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NONLINEAR UNILATERAL PROBLEMS IN ORLICZ SPACES

Abstract. We prove the existence of solutions of the unilateral problem for equations of the type $Au - \operatorname{div}\phi(u) = \mu$ in Orlicz spaces, where A is a Leray–Lions operator defined on $\mathcal{D}(A) \subset W_0^1 L_M(\Omega)$, $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$ and $\phi \in C^0(\mathbb{R}, \mathbb{R}^N)$.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^N$ with the segment property. Consider the following nonlinear Dirichlet problem:

$$Au - \operatorname{div}\phi(u) = \mu, \tag{1.1}$$

where $Au = -\operatorname{div} a(x, u, \nabla u)$ is a Leray–Lions operators defined on its domain $\mathcal{D}(A) = \{u \in W_0^1 L_M(\Omega) : a(x, u, \nabla u) \in (L_{\overline{M}}(\Omega))^N\}$ into $W^{-1}E_{\overline{M}}(\Omega)$, with M an N-function and $\phi \in \mathcal{C}^0(\mathbb{R}, \mathbb{R}^N)$. The right-hand side μ is assumed to belong to $L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$.

In the variational case (i.e. where $\mu \in W^{-1}E_{\overline{M}}(\Omega)$), J.-P. Gossez and V. Mustonen [14] solved (1.1) in the case where $\phi = 0$. The case where $\mu \in L^1(\Omega)$ is treated in [5, 6].

In [6], the authors deal with the case $\phi = 0$. They prove the existence and uniqueness of solution for the associated unilateral problem but under some restriction on the N-function M (the Δ_2-condition), and in [5] they etablish the existence of an entropy solution of (1.1) without any restriction on M.

It is our purpose in this paper to prove the existence of solution for the unilateral problem associated to the equation (1.1) in the setting of Orlicz spaces for general N-functions M.

Let us also mention the works of Elmahi and Meskine [11, 12] who studied the existence of solutions for equations of the form $-\operatorname{div} a(x, u, \nabla u) +$

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$g(x, u, \nabla u) = f$ where g is a nonlinearity having natural growth and satisfying the sign condition, and the term f belongs either to $W^{-1}E_{\overline{M}}(\Omega)$ or to $L^1(\Omega)$. These results are generalized in [2, 3].

Let us briefly summarize the contents of the paper. After a section devoted to developing the necessary preliminaries, we introduce some technical lemmas (Section 2). In Section 3, we give our main result and we prove it in Section 4.

2. Preliminaries and some technical lemmas. In this section we list briefly some definitions and well known facts about N-functions and Orlicz–Sobolev spaces. Standard references are [1, 7, 15].

2.1. Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-*function*, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$.

Equivalently, M admits a representation $M(t) = \int_0^t a(s)\,ds$ where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing, right continuous function, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)\,ds$, where $\bar{a} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{a}(t) = \sup\{s : a(s) \le t\}$.

The N-function M is said to satisfy the Δ_2-*condition* if, for some k,

$$M(2t) \le kM(t) \quad \forall t \ge 0. \tag{2.1}$$

It is readily seen that this is the case if and only if for every $r > 0$ there exists a positive constant $k = k(r)$ such that for all $t > 0$,

$$M(rt) \le kM(t) \quad \forall t \ge 0. \tag{2.2}$$

If (2.1) and (2.2) hold only for $t \ge t_0$ for some $t_0 > 0$ then M is said to satisfy the Δ_2-*condition near infinity*.

We extend N-functions to even functions on all of $\mathbb{R}$.

Moreover, we have the following Young inequality:

$$\forall s, t \ge 0, \quad st \le M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. We say that P *grows essentially less rapidly than* Q *near infinity*, written $P \ll Q$, if for every $\varepsilon > 0$, $P(t)/Q(\varepsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} Q^{-1}(t)/P^{-1}(t) = 0$.

2.2. Let M be an N-function and $\Omega \subset \mathbb{R}^N$ be an open and bounded set. The *Orlicz class* $\mathcal{K}_M(\Omega)$ (resp. the *Orlicz space* $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_\Omega M(u(x))\,dx < \infty \quad \Big(\text{resp. } \int_\Omega M(u(x)/\lambda)\,dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M(u(x)/\lambda)\,dx \le 1\Big\}$$

and $\mathcal{K}_M(\Omega)$ is a convex subset of $L_M(\Omega)$ but not necessarily a linear space.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The dual space of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

Let X and Y be arbitrary Banach spaces with a bilinear bicontinuous pairing $\langle\,,\rangle_{X,Y}$. We say that a sequence $\{u_n\}\subset X$ converges to $u\in X$ with respect to the topology $\sigma(X,Y)$, written $u_n\to u$ $(\sigma(X,Y))$ in X, if $\langle u_n, v\rangle \to \langle u, v\rangle$ for all $v\in Y$. For example, if $X = L_M(\Omega)$ and $Y = L_{\overline{M}}(\Omega)$, then the pairing is defined by $\langle u, v\rangle = \int_\Omega u(x)v(x)\,dx$ for all $u\in X$, $v\in Y$.

2.3. We now turn to the Orlicz–Sobolev space $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$], which is the space of all functions u such that u and its distributional derivatives of order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$, and $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}\subset L_M(\Omega)$ converges to $u\in L_M(\Omega)$ in the modular sense, and write $u_n\to u$ (mod) in $L_M(\Omega)$, if for some $\lambda>0$,

$$\int_\Omega M(|u_n(x)-u(x)|/\lambda)\,dx \to 0 \quad \text{as } n\to\infty.$$

If M satisfies the Δ_2-condition (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence (see [15]).

We say that a sequence $\{u_n\}\subset W^1L_M(\Omega)$ converges to $u\in W^1L_M(\Omega)$ in the modular sense, and write $u_n\to u$ (mod) in $W^1L_M(\Omega)$, if there exists $\lambda>0$ such that

$$\int_\Omega M(|D^\alpha u_n(x) - D^\alpha u(x)|/\lambda)\,dx \to 0 \quad \text{for all } |\alpha|\le 1 \text{ as } n\to\infty.$$

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm.

We recall some lemmas introduced in [7] which will be used later.

LEMMA 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Let M be an N-function and let $u \in W^1 L_M(\Omega)$ (resp. $u \in W^1 E_M(\Omega)$). Then $F(u) \in W^1 L_M(\Omega)$ (resp. $F(u) \in W^1 E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \dfrac{\partial}{\partial x_i} u & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

LEMMA 2.2. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian with $F(0) = 0$. Suppose that the set of discontinuity points of F' is finite. Let M be an N-function. Then the mapping $T_F : W^1 L_M(\Omega) \to W^1 L_M(\Omega)$ defined by $T_F(u) = F(u)$ is sequentially continuous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

We now give the following lemma which concerns operators of the Nemytskiĭ type in Orlicz spaces (see [7]).

