\to\infty} \psi'(t) = 0.$$
We get
$$\begin{aligned} C(1-v_1, v_2, \widetilde{u}) - C(1, v_2, \widetilde{u}) &= \psi(\varphi(1-v_1) + \varphi(v_2) + \cdots) - \psi(\varphi(v_2) + \cdots) \\ &= \psi'(\varphi(v_2) + \cdots)\varphi'(1)(-v_1) + o(v_1) \\ &= o(v_1 + v_2) \end{aligned}$$
(at points where ψ or φ are not differentiable we take the left derivative). Therefore
$$\widehat{C}_e(v) = V_C((1-v_1, 0, \ldots), (1, v_2, \ldots)) = o(v_1 + v_2),$$
and
$$\frac{\widehat{C}_e(tv)}{t} = \frac{o(t)}{t} \to 0.$$

7.4. *Multivariate extreme value copulas.* In modelling the multivariate extreme value distribution many authors based on the following family of copulas ([12], [6]):
$$C_W(u_1, \ldots, u_n) = \begin{cases} \exp(-W(-\ln(u_1), \ldots, -\ln(u_n))) & \text{if } \prod u_i > 0, \\ 0 & \text{if } \prod u_i = 0, \end{cases}$$
$$W(z_1, \ldots, z_n) = \int_{\Delta} \max(w_1 z_1, \ldots, w_n z_n)\, dH(w),$$
where Δ is the unit simplex in $\mathbb{R}^n$, $w_1 + \cdots + w_n = 1$, and H is a positive dependence measure subject to
$$\int_{\Delta} w_j\, dH(w) = 1, \qquad j = 1, \ldots, n.$$
We shall show that the copulas of this family have uniform expansions at all corners.

Proposition 12. *Every copula C_W has a nontrivial upper tail expansion with leading term*
$$L(u) = (-1)^{n+1} V_W(0, u) = \int_{\Delta} \min(w_1 u_1, \ldots, w_n u_n)\, dH(w).$$

Proof. Let e be the upper vertex of the unit rectangle, $e = (1, \ldots, 1)$. The copula $\widehat{C}_e$ associated to C_W at e equals
$$\widehat{C}_e(u) = V_{C_W}(e - u, e).$$
Therefore
$$\lim_{t\to 0^+} \frac{\widehat{C}_e(tu)}{t} = \lim_{t\to 0^+} \frac{V_{C_W}(e - tu, e)}{t} = \lim_{t\to 0^+} \frac{(-1)^d V_{\exp(-W())}(0, tu)}{t}$$

$$= \lim_{t\to 0^+} \frac{(-1)^d V_{1-W()}(0,tu)}{t} = \lim_{t\to 0^+} \frac{(-1)^{n+1} V_{tW()}(0,u)}{t} = (-1)^{n+1} V_W(0,u).$$

Thus $\widehat{C}_e$ has a uniform expansion. Furthermore

$$L(u) = (-1)^{n+1} V_W(0,u) = (-1)^{n+1} \int_{\Delta} V_{\max}(0,(w_1u_1,\ldots,w_nu_n))\, dH(w)$$
$$= \int_{\Delta} \min(w_1u_1,\ldots,w_nu_n)\, dH(w).$$

PROPOSITION 13. *If $W(1,\ldots,1) > 1$ then the copula C_W has a trivial lower tail expansion.*

Proof. We have

$$\lim_{t\to 0^+} \frac{C_W(tu)}{t} = \lim_{t\to 0^+} \frac{\exp(-W(-\ln(tu_1),\ldots,-\ln(tu_n)))}{t}$$
$$= \lim_{t\to 0^+} \frac{\exp(-W(-\ln(t)(1+\ln(t)^{-1}\ln(u_1)),\ldots))}{t}$$
$$= \lim_{t\to 0^+} \frac{\exp(\ln(t)W(1+\ln(t)^{-1}\ln(u_1),\ldots))}{t}$$
$$= \lim_{t\to 0^+} t^{W(1+\ln(t)^{-1}\ln(u_1),\ldots)-1} = 0.$$

REMARK 5. The exceptional case $W(1,\ldots,1) = 1$ occurs only when the measure H is concentrated at one point (the centre of the simplex). In this case $W(z) = \max(z_i)$ and $C_W(u) = \min(u_i)$, hence the lower tail expansion is uniform.

