APPLICATIONES MATHEMATICAE
33,2 (2006), pp. 243–252

Daniel Berend and Luba Bromberg (Beer-Sheva)

UNIFORM DECOMPOSITIONS OF POLYTOPES

Abstract. We design a method of decomposing convex polytopes into simpler polytopes. This decomposition yields a way of calculating exactly the volume of the polytope, or, more generally, multiple integrals over the polytope, which is equivalent to the way suggested in [9], based on Fourier–Motzkin elimination ([10, pp. 155–157]). Our method is applicable for finding uniform decompositions of certain natural families of polytopes. Moreover, this allows us to find algorithmically an analytic expression for the distribution function of a random variable of the form $\sum_{i=1}^{d} c_i X_i$, where $(X_1, \ldots, X_d)$ is a random vector, uniformly distributed in a polytope.

1. Introduction. The indefinite integral of a function is in general "smoother" than the function itself. However, it is also usually more difficult to express. Thus, the integral of an elementary function is usually non-elementary. The value of a definite integral may not be a "recognizable" number even if the function is quite simple. The situation is even more difficult for multiple integrals; these can be seldom evaluated exactly. Therefore, there is an abundance of methods for approximating the values of definite integrals.

One situation where multiple integrals may be exactly calculated is where the region of integration is a polytope and the function very special. For example, if the function is constant, the problem reduces to the computation of the volume of the polytope. For the problem of calculating more general multiple integrals see, for example, [2] and [8]. While the problem of finding the exact volume of a polytope is $\sharp P$-hard [4], which implies that no efficient algorithm should be expected, there are nevertheless several algorithms for it. These algorithms start with decomposing the given polytope into a union

2000 *Mathematics Subject Classification*: 65D18, 60-08.
Key words and phrases: decomposition of polyhedra, repetitive polyhedron, calculating volume.

(or signed union [3]) of simpler polytopes, usually simplices. These polytopes are disjoint up to lower dimensional faces, and the volume of each is easy to calculate. We refer to [6] for updated information on the status of this and numerous other problems concerning polytopes.

Our interest in the problem started from trying to generalize a certain result, related to a problem in physics, where a certain distribution function has to be evaluated [1, Th. 1]. The evaluation of distribution (and density) functions often reduces to a problem of integration. For example, if X and Y are independent random variables, with densities f_X and f_Y, respectively, then the density of the sum $X+Y$ is given by

$$f_{X+Y}(t) = \int\limits_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx, \qquad -\infty < t < \infty.$$

The analytic expression for $f_{X+Y}(t)$, if there is one, is usually quite cumbersome, as it may be given by distinct formulas in distinct intervals. (See, for example, the formula for the density of the sum of n independent $U(0,1)$ variables [5, p. 27].)

In this paper we design a method for decomposing a polytope which, in principle, transforms the problem of calculating a multiple integral on a polytope into that of calculating the sum of (a huge number of) repeated integrals. Although very different looking, our method is essentially equivalent to the one presented in [9], which is based on the Fourier–Motzkin elimination method. (However, it may be more amenable to certain computational improvements; see Remark 3.1 below.) It enables us tackling also the following problem: Given a polytope $P \subseteq \mathbb{R}^d$ and a linear function L of d arguments, find an expression for the volume of the sub-polytope P, consisting of those points of P at which the value of L does not exceed t, as a function of t. Clearly, this solves the problem of finding the distribution function of a random variable of the form $L(\vec{X})$, where L is the linear function of several variables and $\vec{X}$ a random vector, uniformly distributed in some polytope.

In Section 2 we present the main results. Section 3 is devoted to the proofs.

We would like to express our gratitude to A. Barvinok, A. Enge, M. Goldberg, V. Kaibel, K. Kedem, M. Pfetsch and T. Scot for the helpful information they provided us on the subject matter.