LEMMA 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. Let M, P and Q be N-functions such that $Q \ll P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$,*

$$|f(x,s)| \le c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c \in E_Q(\Omega)$. Then the Nemytskiĭ operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), 1/k_2) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2\}$ into $E_Q(\Omega)$.

We introduce the function spaces we will need later.

For an N-function M, $\mathcal{T}_0^{1,M}(\Omega)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for all $k > 0$ the truncated functions $T_k(u)$ are in $W_0^1 L_M(\Omega)$, where $T_k(s) = \max(-k, \min(k, s))$.

We give the following lemma which is a generalization of [4, Lemma 2.1] to Orlicz spaces and whose proof is a slight modification of the one in the L^p case.

LEMMA 2.4. *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v \chi_{\{|u|<k\}}, \qquad \text{almost everywhere in } \Omega, \text{ for every } k > 0.$$

We will call v the gradient of u, and write $v = \nabla u$.

LEMMA 2.5. *Let $\lambda \in \mathbb{R}$ and let $u, v \in \mathcal{T}_0^{1,M}(\Omega)$ be finite almost everywhere. Then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \qquad \text{a.e. in } \Omega,$$

where ∇ is the gradient introduced in Lemma 2.4.

The proof of this lemma is similar to the proof of [10, Lemma 2.12] in the L^p case.

Below, we will use the following technical lemmas.

Lemma 2.6 ([7]). *Let $f_n, f, \gamma \in L^1(\Omega)$ be such that*

(i) $f_n \geq \gamma$ *a.e. in* Ω,
(ii) $f_n \to f$ *a.e. in* Ω,
(iii) $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$.

Then $f_n \to f$ strongly in $L^1(\Omega)$.

Lemma 2.7 ([5]). *Let Ω be an open bounded subset of $\mathbb{R}^N$ with the segment property. If $u \in W_0^1 L_M(\Omega)$, then*

$$\int_\Omega \operatorname{div} u \, dx = 0.$$

3. Statement of main results

3.1. *Basic assumptions.* Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property and M be an N-function.

Given a measurable obstacle function $\psi : \Omega \to \overline{\mathbb{R}}$, we consider the set

$$K_\psi = \{u \in W_0^1 L_M(\Omega) : u \geq \psi \text{ a.e. in } \Omega\}.$$

This convex set is sequentially $\sigma(\prod L_M, \prod E_{\overline{M}})$ closed in $W_0^1 L_M(\Omega)$ (see [14]). We now state our hypotheses on the differential operator A defined by

$$Au = -\operatorname{div}(a(x,u,\nabla u)).$$

(A_1) $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.

(A_2) There exists an N-function P with $P \ll M$, a function $c \in E_{\overline{M}}(\Omega)$, and positive constants k_1, k_2, k_3, k_4 such that

$$|a(x,s,\zeta)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|)$$

for a.e. x in Ω and for all $s \in \mathbb{R}$, $\zeta \in \mathbb{R}^N$.

(A_3) For a.e. x in Ω, $s \in \mathbb{R}$ and ζ, ζ' in $\mathbb{R}^N$ with $\zeta' \neq \zeta$,

$$[a(x,s,\zeta) - a(x,s,\zeta')](\zeta - \zeta') > 0.$$

(A_4) For a.e. x in Ω and all $\zeta \in \mathbb{R}^N$,

$$a(x,s,\zeta)\zeta \geq \alpha M(|\zeta|/\nu).$$

(A_5) For each $v \in K_\psi \cap L^\infty(\Omega)$ there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that

$$v_j \to v \quad \text{for the modular convergence.}$$

Finally, we suppose that

$$\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega), \tag{3.1}$$

$$\phi \in C^0(\mathbb{R}, \mathbb{R}^N), \tag{3.2}$$

$$K_\psi \cap L^\infty(\Omega) \neq \emptyset. \tag{3.3}$$

REMARK 3.1. Condition (A_5) holds if one of the following conditions is satisfied:

(a) There exists $\overline{\psi} \in K_\psi$ such that $\psi - \overline{\psi}$ is continuous in Ω (see [14, Proposition 9]).
(b) $\psi \in W_0^1 E_M(\Omega)$ (see [14, Proposition 10]).
(c) The N-function M satisfies the Δ_2-condition near infinity.
(d) $\psi = -\infty$ (i.e., $K_\psi = W_0^1 L_M(\Omega)$).

3.2. *Principal result.* Since $\mu \in L^1(\Omega) + W^{-1}E_{\overline{M}}(\Omega)$, it can be written as follows:

$$\mu = f - \operatorname{div} F \quad \text{ with } f \in L^1(\Omega),\ F \in (E_{\overline{M}}(\Omega))^N.$$

We consider the following unilateral problem:

$$\begin{cases} u \in \mathcal{T}_0^{1,M}(\Omega), \quad u \geq \psi \text{ a.e. in } \Omega, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \qquad \leq \int_\Omega f T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx, \\ \qquad\qquad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0. \end{cases} \tag{3.4}$$

We prove the following existence result.

THEOREM 3.1. *Under the assumptions* (A_1)–(A_5) *and* (3.1)–(3.3), *there exists at least one solution of the problem* (3.4).

REMARK 3.2. In the previous result, we cannot replace $K_\psi \cap L^\infty(\Omega)$ by just K_ψ, since in general the integral $\int_\Omega \phi(u)\nabla T_k(u-v)\,dx$ may not have a meaning.

REMARK 3.3. If we take $M(t) = |t|^p$ in the previous statement, we obtain an existence result in the classical Sobolev spaces (which seems to be new).

REMARK 3.4. The statement of Theorem 3.1 holds when $\mu \in L^1(\Omega) + W^{-1}L_{\overline{M}}(\Omega)$.

It suffices to approximate $\mu = f - \operatorname{div} F$ with $F = (F_1, \ldots, F_N) \in (L_{\overline{M}}(\Omega))^N$ by $\mu_n = f_n - \operatorname{div} F_n$ where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$ and $F_n = (T_n(F_1), \ldots, T_n(F_N))$.

We write $\varepsilon(n,i,j)$ for any quantity such that

$$\lim_{j\to\infty}\lim_{i\to\infty}\lim_{n\to\infty} \varepsilon(n,i,j) = 0.$$

The notations $\varepsilon(n,j)$ etc. are defined similarly. Finally, we denote (for example) by $\varepsilon_j(n,i)$ a quantity that depends on n,i,j and is such that

$$\lim_{i\to\infty}\lim_{n\to\infty}\varepsilon_j(n,i)=0$$

for any fixed value of j.

4. Proof of principal result. Without loss the generality we take $\nu=1$ in condition (A_4). We fix a function $v_0\in K_\psi\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ (such a function exists by (A_5) and (3.3)).