PROPOSITION 14. *At any intermediate vertex e the MEV copula C_W has a trivial tail expansion.*

Proof. Assume that $e_1 = 1$ and $e_2 = 0$. Then

$$0 \le C(1,v_2,\widetilde{u}) - C(1-v_1,v_2,\widetilde{u})$$
$$= C(1,v_2,\widetilde{u})(1-\exp(-W(-\ln(1-v_1),\ldots)+W(0,\ldots)))$$
$$\le v_2(1-\exp(\ln(1-v_1))) = v_2v_1.$$

Therefore

$$\widehat{C}_e(v) = V_{C_W}((1-v_1,0,\ldots),(1,v_2,\ldots)) = O(v_1v_2),$$

and

$$\frac{\widehat{C}_e(tv)}{t} = \frac{O(t^2)}{t} \to 0.$$

8. Tests and estimators for nontrivial tail expansions. In practice one is not only interested whether an empirical distribution has a uniform tail expansion, but also how big part of the tail can be approximated by the leading part of the tail expansion. Therefore in this section we shall discuss the tests which tell us whether past a given benchmark γ the tail of an empirical copula C_e is close enough to some n-nondecreasing, grounded function L_e homogeneous of degree 1.

Model. $\mathcal{X}_i$, $i = 1, \ldots, n$, are random variables defined on the same probability space $(\Omega, \mathcal{M}, \mathbb{P})$, with continuous distribution functions F_i and a copula C. Assume that we have a sample consisting of N n-tuples of observations $x_{t,i}$, $t = 1, \ldots, N$, $i = 1, \ldots, n$.

We consider a new random variable

$$\mathcal{R} = \sum_{i=1}^{n} F_i(\mathcal{X}_i).$$

THEOREM 4. *If C has a nontrivial uniform lower tail expansion then for $\gamma \to 0$ the conditional distributions of $\gamma^{-1}\mathcal{R} \,|\, \mathcal{R} \le \gamma$ converge in distribution to the uniform distribution on the unit interval.*

Proof. Let F_γ be the distribution functions of the conditional distributions of $\gamma^{-1}\mathcal{R} \,|\, \mathcal{R} \le \gamma$. For $0 < x < 1$ we have

$$\begin{aligned} F_\gamma(x) &= \mathbb{P}(\gamma^{-1}\mathcal{R} \le x \,|\, \mathcal{R} \le \gamma) = \frac{\mathbb{P}(\mathcal{R} \le x\gamma)}{\mathbb{P}(\mathcal{R} \le \gamma)} \\ &= \frac{\mu_C(\bigcup_{t\le 1} tx\gamma\Delta)}{\mu_C(\bigcup_{t\le 1} t\gamma\Delta)} = \frac{\mu_\gamma(\bigcup_{t\le 1} tx\Delta)}{\mu_\gamma(\bigcup_{t\le 1} t\Delta)}. \end{aligned}$$

Since $\mu_L(\partial(\bigcup_{t\le 1} t\gamma\Delta)) = \mu_L(\partial(\bigcup_{t\le 1} tx\gamma\Delta)) = 0$, from Theorem 1 and [1, Theorem 29.1] or [16, Corollary IV.7.2] we get

$$\lim_{\gamma\to 0} F_\gamma(x) = \frac{\mu_L(\bigcup_{t\le 1} tx\Delta)}{\mu_L(\bigcup_{t\le 1} t\Delta)} = \frac{x\mu_L(\bigcup_{t\le 1} t\Delta)}{\mu_L(\bigcup_{t\le 1} t\Delta)} = x.$$

This finishes the proof.