2. Main results. Methods for finding the exact volume of a polytope usually fall into one of three categories: those starting with the *half-space* (H-) representation of the polytope, those starting from its *vertex* (V-) representation, and those requiring both representations. Our method belongs to the first of these. Thus, let $P \subseteq \mathbb{R}^d$ be a *polyhedron*, given in H-representation, that is,

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq b\} \tag{1}$$

for some $m \times d$ matrix A and m-vector b. Of course, calculation of volumes is relevant only for *polytopes*, that is, bounded polyhedra, but multiple integration makes sense for general polyhedra as well.

It will be usually convenient for us to move the components of the vector b in the definition of P to the left-hand side, and write the system of inequalities (1) defining P in the form

$$\begin{cases} f_1(x_1, \dots, x_d) \leq 0, \\ \qquad\quad \dots \\ f_m(x_1, \dots, x_d) \leq 0. \end{cases} \tag{2}$$

Here $f_i(x_1, \dots, x_d) = A_i\mathbf{x} - b_i$, where $\mathbf{x} = (x_1, \dots, x_d)$ and A_i is the ith row of A. Also, for $1 \leq i \leq d$, we write $\overline{x}_i = (x_1, \dots, x_i)$.

Definition 2.1. (a) A polytope $P \subseteq \mathbb{R}^d$ is *repetitive* if it may be represented in the form

$$P = \{(x_1, \dots, x_d) : a \leq x_1 \leq b,\ f_1^-(x_1) \leq x_2 \leq f_1^+(x_1), \dots, \\ f_{d-1}^-(\overline{x}_{d-1}) \leq x_d \leq f_{d-1}^+(\overline{x}_{d-1})\} \tag{3}$$

for appropriate $a, b \in \mathbb{R}$ and linear functions $f_i^+, f_i^- : \mathbb{R}^i \to \mathbb{R}, 1 \leq i \leq d-1$, satisfying $f_i^-(\overline{x}_i) \leq f_i^+(\overline{x}_i)$ for every $\overline{x}_i$ in the projection of P in (the space corresponding to the first i coordinates) $\mathbb{R}^i$, $1 \leq i \leq d-1$.

(b) A polyhedron is *repetitive* if it can be represented as in (3), where some of the constraints $x_i \leq f_i^+(\overline{x}_{i-1})$ (or $x_i \geq f_i^-(\overline{x}_{i-1})$) may be omitted (and the same for the constraints $x_1 \geq a$ and $x_1 \leq b$).

Remark 2.1. The property of being repetitive depends on the way the polytope is embedded in $\mathbb{R}^d$, and is not an inherent property. For example, a rectangle with edges parallel to the coordinate axes is repetitive according to our definition, but after a rotation it may become non-repetitive. Also, the property depends on the order in which the coordinates are taken. For example, the triangle defined by the system

$$\begin{cases} -1 \leq x \leq 1, \\ 0 \leq y \leq 1, \\ |x| + y \leq 1, \end{cases}$$

is repetitive if we start with the variable y, but not if we start with x.

Theorem 2.1. *Any polyhedron P is effectively decomposable into a union of finitely many repetitive polyhedra, the intersection of any two of which is contained in a $(d-1)$-dimensional polytope.*

Remark 2.2. As will be apparent from the proof, the number of polyhedra in the decomposition is in general of the order of magnitude of $(m^2/4)^d$. However, for the application to calculating multiple integrals, it will be possible to reduce the computation to about m^d repetitive polyhedra. (For details, see Remark 3.1 infra.) As elaborated in [3], each method works well for certain types of polyhedra and worse for others. Our method is clearly much superior to methods based on decomposition into simplices for, say, axis-parallel boxes or, more generally, repetitive polyhedra. It is much worse, however, for simplices.