4.1. *Approximate problem.* We consider the sequence of approximate problems

$$\begin{cases} u_n\in K_\psi, \\ \langle Au_n,u_n-v\rangle+\int_\Omega \phi(T_n(u_n))\nabla(u_n-v)\,dx \\ \qquad\qquad \le \int_\Omega f_n(u_n-v)\,dx+\int_\Omega F\nabla(u_n-v)\,dx \quad \forall v\in K_\psi. \end{cases} \tag{4.1}$$

where f_n is a regular function such that f_n strongly converges to f in $L^1(\Omega)$. This approximate problem has a solution by the classical result of [14].

4.2. *Some intermediate results.* Let us prove the following lemma which is needed below:

LEMMA 4.1. *Assume that (A_1)–(A_4) are satisfied, and let $(z_n)_n$ be a sequence in $W_0^1L_M(\Omega)$ such that*

(a) $z_n\rightharpoonup z$ *in* $W_0^1L_M(\Omega)$ *for* $\sigma(\prod L_M(\Omega),\prod E_{\overline{M}}(\Omega))$;
(b) $(a(x,z_n,\nabla z_n))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$;
(c) $\int_\Omega[a(x,z_n,\nabla z_n)-a(x,z_n,\nabla z\chi_s)][\nabla z_n-\nabla z\chi_s]\,dx\to 0$ *as* $n,\ s\to\infty$ (*where χ_s is the characteristic function of* $\Omega_s=\{x\in\Omega:|\nabla z|\le s\}$).

Then

$$M(|\nabla z_n|)\to M(|\nabla z|)\quad \text{in } L^1(\Omega).$$

REMARK 4.1. Condition (b) is not necessary if M satisfies the Δ_2-condition.

Indeed, (a) implies that $(z_n)_n$ is bounded in $W_0^1L_M(\Omega)$, hence there exist two positive constants λ, C such that

$$\int_\Omega M(\lambda|\nabla z_n|)\,dx\le C. \tag{4.2}$$

On the other hand, let Q be an N-function such that $M\ll Q$ and the continuous embedding $W_0^1L_M(\Omega)\subset E_Q(\Omega)$ holds (see [13]). Let $\varepsilon>0$. Then there exists $C_\varepsilon>0$, as in [7], such that

$$|a(x,s,\zeta)|\le c(x)+C_\varepsilon+k_1\overline{M}^{-1}Q(\varepsilon|s|)+k_3\overline{M}^{-1}M(\varepsilon|\zeta|) \tag{4.3}$$

for a.e. $x \in \Omega$ and all $(s,\zeta) \in \mathbb{R} \times \mathbb{R}^N$. From (4.2) and (4.3) we deduce that $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$.

Proof of Lemma 4.1. Fix $r > 0$ and let $s > r$. We have

$$\begin{aligned}0 &\le \int_{\Omega_r} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx \\ &\le \int_{\Omega_s} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx \\ &= \int_{\Omega_s} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \\ &\le \int_{\Omega} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx.\end{aligned}$$

Together with (c) this implies

$$\lim_{n\to\infty} \int_{\Omega_r} [a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z)][\nabla z_n - \nabla z]\,dx = 0.$$

So, as in [13],

$$\nabla z_n \to \nabla z \quad \text{a.e. in } \Omega. \tag{4.4}$$

On the one hand, we have

$$\begin{aligned}\int_{\Omega} a(x,z_n,\nabla z_n)\nabla z_n\,dx = \int_{\Omega}&[a(x,z_n,\nabla z_n) - a(x,z_n,\nabla z\chi_s)] \\ &\times [\nabla z_n - \nabla z\chi_s]\,dx \\ &+ \int_{\Omega} a(x,z_n,\nabla z\chi_s)(\nabla z_n - \nabla z\chi_s)\,dx \\ &+ \int_{\Omega} a(x,z_n,\nabla z_n)\nabla z\chi_s\,dx.\end{aligned} \tag{4.5}$$

Since $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, from (4.4), we obtain

$$a(x,z_n,\nabla z_n) \rightharpoonup a(x,z,\nabla z) \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\textstyle\prod L_{\overline{M}}, \prod E_M).$$

Consequently,

$$\int_{\Omega} a(x,z_n,\nabla z_n)\nabla z\chi_s\,dx \to \int_{\Omega} a(x,z,\nabla z)\nabla z\chi_s\,dx \tag{4.6}$$

as $n \to \infty$. Letting also $s \to \infty$, we get

$$\int_{\Omega} a(x,z,\nabla z)\nabla z\chi_s\,dx \to \int_{\Omega} a(x,z,\nabla z)\nabla z\,dx. \tag{4.7}$$

On the other hand, it is easy to see that the second term on the right hand side of (4.5) tends to 0 as $n, s \to \infty$.

Moreover, from (c), (4.6) and (4.7) we have

$$\lim_{n\to\infty}\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n\,dx=\int_\Omega a(x,z,\nabla z)\nabla z\,dx.$$

Finally, using (A_4), by Lemma 2.6 and Vitali's theorem, one obtains the assertion.

PROPOSITION 4.1. *Assume that* (A_1)–(A_5) *and* (3.1)–(3.3) *hold and let* u_n *be a solution of the approximate problem* (4.1). *Then for all* $k>0$, *there exists a constant* $c(k)$ (*which does not depend on* n) *such that*

$$\int_\Omega M(|\nabla T_k(u_n)|)\le c(k).$$

Proof. Let $k>0$. Taking $u_n-T_k(u_n-v_0)$ as a test function in (4.1), we obtain, for n large enough,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx+\int_\Omega \phi(u_n)\nabla T_k(u_n-v_0)\,dx \le\int_\Omega f_nT_k(u_n-v_0)\,dx+\int_\Omega F\nabla T_k(u_n-v_0)\,dx.$$

Since $\nabla T_k(u_n-v_0)$ is identically zero on the set where $|u_n-v_0|>k$, we can write

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx &\le \int_{\{|u_n-v_0|\le k\}}|\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla u_n|\,dx\\ &\quad+\int_{\{|u_n-v_0|\le k\}}|\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla v_0|\,dx\\ &\quad+\int_\Omega f_nT_k(u_n-v_0)\,dx+\int_\Omega F\nabla T_k(u_n-v_0)\,dx.\end{aligned}$$

Now observe that (for $0<c<1$)

$$\begin{aligned}\int_{\{|u_n-v_0|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx &\le c\int_{\{|u_n-v_0|\le k\}} a(x,u_n,\nabla u_n)\,\frac{\nabla v_0}{c}\,dx\\ &\quad+\int_{\{|u_n-v_0|\le k\}}|\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla u_n|\,dx\\ &\quad+\int_{\{|u_n-v_0|\le k\}}|\phi(T_{k+\|v_0\|_\infty}(u_n))|\,|\nabla v_0|\,dx+\int_\Omega f_nT_k(u_n-v_0)\,dx\\ &\quad+\int_\Omega F\nabla T_k(u_n-v_0)\,dx.\end{aligned}\tag{4.8}$$