Thus we should first check whether, for a given benchmark γ, the conditional distribution $\mathcal{R} \,|\, \mathcal{R} \le \gamma$ is uniform on the interval $[0, \gamma]$. This can be easily done with the help of some classical test like the Kolmogorov test. Note that if we approximate F_i's by empirical distributions then we get a sample $r_1, \ldots, r_N$ from $\mathcal{R}$, where r_t are just sums of ranks divided by the size of the sample,

$$r_t = \frac{\operatorname{rank}(x_{t,1}) + \cdots + \operatorname{rank}(x_{t,n})}{N}.$$

Let $\mathcal{Q}$ be an n-dimensional random variable with values in the unit simplex Δ,

$$\mathcal{Q}_i = \frac{F_i(\mathcal{X}_i)}{\mathcal{R}}.$$

THEOREM 5. *If C has a nontrivial uniform lower tail expansion then for $\gamma \to 0$ the conditional distributions of $\mathcal{Q} \,|\, \mathcal{R} \le \gamma$ converge in distribution to the factor measure μ_Δ divided by $\mu_\Delta(\Delta)$.*

Proof. Since C has a nontrivial tail expansion, we have $\mu_\Delta(\Delta) > 0$. We denote by $\mathbb{P}_\Delta$ the probabilility measure on $\mathbb{R}^n$ induced by the factor measure μ_Δ,

$$\mathbb{P}_\Delta(A) = \frac{\mu_\Delta(A \cap \Delta)}{\mu_\Delta(\Delta)}.$$

Let $\mathbb{P}_\gamma$ be the conditional distribution of $\mathcal{Q} \,|\, \mathcal{R} \le \gamma$. For any open subset G of $\mathbb{R}^n$ we have

$$\begin{aligned}\mathbb{P}_\gamma(G) &= \mathbb{P}_\gamma(G \cap \Delta) = \mathbb{P}(\mathcal{Q} \in G \cap \Delta \,|\, \mathcal{R} \le \gamma) = \frac{\mathbb{P}(\mathcal{Q} \in G \cap \Delta \wedge \mathcal{R} \le \gamma)}{\mathbb{P}(\mathcal{R} \le \gamma)} \\ &= \frac{\mathbb{P}(\mathcal{R}\mathcal{Q} \in \bigcup_{t \le 1} t\gamma G \cap \Delta)}{\mathbb{P}(\mathcal{R} \le \gamma)} = \frac{\mu_C(\gamma \bigcup_{t \le 1} t(G \cap \Delta))}{\mu_C(\gamma \bigcup_{t \le 1} t\Delta)} = \frac{\mu_\gamma(\bigcup_{t \le 1} t(G \cap \Delta))}{\mu_\gamma(\bigcup_{t \le 1} t\Delta)}.\end{aligned}$$

Since $\mu_L(\partial(\bigcup_{t \le 1} t\gamma\Delta)) = 0$, we get (cf. [1, Theorem 29.1])

$$\begin{aligned}\liminf_{\gamma \to 0} \mathbb{P}_\gamma(G) &= \liminf_{\gamma \to 0} \frac{\mu_\gamma(\bigcup_{t \le 1} t(G \cap \Delta))}{\mu_\gamma(\bigcup_{t \le 1} t\Delta)} \ge \frac{\liminf_{\gamma \to 0} \mu_\gamma(\bigcup_{t < 1} t(G \cap \Delta))}{\lim_{\gamma \to 0} \mu_\gamma(\bigcup_{t \le 1} t\Delta)} \\ &\ge \frac{\mu_L(\bigcup_{t < 1} t(G \cap \Delta))}{\mu_L(\bigcup_{t \le 1} t\Delta)} = \frac{\mu_\Delta(G \cap \Delta)}{\mu_\Delta(\Delta)} = \mathbb{P}_\Delta(G).\end{aligned}$$

This finishes the proof.

Thus we should next check whether, for a given benchmark γ, the conditional distributions of $\mathcal{Q} \,|\, \mathcal{R} \le t$, $0 < t \le \gamma$, depend on t.