The decomposition shows immediately whether or not P is a polytope. If it is one, we can find its volume $\mathrm{Vol}(P)$ by summing the volumes of the repetitive polytopes appearing in the decomposition of P. The volume of a repetitive polytope as in (3) is calculated in a straightforward manner:

$$\mathrm{Vol}(P) = \int_a^b \int_{f_1^-(x_1)}^{f_1^+(x_1)} \cdots \int_{f_{d-1}^-(\overline{x}_{d-1})}^{f_{d-1}^+(\overline{x}_{d-1})} dx_d \ldots dx_2\, dx_1.$$

Sometimes, we may want to find the volumes of all members in a parameterized family of polytopes as a function of the parameter. For example, consider the following situation. Let $(X_1, \ldots, X_d)$ be a d-dimensional random variable, uniformly distributed in a polytope P of positive volume in $\mathbb{R}^d$. That is, the probability of $(X_1, \ldots, X_d)$ to assume a value in some (measurable) set $A \subseteq P$ is $\mathrm{Vol}(A)/\mathrm{Vol}(P)$. Consider a 1-dimensional random variable of the form $T = c_1X_1 + \cdots + c_dX_d$ for some constants $c_1, \ldots, c_d$. To find the value of the distribution function $F_T(t)$ at any point t, we need to find the ratio

$$\frac{\mathrm{Vol}(P \cap \{\mathbf{x} \in \mathbb{R}^d : c_1x_1 + \cdots + c_dx_d \le t\})}{\mathrm{Vol}(P)}. \tag{4}$$

The calculation of this expression for any specific value of t presents no difficulty. However, we would like to obtain an explicit formula for it as a function of t.

Example 2.1. Let $P = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \ge 0,\ x + y + z \le 1\}$ and $T = X + 2Y + 3Z$. We would like to express $F_T(t)$ as a function of t.

To this end, it is natural to look for a way of writing the polyhedron

$$P \cap \{\mathbf{x} \in \mathbb{R}^d : c_1x_1 + \cdots + c_dx_d \le t\},$$

appearing in the numerator of (4), as a union of repetitive polyhedra in a uniform way as t varies. In general, given a polyhedron $P \subseteq \mathbb{R}^d$ and a fixed linear function $L : \mathbb{R}^d \to \mathbb{R}$, denote

$$P_L(t) = P \cap \{\mathbf{x} \in \mathbb{R}^d : L(\mathbf{x}) \le t\}.$$

EXAMPLE 2.2. Let P and T be as in Example 2.1. Thus, putting $L(x,y,z) = x + 2y + 3z$, we want to express $P_L(t)$ as a function of t. Let

$$
\begin{aligned}
P_{1,t} &= \{ && 1 \le x \le t, && 0 \le y \le \tfrac{t-x}{2}, && 0 \le z \le \tfrac{t-x-2y}{3} \},\\
P_{2,t} &= \{ && 2-t \le x \le \tfrac{3-t}{2}, && 3-t-2x \le y \le 1-x, && 0 \le z \le 1-x-y \},\\
P_{3,t} &= \{ && 0 \le x \le 2-t, && 0 \le y \le \tfrac{t-x}{2}, && 0 \le z \le \tfrac{t-x-2y}{3} \},\\
P_{4,t} &= \{ && 2-t \le x \le \tfrac{3-t}{2}, && 0 \le y \le 3-t-2x, && 0 \le z \le \tfrac{t-x-2y}{3} \},\\
P_{5,t} &= \{ && \tfrac{3-t}{2} \le x \le 1, && 0 \le y \le 1-x, && 0 \le z \le 1-x-y \},\\
P_{6,t} &= \{ && 0 \le x \le \tfrac{3-t}{2}, && 3-t-2x \le y \le 1-x, && 0 \le z \le 1-x-y \},\\
P_{7,t} &= \{ && 0 \le x \le 1, && 0 \le y \le 1-x, && 0 \le z \le 1-x-y \}.
\end{aligned}
$$

One can verify that

$$
P_L(t) = \begin{cases} \emptyset, & t < 0,\\ P_{1,t}, & 0 \le t \le 1,\\ P_{2,t} \cup P_{3,t} \cup P_{4,t} \cup P_{5,t}, & 1 < t \le 2,\\ P_{5,t} \cup P_{6,t}, & 2 < t \le 3,\\ P_{7,t}, & 3 < t. \end{cases} \tag{5}
$$