By using (A_3), we get

$$c \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n) \frac{\nabla v_0}{c} \, dx$$

$$\le c\left\{ \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx - \int_{\{|u_n - v_0| \le k\}} a\left(x, u_n, \frac{\nabla v_0}{c}\right) \left(\nabla u_n - \frac{\nabla v_0}{c}\right) dx \right\},$$

which yields, thanks to (4.8),

$$\begin{aligned}(1-c) &\int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \\ \le &\int_{\{|u_n - v_0| \le k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))| \, |\nabla u_n| \, dx \\ &+ \int_{\{|u_n - v_0| \le k\}} |\phi(T_{k+\|v_0\|_\infty}(u_n))| \, |\nabla v_0| \, dx + \int_\Omega f_n T_k(u_n - v_0) \, dx \\ &+ \int_\Omega F \nabla T_k(u_n - v_0) \, dx - c \int_{\{|u_n - v_0| \le k\}} a\left(x, u_n, \frac{\nabla v_0}{c}\right) \left(\nabla u_n - \frac{\nabla v_0}{c}\right) dx.\end{aligned}$$

Since $\frac{\nabla v_0}{c} \in (E_M(\Omega))^N$, using (A_2) and the Young inequality, we have

$$(1-c) \int_{\{|u_n - v_0| \le k\}} a(x, u_n, \nabla u_n) \nabla u_n \, dx \le \frac{\alpha(1-c)}{2} \int_{\{|u_n - v_0| \le k\}} M(|\nabla u_n|) \, dx + c_3(k) \tag{4.9}$$

where $c_3(k)$ is a positive constant which depends only on k.

Using also (A_4) we obtain

$$\frac{\alpha(1-c)}{2} \int_{\{|u_n - v_0| \le k\}} M(|\nabla u_n|) \, dx \le c_3(k).$$

Moreover, from $\{|u_n| \le k\} \subset \{|u_n - v_0| \le k + \|v_0\|_\infty\}$, we conclude that

$$\int_\Omega M(|\nabla T_k(u_n)|) \, dx \le c_4(k). \tag{4.10}$$

PROPOSITION 4.2. *Assume that* (A_1)–(A_5) *and* (3.1)–(3.3) *hold*, *and let* u_n *be a solution of the approximate problem* (4.1). *Then there exists a measurable function* u *such that for all* $k > 0$ *we have* (*for a subsequence still denoted by* u_n),

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $T_k(u_n) \rightharpoonup T_k(u)$ *weakly in* $W_0^1 L_M(\Omega)$ *for* $\sigma(\prod L_M, \prod E_{\overline{M}})$,
$T_k(u_n) \to T_k(u)$ *strongly in* $E_M(\Omega)$ *and a.e. in* Ω.

Before proving this proposition, we begin with the following estimate:

Lemma 4.2. *If* u_n *is a solution of* (4.1), *then for* $k > h > \|v_0\|_\infty$, *we have*

$$\int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx \le kC,$$

where C *is a constant that does not depend of* n, k *and* h.

Proof. By Proposition 4.1, there exists some $v_k \in W_0^1 L_M(\Omega)$ such that

$$\begin{aligned} &T_k(u_n) \rightharpoonup v_k \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\textstyle\prod L_M, \prod E_{\overline{M}}),\\ &T_k(u_n) \to v_k \quad \text{strongly in } E_M(\Omega) \text{ and a.e. in } \Omega. \end{aligned} \tag{4.11}$$

On the other hand, let $k > h \ge \|v_0\|_\infty$. By using $v = u_n - T_k(u_n - T_h(u_n))$ as a test function in (4.1) we obtain

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\,dx + \int_\Omega \phi(T_n(u_n))\nabla T_k(u_n - T_h(u_n))\,dx \\ \le \int_\Omega f_n T_k(u_n - T_h(u_n))\,dx + \int_\Omega F\nabla T_k(u_n - T_h(u_n))\,dx. \end{aligned}$$

The second term on the left hand side vanishes for n large enough. Indeed, by Lemma 2.7,

$$\begin{aligned} \int_\Omega \phi(T_n(u_n))\nabla T_k(u_n - T_h(u_n))\,dx &= \int_\Omega \phi(u_n)\nabla T_k(u_n - T_h(u_n))\,dx \\ &= \int_\Omega \operatorname{div}\Big[\int_0^{u_n} \phi(s)\chi_{\{h\le|s|\le k+h\}}\,ds\Big]\,dx = 0 \end{aligned}$$

(since $\int_0^{u_n} \phi(s)\chi_{\{h\le|s|\le k+h\}}\,ds$ lies in $W_0^1 L_M(\Omega)$). Thus,

$$\begin{aligned} \int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - T_h(u_n))\,dx \\ \le \int_\Omega f_n T_k(u_n - T_h(u_n))\,dx + \int_\Omega F\nabla T_k(u_n - T_h(u_n))\,dx, \end{aligned}$$

which yields the conclusion by using (A_4) and Young's inequality.

Proof of Proposition 4.2. (i) We prove that u_n converges to some function u in measure (and hence a.e. by passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure.

Let $k > h > \|v_0\|_\infty$ be large enough. Thanks to Lemma 5.7 of [13], there exist two positive constants C_7 and C_8 independent of k and h such that

$$\int_\Omega M(C_7|T_k(u_n - T_h(u_n))|)\,dx \le C_8 \int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx.$$

By Lemma 4.2 this yields

$$\begin{aligned} M(C_7 k)\,\mathrm{meas}\{|u_n - T_h(u_n)| > k\} &= \int_{\{|u_n - T_h(u_n)|>k\}} M(C_7|T_k(u_n - T_h(u_n))|)\,dx \\ &\le C_8 \int_\Omega M(|\nabla T_k(u_n - T_h(u_n))|)\,dx \le kC_9. \end{aligned}$$

Consequently,

$$\mathrm{meas}(\{|u_n - T_h(u_n)| > k\}) \le \frac{kC_9}{M(kC_7)}$$

for all n and all $k > h > \|v_0\|_\infty$. Hence,

$$\mathrm{meas}\{|u_n| > k\} \le \mathrm{meas}\{|u_n - T_h(u_n)| > k-h\} \le \frac{(k-h)C_9}{M((k-h)C_7)} \quad \text{for all } n.$$

Therefore, since $t/M(t) \to 0$ as $t \to \infty$, we obtain

$$\mathrm{meas}\{|u_n| > k\} \to 0 \quad \text{uniformly in } n \text{ as } k \to \infty. \tag{4.12}$$

Now, for $\lambda > 0$, we have

$$\begin{aligned} \mathrm{meas}(\{|u_n - u_m| > \lambda\}) \le\ &\mathrm{meas}(\{|u_n| > k\}) + \mathrm{meas}(\{|u_m| > k\}) \\ &+ \mathrm{meas}(\{|T_k(u_n) - T_k(u_m)| > \lambda\}). \end{aligned} \tag{4.13}$$

From (4.11), we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon > 0$. By (4.12), (4.13) and the fact that $T_k(u_n)$ is a Cauchy sequence in measure, there exists some $k(\varepsilon) > 0$ such that $\mathrm{meas}(\{|u_n - u_m| > \lambda\}) < \varepsilon$ for all $n, m \ge n_0(k(\varepsilon), \lambda)$. This proves that $(u_n)_n$ is a Cauchy sequence in measure in Ω, thus it converges almost everywhere to some measurable function u.