Furthermore, we know that the factor measure is concentrated at the centre of the simplex (Theorem 3). This should also be tested.

THEOREM 6. *If C has a nontrivial uniform lower tail expansion then*

$$\lim_{\gamma \to 0} E\left(\frac{1}{\gamma \mathcal{Q}_i} \mathbb{I}_{\mathcal{R} \le \gamma}\right) \le 1.$$

Proof. We have

$$\begin{aligned}\lim_{\gamma \to 0} E\left(\frac{1}{\gamma \mathcal{Q}_i} \mathbb{I}_{\mathcal{R} \le \gamma}\right) &= \gamma^{-1} E(\mathcal{Q}_i^{-1} \,|\, \mathcal{R} \le 1) \mathbb{P}(\mathcal{R} \le \gamma) \\ &= \gamma^{-1} E(\mathcal{Q}_i^{-1} \,|\, \mathcal{R} \le 1) \mu_C\Big(\bigcup_{t \le 1} t\gamma\Delta\Big) \\ &= E(\mathcal{Q}_i^{-1} \,|\, \mathcal{R} \le 1) \mu_\gamma\Big(\bigcup_{t \le 1} t\Delta\Big) \\ &\xrightarrow{\gamma \to 0} E_{\mathbb{P}_\Delta}(\mathcal{Q}_i) \mu_\Delta(\Delta) = \int_\Delta \frac{1}{q_i}\, d\mu_\Delta(q) \le 1.\end{aligned}$$

So the required test should check whether for a benchmark γ the expected values of $\gamma^{-1}\mathcal{Q}_i^{-1}\mathbb{I}_{\mathcal{R}\le\gamma}$ are not greater than 1. Note that the statistics

$$T_{\gamma,i} = \frac{1}{\gamma N}\sum_{t:\, r_t\le\gamma}\frac{1}{q_{t,i}}, \quad \text{where} \quad q_{t,i} = \frac{\operatorname{rank}(x_{t,i})}{N r_t},$$

are natural estimators of these expected values.

Having selected a proper benchmark γ, we may start to estimate the leading part L. We first estimate the factor measure μ_Δ. The simplest approach is to take either the discrete estimator or the kernel estimator. To every $t = 1,\dots,N$, we associate a measure $\widehat{\mu}_t$ on the unit simplex Δ as follows.

If $r_t > \gamma$, we put $\widehat{\mu}_t = 0$.

If $r_t \le \gamma$ then either $\widehat{\mu}_t$ is the measure concentrated at a point q_t, $q_{t,i} = \operatorname{rank}(x_{t,i})/N r_t$, such that $\widehat{\mu}_t(\{q_t\}) = 1/\gamma N$, or its kernel smoothing.

The estimator $\widehat{\mu}_{\Delta,\gamma,N}$ is the sum of $\widehat{\mu}_t$'s,

$$\widehat{\mu}_{\Delta,\gamma,N} = \sum_{t=1}^{N}\widehat{\mu}_t.$$

Having estimated μ_Δ, we have natural estimators of μ_L and L. Namely we take as $\widehat{\mu}_{L,\gamma,N}$ the product of $\widehat{\mu}_{\Delta,\gamma,N}$ and the Lebesgue measure m on the real half-line. For any Borel set $A \subset [0,\infty)^n$, we get

$$\widehat{\mu}_{L,\gamma,N}(A) = \int_\Delta m(\Xi_q^{-1}(A))\, d\widehat{\mu}_{\Delta,\gamma,N}(q).$$

Finally, we put

$$\widehat{L}_{\gamma,N}(u) = \widehat{\mu}_{L,\gamma,N}(I(0,u)).$$

By construction the estimator $\widehat{L}$ is homogeneous of degree 1, grounded and n-nondecreasing. Moreover if for all i the statistics $T_{\gamma,i}$ are not greater than 1 then the discrete estimator $\widehat{L}_{\gamma,N}$ is the leading part of some copula (see Theorem 3).

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