We want to develop an algorithm for obtaining representations of $P_L(t)$ of the form (5). More precisely, we start with

DEFINITION 2.2. Let $\{P_t : t \in I\}$ be a family of polyhedra, where I is some interval (finite or infinite). The family is *uniformly repetitive* if there exist linear functions $f_i^+, f_i^- : \mathbb{R}^{i+1} \to \mathbb{R}$, $0 \le i \le d-1$, satisfying $f_i^-(t, \overline{x}_i) \le f_i^+(t, \overline{x}_i)$ for every $t \in I$ and $\overline{x}_i$ in the projection of P_t onto (the space corresponding to the first i coordinates) $\mathbb{R}^i$, such that

$$
\begin{aligned}
P_t = \{\mathbf{x} \in \mathbb{R}^d : f_0^-(t) \le x_1 \le f_0^+(t),\ f_1^-(t, x_1) \le x_2 \le f_1^+(t, x_1), \ldots,\\
f_{d-1}^-(t, \overline{x}_{d-1}) \le x_d \le f_{d-1}^+(t, \overline{x}_{d-1})\}
\end{aligned} \tag{6}
$$

(where some of the functions f_i^+ or f_i^- may be replaced by ∞ or $-\infty$, respectively).

THEOREM 2.2. *Let $P \subseteq \mathbb{R}^d$ be a polyhedron and $L : \mathbb{R}^d \to \mathbb{R}$ a linear function. Then we can effectively find a decomposition of $\mathbb{R}$, say $\mathbb{R} = \bigcup_{j=1}^k I_j$, into a union of finitely many (finite and infinite) intervals, and uniformly repetitive families $\{P_{j,i,t} : t \in I_j\}$, $1 \le j \le k$, $1 \le i \le l_j$, such that*

$$
P_L(t) = \bigcup_{i=1}^{l_j} P_{j,i,t}, \qquad t \in I_j,\ 1 \le j \le k. \tag{7}
$$

EXAMPLE 2.3. Let P and T be as in Examples 2.1 and 2.2. Employing (5), it is easy to verify that the distribution function of T is given by

$$F_T(t) = \begin{cases} 0, & t \le 0, \\ t^3/3, & 0 < t \le 1, \\ 1/2 - 3t/2 + 3t^2/2 - t^3/3, & 1 < t \le 2, \\ -7/2 + 9t/2 - 3t^2/2 + t^3/6, & 2 < t \le 3, \\ 1, & 3 < t. \end{cases} \tag{8}$$

Now we shall see how a formula like (8) may be obtained algorithmically in general. To state our result, we need

DEFINITION 2.3. A function $g : \mathbb{R} \to \mathbb{R}$ is *piecewise polynomial* if there exist intervals (finite or infinite) $I_j \subseteq \mathbb{R}$ and polynomials Q_j, $1 \le j \le k$, such that

$$g(x) = Q_j(x), \qquad x \in I_j,\ 1 \le j \le k.$$

The *degree* of g is $\max_{1 \le j \le k} \deg Q_j$.

REMARK 2.3. To avoid trivialities, we require that, if the length of some interval I_j is 0, then the corresponding polynomial Q_j is constant.

THEOREM 2.3. *Let $(X_1, \dots, X_d)$ be a d-dimensional random variable, uniformly distributed in a polytope P of positive volume in $\mathbb{R}^d$. Given any constants $c_1, \dots, c_d$, the distribution function of the 1-dimensional random variable $T = c_1X_1 + \cdots + c_dX_d$ is a continuous piecewise polynomial function of degree at most d, and can be effectively computed.*

3. Proofs

Proof of Theorem 2.1. We use induction on the dimension d. For $d = 1$ the polyhedron is an interval (finite or infinite), and we easily find its endpoints to write $P = \{x : a \le x \le b\}$ (where possibly $a = -\infty$ or $b = \infty$).