(ii) It suffices to combine assertion (i) and (4.11).

Proposition 4.3. *Assume that* (A_1)–(A_5) *and* (3.1)–(3.3) *hold and let* u_n *be a solution of the approximate problem* (4.1). *Then for all* $k > 0$,

(i) $(a(x, T_k(u_n), \nabla T_k(u_n)))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$,
(ii) $\nabla u_n \to \nabla u$ *a.e. in* Ω.

Proof. (i) Let $w \in (E_M(\Omega))^N$. By condition (A_3) we have

$$(a(x, u_n, \nabla u_n) - a(x, u_n, w))(\nabla u_n - w) \ge 0.$$

Consequently,

$$\int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)w\,dx \le \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int_{\{|u_n|\le k\}} a(x,u_n,w)(w-\nabla u_n)\,dx. \tag{4.14}$$

Combining (4.9) and (4.10) and using the fact that $\{|u_n|\le k\}\subset\{|u_n-v_0|\le k+\|v_0\|_\infty\}$, we get

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \le C_{10},$$

where C_{10} is a positive constant.

On the other hand, by (A_2) we have

$$|a(x,T_k(u_n),w)| \le c(x)+k_1\overline{P}^{-1}M(k_2|T_k(u)|)+k_3\overline{M}^{-1}M(k_4|w|).$$

Therefore,

$$\int_\Omega \overline{M}\biggl(\frac{a(x,T_k(u_n),w)}{\lambda}\biggr)dx \le \int_\Omega \overline{M}\biggl(\frac{c(x)}{\lambda}\biggr)+\int_\Omega \frac{k_3}{\lambda}M(k_4|w|)+C_{11}\le C_{12}$$

when $\lambda>0$ is large enough. Hence $\{a(x,T_k(u_n),w)\}$ is bounded in $(L_{\overline{M}}(\Omega))^N$. This implies that the second term on the right in (4.14) is also bounded. By the theorem of Banach–Steinhaus, the sequence $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ remains bounded in $(L_{\overline{M}}(\Omega))^N$.

(ii) Let $k>\|v_0\|_\infty$. By (A_5) there exists a sequence $v_j\in K_\psi\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ such that

$$v_j\to T_k(u)\quad(\mathrm{mod})\text{ in } W_0^1L_M(\Omega). \tag{4.15}$$

Fix r and let $s>r$. Let $\Omega_r=\{x\in\Omega:|\nabla T_k(u(x))|\le r\}$ and denote by χ_r the characteristic function of Ω_r. Consider the expression

$$I_{n,r}=\int_{\Omega_r}\{[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\times[\nabla T_k(u_n)-\nabla T_k(u)]\}^\theta\,dx$$

where $0<\theta<1$. Let A_n be the expression in braces above. Then for any $0<\eta<1$,

$$I_{n,r}=\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}}A_n^\theta\,dx+\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|>\eta\}}A_n^\theta\,dx.$$

Since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, while $\nabla T_k(u_n)$ is bounded in $(L_M(\Omega))^N$, by applying Hölder's inequality, we obtain

$$I_{n,r}\le c_1\Bigl(\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}}A_n\,dx\Bigr)^\theta + c_2\,\mathrm{meas}\{x:|T_k(u_n)-T_k(v_j)|>\eta\}^{1-\theta}. \tag{4.16}$$

Now observe that

$$(4.17)\quad \int\limits_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} A_n\,dx$$
$$\le \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx$$
$$= \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(v_j))\,dx$$
$$- \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u)\chi_s)(\nabla T_k(u_n) - \nabla T_k(u)\chi_s)\,dx$$
$$+ \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\,dx$$
$$- \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s\,dx,$$

and since $(a(x,T_k(u_n),\nabla T_k(u_n)))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ by Proposition 4.3(i), there exists $\varrho_k \in (L_{\overline{M}}(\Omega))^N$ such that

$$(4.18)\quad a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \varrho_k \quad \text{weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}})$$

as $n\to\infty$. Letting n tend to infinity, we obtain

$$\int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s\,dx = \int\limits_{\{|T_k(u)-T_k(v_j)|\le\eta\}} \varrho_k \nabla T_k(u)\chi_s\,dx + \varepsilon(n),$$
$$\int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\,dx = \int\limits_{\{|T_k(u)-T_k(v_j)|\le\eta\}} \varrho_k \nabla T_k(v_j)\,dx + \varepsilon(n),$$

and hence by letting $j\to\infty$ and using (4.15),

$$(4.19)\quad \int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)\chi_s\,dx = \int\limits_{\Omega} \varrho_k \nabla T_k(u)\chi_s\,dx + \varepsilon(n,j),$$

and

$$\int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\,dx = \int\limits_\Omega \varrho_k\nabla T_k(u)\,dx+\varepsilon(n,j). \tag{4.20}$$

Starting with the second term on the right hand side of (4.17), we have, by letting $n\to\infty$,

$$\int\limits_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx=\varepsilon(n) \tag{4.21}$$

since

$$a(x,T_k(u_n),\nabla T_k(u)\chi_s)\chi_{\{|T_k(u_n)-T_k(v_j)|\le\eta\}} \to a(x,T_k(u),\nabla T_k(u)\chi_s)\chi_{\{|T_k(u)-T_k(v_j)|\le\eta\}}$$

strongly in $(E_{\overline{M}}(\Omega))^N$ by using Lemma 2.3 while $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u_n)$ weakly in $(L_M(\Omega))^N$ by (4.11) and Proposition 4.3(ii).

We now study the first term on the right hand side of (4.17). By using the test function $u_n-T_\eta(u_n-T_k(v_j))$ in (4.1), we get

$$\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx+\int\limits_\Omega \phi(u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx \le \int\limits_\Omega f_nT_\eta(u_n-T_k(v_j))\,dx+\int\limits_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx. \tag{4.22}$$

Splitting the first integral on the left hand side into the regions where $|u_n|\le k$ and $|u_n|>k$, we can write

$$\begin{aligned}\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx &= \int\limits_{\{|u_n|\le k\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx\\ &\quad+\int\limits_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx,\end{aligned}$$

which implies, by using (A_4),

$$\begin{aligned}\int\limits_\Omega a(x,u_n,\nabla u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx &\ge \int\limits_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx\\ &\quad-\int\limits_{\{|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|\,|\nabla v_j|\,dx.\end{aligned} \tag{4.23}$$

Combining (4.22) and (4.23), we deduce

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx \\ +\int_\Omega \phi(u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx \\ \le \int_{\{|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|\,|\nabla v_j|\,dx \\ +\int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx + c_1\eta.\end{aligned}$$

Using the boundedness of $\{|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|\}_n$ in $L_{\overline{M}}(\Omega)$ and reasoning as above, it is easy to see that

$$\int_{\{|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|\,|\nabla v_j|\,dx = \int_{\{|u|>k\}} h_k|\nabla v_j|\,dx,$$

where h_k is some function in $L_{\overline{M}}(\Omega)$ such that

$$|a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))| \rightharpoonup h_k \quad \text{for } \sigma(L_{\overline{M}}(\Omega),E_M(\Omega)) \text{ as } n\to\infty.$$

Moreover, by (4.15) and the fact that $h_k\chi_{\{|u|>k\}}\in L_M(\Omega)$, we get

$$\int_{\{|u_n|>k\}} |a(x,T_{k+1}(u_n),\nabla T_{k+1}(u_n))|\,|\nabla v_j|\,dx = \varepsilon(n,j).$$

Similarly, we have

$$\int_\Omega \phi(u_n)\nabla T_\eta(u_n-T_k(v_j))\,dx = \int_\Omega \phi(u)\nabla T_\eta(u-T_k(u))\,dx + \varepsilon(n,j) = \varepsilon(n,j),$$

$$\begin{aligned}\int_\Omega F\nabla T_\eta(u_n-T_k(v_j))\,dx &= \int_{\{|u-T_k(u)|\le\eta,\,|u|>k\}} F\nabla T_\eta(u-T_k(u))\,dx + \varepsilon(n,j) \\ &\le c_3\|F\chi_{\{|u-T_k(u)|\le\eta,\,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M + \varepsilon(n,j).\end{aligned}$$

Consequently, we deduce

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_\eta(T_k(u_n)-T_k(v_j))\,dx \\ \le c_3\|F\chi_{\{|u-T_k(u)|\le\eta,\,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M + \varepsilon(n,j).\end{aligned}\tag{4.24}$$

Hence, from (4.17), (4.19), (4.20), (4.21) and (4.24), we get

$$\begin{aligned}\int_{\Omega_r\cap\{|T_k(u_n)-T_k(v_j)|\le\eta\}} A_n\,dx \\ \le c_3\|F\chi_{\{|u-T_k(u)|\le\eta,\,|u|>k\}}\|_{\overline{M}}\|\nabla T_1(u-T_k(u))\|_M \\ +\int_{\Omega\setminus\Omega_s} \varrho_k\nabla T_k(u)\,dx + C_1\eta + \varepsilon(n,j).\end{aligned}\tag{4.25}$$

Finally, in virtue of (4.16) and (4.2), we deduce

$$I_{n,r} \le C_4 \operatorname{meas}\{x : |T_k(u_n) - T_k(v_j)| > \eta\}^{1-\theta} + \Big\{ C_2 \int_{\Omega \setminus \Omega_s} \varrho_k \nabla T_k(u)\, dx + c_3 \|F\chi_{\{|u-T_k(u)|\le \eta,\, |u|>k\}}\|_{\overline{M}} \|\nabla T_1(u - T_k(u))\|_M + C_1\eta + \varepsilon(n,j)\Big\}^{\theta}.$$

Consequently,

$$\limsup_{n\to\infty} I_{n,r} \le c_4 \operatorname{meas}\{x : |T_k(u) - T_k(v_j)| > \eta\}^{1-\theta} + \Big\{ C_2 \int_{\Omega \setminus \Omega_s} h_k \nabla T_k(u)\, dx + c_3 \|F\chi_{\{|u-T_k(u)|\le \eta,\, |u|>k\}}\|_{\overline{M}} \|\nabla T_1(u - T_k(u))\|_M + C_1\eta + \varepsilon(n,j)\Big\}^{\theta},$$

in which we let successively $j \to \infty$, $s \to \infty$ and $\eta \to 0$ to obtain

$$\limsup_{n\to\infty} I_{n,r} = 0.$$

As in [13], this implies that there exists a subsequence also denoted by u_n such that $\nabla u_n \to \nabla u$ a.e. in Ω.

PROPOSITION 4.4. *Assume that* (A_1)–(A_5) *and* (3.1)–(3.3) *hold and let* u_n *be a solution of the approximate problem* (4.1). *Then for all* $k > 0$,

$$M(|\nabla T_k(u_n)|) \to M(|\nabla T_k(u)|) \quad \text{in } L^1(\Omega).$$

Proof. We fix $k > \|v_0\|_\infty$. Then by (A_5) there exists a sequence $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$. We define

$$\begin{aligned} w_{n,j}^h &= T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(v_j)),\\ w_j^h &= T_{2k}(u - T_h(u) + T_k(u) - T_k(v_j)),\\ w^h &= T_{2k}(u - T_h(u)), \end{aligned}$$

where $h > 2k$.

We choose $v = u_n - w_{n,j}^h$ as a test function in (4.1) to obtain

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla w_{n,j}^h\, dx + \int_\Omega \phi(u_n) \nabla w_{n,j}^h\, dx \le \int_\Omega f_n w_{n,j}^h\, dx + \int_\Omega F \nabla w_{n,j}^h\, dx. \tag{4.26}$$

By the strong convergence of f_n and since $w_{n,j}^h$ converges to w_j^h in the weak* topology of $L^\infty(\Omega)$ as $n \to \infty$, we have

$$\int_\Omega f_n w_{n,j}^h\, dx = \int_\Omega f w_j^h\, dx + \varepsilon(n) = \int_\Omega f w^h\, dx + \varepsilon(n,j).$$

The last passage is due to the fact that w_j^h converges to zero in the weak* topology of $L^\infty(\Omega)$ as $j \to \infty$.