Assume the theorem holds for $(d-1)$-dimensional polyhedra, and let P be a d-dimensional polyhedron given, say, by (2). Without loss of generality, any two of the f_i's (not including free terms) are linearly independent.

Reordering the f_i's, we may assume that, for some $0 \le k \le l \le m$, the coefficient of x_d in $f_1, \dots, f_k$ is positive, in $f_{k+1}, \dots, f_l$ negative, and in $f_{l+1}, \dots, f_m$ zero. The first k inequalities provide a "ceiling" for x_d,

$$\begin{cases} x_d \le g_1^+(x_1, \dots, x_{d-1}) = g_1^+(\overline{x}_{d-1}), \\ \qquad\qquad\cdots \\ x_d \le g_k^+(x_1, \dots, x_{d-1}) = g_k^+(\overline{x}_{d-1}), \end{cases}$$

and the next $l-k$ provide a "floor",

$$\begin{cases} x_d \geq g_{k+1}^-(x_1,\dots,x_{d-1}) = g_{k+1}^-(\overline{x}_{d-1}), \\ \qquad\qquad\qquad \dots \\ x_d \geq g_l^-(x_1,\dots,x_{d-1}) = g_l^-(\overline{x}_{d-1}). \end{cases}$$

Given a point $(x_1,\dots,x_{d-1})$, the maximal value of x_d (if any) for which $(x_1,\dots,x_d)$ belongs to P is one of the values $g_1^+(\overline{x}_{d-1}),\dots,g_k^+(\overline{x}_{d-1})$ (the minimal of them, i.e., $g^+(\overline{x}_{d-1}) = \min\{g_1^+(\overline{x}_{d-1}),\dots,g_k^+(\overline{x}_{d-1})\}$).

We want to split P into subsets, according to which of the g_i^+'s is the actual ceiling:

$$P_i^+ = \{(x_1,\dots,x_d) \in P : g_i^+(\overline{x}_{d-1}) \leq g_{i'}^+(\overline{x}_{d-1}),\, i' = 1,\dots,k\}, \qquad 1 \leq i \leq k.$$

Similarly, we may split P according to the prevalent floor:

$$P_j^- = \{(x_1,\dots,x_d) \in P : g_j^-(\overline{x}_{d-1}) \geq g_{j'}^-(\overline{x}_{d-1}),\, j' = k+1,\dots,l\}, \qquad k+1 \leq j \leq l.$$

Finally, put

$$P_{ij} = P_i^+ \cap P_j^-, \qquad 1 \leq i \leq k,\, k+1 \leq j \leq l.$$

Obviously, $P = \bigcup_{i=1}^k \bigcup_{j=k+1}^l P_{ij}$, and the intersection of any two of the P_{ij}'s is contained in some hyperplane in $\mathbb{R}^d$. (If $k=0$ or $l=k$, namely x_d is unbounded from above or below, the splitting of P is only according to the prevalent floor or ceiling, respectively. If $l=k=0$, then we do not split P at all. In the following we shall disregard these simpler cases.)

Denote by $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ the projection given by $\pi(\overline{x}_d) = \overline{x}_{d-1}$. The set $\pi(P_{ij})$ is the polyhedron in $\mathbb{R}^{d-1}$ determined by the $(m-1)$ inequalities

$$\begin{array}{ll} g_j^-(\overline{x}_{d-1}) \leq g_i^+(\overline{x}_{d-1}), & \\ g_i^+(\overline{x}_{d-1}) \leq g_{i'}^+(\overline{x}_{d-1}), & i' = 1,2,\dots,i-1,i+1,\dots,k, \\ g_j^-(\overline{x}_{d-1}) \geq g_{j'}^-(\overline{x}_{d-1}), & j' = k+1,k+2,\dots,j-1,j+1,\dots,l, \\ f_s(\overline{x}_d) \leq 0, & l+1 \leq s \leq m. \end{array}$$

(Recall that the f_s's do not depend on the dth coordinate, so the inequalities involving them make sense in $\mathbb{R}^{d-1}$.)