Finally, letting $h \to \infty$, Lebesgue's theorem yields $\int_\Omega f w_{n,j}^h \, dx \to 0$, so

$$\int_\Omega f_n w_{n,j}^h \, dx = \varepsilon(n, j, h).$$

We now study the first integral on the left hand side of (4.26):

$$(4.27) \quad \int_\Omega a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \, dx = \int_{\{|u_n| \le k\}} a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \, dx + \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \, dx.$$

Set $m = 4k + h$. By using (A_3) and the fact that $\nabla w_{n,j}^h = 0$ if $|u_n(x)| > m$, we get

$$(4.28) \quad \int_{\{|u_n| > k\}} a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \, dx \ge - \int_{\{|u_n| > k\}} |a(x, T_m(u_n), \nabla T_m(u_n))| \, |\nabla v_j| \, dx.$$

Since $(|a(x, T_m(u_n), \nabla T_m(u_n))|)_n$ is a bounded sequence in $L_{\overline{M}}(\Omega)$, for some subsequence still denoted u_n and for some $l_m \in L_{\overline{M}}(\Omega)$ we have

$$|a(x, T_m(u_n), \nabla T_m(u_n))| \rightharpoonup l_m \quad \text{in } L_{\overline{M}}(\Omega) \text{ for } \sigma(L_{\overline{M}}(\Omega), E_M(\Omega))$$

as $n \to \infty$, and since $\nabla v_j \chi_{\{|u_n| > k\}} \to \nabla v_j \chi_{\{|u| > k\}}$ strongly in $E_M(\Omega)$ as $n \to \infty$, we get

$$- \int_{\{|u_n| > k\}} |a(x, T_m(u_n), \nabla T_m(u_n))| \, |\nabla v_j| \, dx = - \int_{\{|u| > k\}} l_m |\nabla v_j| \, dx + \varepsilon(n).$$

We let $j \to \infty$ to obtain

$$(4.29) \quad - \int_{\{|u| > k\}} l_m |\nabla v_j| \, dx = - \int_{\{|u| > k\}} l_m |\nabla T_k(u)| \, dx + \varepsilon(n, j) = \varepsilon(n, j).$$

Combining (4.27)–(4.29), we deduce

$$\int_\Omega a(x, u_n, \nabla u_n) \nabla w_{n,j}^h \, dx \ge \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n)) [\nabla T_k(u_n) - \nabla T_k(v_j)] \, dx + \varepsilon(n, h) + \varepsilon(n, j) + \varepsilon_h(n, j),$$

which implies that

$$
\begin{aligned}
(4.30)\qquad &\int_\Omega a(x,u_n,\nabla u_n)\nabla w^h_{n,j}\,dx \\
&\quad\ge \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)] \\
&\qquad\times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s]\,dx \\
&\qquad+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s]\,dx \\
&\qquad- \int_{\Omega\setminus\Omega^j_s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\,dx \\
&\qquad+ \varepsilon(n,h) + \varepsilon(n,j) + \varepsilon_h(n,j)
\end{aligned}
$$

where χ^j_s is the characteristic function of $\Omega^j_s = \{x\in\Omega : |\nabla T_k(v_j)| \le s\}$.

By (4.18) and the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s} \in (E_M(\Omega))^N$, the third term on the right hand side of (4.30) tends to $\int_\Omega \varrho_k \nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s}\,dx$ as $n\to\infty$. Letting $j\to\infty$ we obtain, by (4.15),

$$
(4.31)\qquad \int_\Omega \varrho_k \nabla T_k(v_j)\chi_{\Omega\setminus\Omega^j_s}\,dx = \int_{\Omega\setminus\Omega_s} \varrho_k \nabla T_k(u)\,dx + \varepsilon(n,j).
$$

The second term on the right hand side of (4.30) goes to zero as first $n\to\infty$ and then $j\to\infty$. Indeed, since $a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s) \to a(x,T_k(u),\nabla T_k(v_j)\chi^j_s)$ strongly in $(E_{\overline{M}}(\Omega))^N$ by using (A_2) and the Lebesgue theorem while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $(L_M(\Omega))^N$, we have

$$
\begin{aligned}
&\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s]\,dx \\
&\qquad= \int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi^j_s)[\nabla T_k(u) - \nabla T_k(v_j)\chi^j_s]\,dx + \varepsilon(n).
\end{aligned}
$$

Letting $j\to\infty$, one has

$$
\begin{aligned}
&\int_\Omega a(x,T_k(u),\nabla T_k(v_j)\chi^j_s)[\nabla T_k(u) - \nabla T_k(v_j)\chi^j_s]\,dx \\
&\qquad= \int_\Omega a(x,T_k(u),\nabla T_k(u)\chi_s)[\nabla T_k(u) - \nabla T_k(u)\chi_s]\,dx + \varepsilon(j).
\end{aligned}
$$

Finally,

$$
(4.32)\qquad \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi^j_s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi^j_s]\,dx = \varepsilon_m(n,j).
$$

Combining (4.30)–(4.32), we deduce

$$\begin{aligned}(4.33)\qquad &\int_\Omega a(x,u_n,\nabla u_n)\nabla w_{n,j}^h\,dx\\ &\quad\ge \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx-\int_{\Omega\setminus\Omega_s}\varrho_k\nabla T_k(u)\,dx+\varepsilon(n,j,h).\end{aligned}$$

On the other hand, by using Proposition 4.2(ii) and (4.15) we can easily find that

$$(4.34)\qquad \int_\Omega \phi(u_n)\nabla w_{n,j}^h\,dx=\int_\Omega \phi(u)\nabla T_{2k}(u-T_h(u))\,dx+\varepsilon_h(n,j)=\varepsilon_h(n,j)$$

and

$$(4.35)\qquad \int_\Omega F\nabla w_{n,j}^h\,dx=\int_\Omega F\nabla T_{2k}(u-T_h(u))\,dx+\varepsilon(n,j).$$

Combining (4.26) and (4.33)–(4.35), we get

$$\begin{aligned}(4.36)\qquad &\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\qquad\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\quad\le \int_{\Omega\setminus\Omega_s}\varrho_k\nabla T_k(u)\,dx+\int_\Omega F\nabla T_{2k}(u-T_h(u))\,dx+\varepsilon(n,j,h).\end{aligned}$$

Now, we remark that

$$\begin{aligned}&\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &\qquad-\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\quad=\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &\qquad-\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\,dx\\ &\qquad+\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(v_j)\chi_s^j-\nabla T_k(u)\chi_s]\,dx,\end{aligned}$$

and, as can be easily seen, each integral on the right hand side is of the form

$\varepsilon(n,j)$, implying that

$$(4.37)\qquad \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_s^j)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_s^j]\,dx + \varepsilon(n,j),$$

and thanks to (4.36) and (4.37), we deduce

$$\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)][\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \le \int_{\Omega\setminus\Omega_s} \varrho_k \nabla T_k(u)\,dx + \int_\Omega F\nabla T_{2k}(u - T_h(u))\,dx + \varepsilon(n,j,h).$$

Hence,

$$(4.38)\qquad \limsup_{n\to\infty} \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \le \int_{\Omega\setminus\Omega_s} \varrho_k \nabla T_k(u)\,dx + \int_\Omega F\nabla T_{2k}(u - T_h(u))\,dx + \lim_{n\to\infty} \varepsilon(n,j,h).$$