According to the induction hypothesis, each $\pi(P_{ij})$ may be expressed as a finite union of repetitive polyhedra. For typical i,j, write $\pi(P_{ij}) = \bigcup_{h\in H_{ij}} Q_{ijh}$, where H_{ij} is a finite index set and the Q_{ijh}'s are repetitive and intersect each other in sets of dimension $d-2$ (or smaller). The set $P_{ij} \cap \pi^{-1}(Q_{ijh})$ is also repetitive, as it is determined by the same inequalities defining Q_{ijh} and the additional inequality $g_j^-(\overline{x}_{d-1}) \leq x_d \leq g_i^+(\overline{x}_{d-1})$. Hence the decomposition $P = \bigcup_{i=1}^k \bigcup_{j=k+1}^l \bigcup_{h\in H_{ij}} (P_{ij} \cap \pi^{-1}(Q_{ijh}))$ provides a decomposition of P as required.

Remark 3.1. The number of polyhedra in the decomposition may be (almost) as large as $(m^2/4)^d$. In fact, the first decomposition of P in the proof is into $k(l-k)$ polyhedra with common floors and ceilings. If the m constraints are divided roughly evenly between x_d-floors and x_d-ceilings, then after this stage we have about $m^2/4$ polyhedra. Each of these will be divided at the second stage into about $(m-1)^2/4$ polyhedra, and so forth, leaving us at the end of the process with a union of

$$\frac{m^2}{4} \cdot \frac{(m-1)^2}{4} \cdots \frac{(m-d+1)^2}{4} \approx \left(\frac{m^2}{4}\right)^d$$

polyhedra. Note that there are explicit examples of polytopes for which the decomposition will yield exponentially many components [10, p. 156].

One may try to reduce this number by splitting each region according to that variable which gives as many floors and few ceilings (or vice versa) as possible. In the best case, this might lead to a decomposition into about m^d polyhedra. However, for "random" constraints it is unlikely that the gain achieved by this heuristic will be significant.

If we allow signed decompositions (which is suitable in particular for the application to calculating volumes and multiple integrals), we can do much better. In fact, let P' be the projection of P on the subspace determined by the first $d-1$ coordinates. Suppose P is bounded above by a (piecewise linear) surface g^+ and below by a surface g^-, that is, for each $(x_1, \ldots, x_{d-1}) \in P'$,

$$\{x \in \mathbb{R} : (x_1, \ldots, x_{d-1}, x) \in P\} = [g^-(\overline{x}_{d-1}), g^+(\overline{x}_{d-1})].$$

Set $a_i = \min_{\overline{x}_d \in P} x_i$ for $i = 1, \ldots, d$. (Note that the a_i's are easily found by solving suitable linear programming problems.) Put

$$P^+ = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : (x_1, \ldots, x_{d-1}) \in P',\ a_d \le x_d \le g^+(\overline{x}_{d-1})\},$$
$$P^- = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : (x_1, \ldots, x_{d-1}) \in P',\ a_d \le x_d \le g^-(\overline{x}_{d-1})\}.$$

Then $P = P^+ - P^-$ (up to $(d-1)$-dimensional boundaries). Decomposing P^+ and P^- similarly, we finally arrive at a signed decomposition consisting of about m^d repetitive polyhedra.