We shall now prove that

$$(4.39)\qquad \int_\Omega F\nabla T_{2k}(u - T_h(u))\,dx \to 0 \quad \text{as } h \to \infty.$$

If we take $u_n - T_{2k}(u_n - T_h(u_n))$ as a test function in (4.1) we obtain

$$\int_{\{h\le|u_n|\le 2k+h\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx + \int_\Omega \phi(u_n)\nabla T_{2k}(u_n - T_h(u_n))\,dx \le \int_{\{h\le|u_n|\le 2k+h\}} F\nabla u_n\,dx + \int_\Omega f_n T_{2k}(u_n - T_h(u_n))\,dx.$$

Since $\int_\Omega \phi(u_n)\nabla T_{2k}(u_n - T_h(u_n))\,dx = 0$, we get

$$\int_{\{h\le|u_n|\le 2k+h\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \le \int_{\{h\le|u_n|\le 2k+h\}} F\nabla u_n\,dx + \int_\Omega f_n T_{2k}(u_n - T_h(u_n))\,dx,$$

which yields, thanks to (A_4) and Young's inequality,

$$\frac{\alpha}{2}\int_{\{h\le|u_n|\le 2k+h\}} M(|\nabla u_n|)\,dx \le C_1 \int_{\{|u_n|>h\}} \overline{M}(|F|)\,dx + \int_\Omega f_n T_{2k}(u_n - T_h(u_n))\,dx.$$

Letting $n \to \infty$, by using the Fatou lemma, we get

$$\frac{\alpha}{2} \int_{\{h \le |u| \le 2k+h\}} M(|\nabla u|)\,dx \le C_1(k) \int_{\{|u|>h\}} \overline{M}(|F|)\,dx + \int_{\{|u|>h\}} |f|\,dx.$$

Consequently,

$$\limsup_{h\to\infty} \int_{\{h\le|u|\le 2k+h\}} M(|\nabla u|)\,dx = 0,$$

so that

$$\lim_{h\to\infty} \int_{\Omega} F\nabla T_{2k}(u - T_h(u))\,dx = 0,$$

which implies (4.39).

Thanks to (4.38) and (4.39), we can write

$$\limsup_{n\to\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx \le \int_{\Omega\setminus\Omega_s} \varrho_k \nabla T_k(u)\,dx + \lim_{n\to\infty} \varepsilon(n, j, h),$$

in which we can pass to the limit as $j, h, s \to \infty$ to obtain

$$\limsup_{s\to\infty} \limsup_{n\to\infty} \int_{\Omega} [a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u)\chi_s)] \times [\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx = 0.$$

Finally, Lemma 4.1 yields the conclusion of Proposition 4.4.

4.3. *Proof of Theorem 3.1.* Let $v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$. Taking $u_n - T_k(u_n - v)$ as a test function in (4.1), we can write

$$\int_{\Omega} a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\,dx + \int_{\Omega} \phi(T_n(u_n))\nabla T_k(u_n - v)\,dx \le \int_{\Omega} f_n T_k(u_n - v)\,dx + \int_{\Omega} F_n \nabla T_k(u_n - v)\,dx,$$

which implies that

$$\begin{aligned} (4.40)\quad & \int_{\{|u_n - v|\le k\}} a(x, u_n, \nabla u_n)\nabla u_n\,dx \\ & - \int_{\{|u_n - v|\le k\}} a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla v\,dx \\ & + \int_{\Omega} \phi(T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n - v)\,dx \\ & \le \int_{\Omega} f_n T_k(u_n - v)\,dx + \int_{\Omega} F_n \nabla T_k(u_n - v)\,dx. \end{aligned}$$

By Fatou's lemma and the fact that

$$a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))$$

weakly in $(L_{\overline{M}}(\Omega))^N$ for $\sigma(\prod L_{\overline{M}}, \prod E_M)$ one easily sees that

$$\begin{aligned}(4.41)\qquad \lim_{n\to\infty}\Big\{ &\int_{\{|u_n-v|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx \\ &- \int_{\{|u_n-v|\le k\}} a(x, T_{k+\|v\|_\infty}(u_n), \nabla T_{k+\|v\|_\infty}(u_n))\nabla v\,dx\Big\} \\ \ge &\int_{\{|u-v|\le k\}} a(x,u,\nabla u)\nabla u\,dx \\ &- \int_{\{|u-v|\le k\}} a(x, T_{k+\|v\|_\infty}(u), \nabla T_{k+\|v\|_\infty}(u))\nabla v\,dx.\end{aligned}$$

On the other hand, by using Proposition 4.2, we can easily see that

$$(4.42)\qquad \int_\Omega \phi(T_n(u_n))\nabla T_k(u_n - v)\,dx \to \int_\Omega \phi(u)\nabla T_k(u-v)\,dx,$$

$$(4.43)\qquad \int_\Omega F\nabla T_k(u_n - v)\,dx \to \int_\Omega F\nabla T_k(u-v)\,dx,$$

$$(4.44)\qquad \int_\Omega f_n T_k(u_n - v)\,dx \to \int_\Omega f T_k(u-v)\,dx$$

as $n \to \infty$.

Combining (4.40)–(4.44), we have

$$\begin{aligned}(4.45)\qquad \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \le \int_\Omega f T_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-v)\,dx \\ \forall v \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega).\end{aligned}$$

Now, let $v \in K_\psi \cap L^\infty(\Omega)$. By (A_5) there exists $v_j \in K_\psi \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ such that v_j converges to v in the modular sense. Let $h > \max(\|v_0\|_\infty, \|v\|_\infty)$. Taking $v = T_h(v_j)$ in (4.45), we have

$$\begin{aligned}\int_\Omega a(x,u,\nabla u)\nabla T_k(u - T_h(v_j))\,dx + \int_\Omega \phi(u)\nabla T_k(u - T_h(v_j))\,dx \\ \le \int_\Omega f T_k(u - T_h(v_j))\,dx + \int_\Omega F\nabla T_k(u - T_h(v_j))\,dx.\end{aligned}$$

We can easily pass to the limit as $j \to \infty$ to get

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(v))\,dx + \int_\Omega \phi(u)\nabla T_k(u-T_h(v))\,dx \\ \le \int_\Omega fT_k(u-T_h(v))\,dx + \int_\Omega F\nabla T_k(u-T_h(v))\,dx \quad \forall v \in K_\psi \cap L^\infty(\Omega).$$

Finally, since $h \ge \max(\|v_0\|_\infty, \|v\|_\infty)$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega \phi(u)\nabla T_k(u-v)\,dx \\ \le \int_\Omega fT_k(u-v)\,dx + \int_\Omega F\nabla T_k(u-T_h(v))\,dx \quad \forall v \in K_\psi \cap L^\infty(\Omega),\ \forall k > 0.$$

Thus, the proof of Theorem 3.1 is complete.

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