Proof of Theorem 2.2. Let P be given by (2) and $L(\mathbf{x}) = c_1x_1 + \cdots + c_dx_d$. Write the system of inequalities defining $P_L(t)$ in the form

$$\begin{cases} f_1(x_1, \ldots, x_d) \le 0, \\ \qquad\qquad \cdots \\ f_m(x_1, \ldots, x_d) \le 0, \\ -t + c_1x_1 + \cdots + c_dx_d \le 0. \end{cases} \tag{9}$$

The system (9), with t varying rather than fixed, defines a $(d+1)$-dimensional polyhedron. Denote this polyhedron by $\widetilde{P}$. By Theorem 2.1 we can write

$\widetilde{P}=\bigcup_{i'=1}^{r} P_{i'}$, where the polyhedra $P_{i'}$ intersect in lower dimensional sets and are repetitive:

$$P_{i'}=\{(t,x_1,\dots,x_d): a_{i'}\le t\le b_{i'},\ f^-_{0,i'}(t)\le x_1\le f^+_{0,i'}(t),\dots,\\ f^-_{d-1,i'}(t,x_1,\dots,x_{d-1})\le x_d\le f^+_{d-1,i'}(t,x_1,\dots,x_{d-1})\}. \tag{10}$$

(Note that the $a_{i'}$'s and $b_{i'}$'s, as well as the functions $f^-_{j',i'}$ and $f^+_{j',i'}$, may be infinite.) By splitting the polyhedra $P_{i'}$ if necessary, we may assume that the t-intervals (and rays) $[a_{i'},b_{i'}]$ belonging to distinct $P_{i'}$'s either coincide or are disjoint (except for their endpoints). Denote these intervals by $I_1,\dots,I_k$.

Now group the polyhedra $P_{i'}$ according to the t-intervals they lie over. That is, for $1\le j\le k$, let

$$P_{ji}=\{(t,x_1,\dots,x_d): t\in I_j,\ f^-_{0,i}(t)\le x_1\le f^+_{0,i}(t),\dots,\\ f^-_{d-1,i}(t,\overline{x}_{d-1})\le x_d\le f^+_{d-1,i}(t,\overline{x}_{d-1})\},\quad i=1,\dots,l_j, \tag{11}$$

be the polyhedra $P_{i'}$ lying over I_j.

For $1\le j\le k,\ 1\le i\le l_j$ and any $t\in I_j$, set

$$P_{j,i,t}=\{(x_1,\dots,x_d): f^-_{0,i}(t)\le x_1\le f^+_{0,i}(t),\dots,\\ f^-_{d-1,i}(t,x_1,\dots,x_{d-1})\le x_d\le f^+_{d-1,i}(t,x_1,\dots,x_{d-1})\}. \tag{12}$$

Each of the families $\{P_{j,i,t}: t\in I_j\}$, $1\le j\le k$, $1\le i\le l_j$, is uniformly repetitive, and the required decomposition (7) clearly holds.

Proof of Theorem 2.3. Let

$$L(\mathbf{x})=c_1x_1+\dots+c_dx_d,\qquad \mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d,$$

and denote

$$P_L(t)=P\cap\{\mathbf{x}\in\mathbb{R}^d: L(\mathbf{x})\le t\},\quad -\infty<t<\infty.$$

The distribution function of T is given by

$$F_T(t)=\frac{\mathrm{Vol}(P_L(t))}{\mathrm{Vol}(P)},\qquad -\infty<t<\infty.$$

The denominator is constant and we want to find the numerator. According to Theorem 2.2 we define a decomposition of the t-interval $(-\infty,\infty)$ into a union of finitely many (finite and infinite) intervals I_j, and uniformly repetitive families $P_{j,i,t}$ as in (7). By representation (12) of $P_{j,i,t}$ we can write

$$\mathrm{Vol}(P_L(t))=V_j(t)=\sum_{i=1}^{l_j}\int_{f^-_{0,i}(t)}^{f^+_{0,i}(t)}\dots\int_{f^-_{d-1,i}(t,\overline{x}_d)}^{f^+_{d-1,i}(t,\overline{x}_d)} dx_d\dots dx_1,\qquad t\in I_j,\ 1\le j\le k.$$

(Note that the functions $f^-_{j,i}$ and $f^+_{j,i}$ may be only finite.) Obviously, each

$V_j(t)$ is a polynomial of degree at most d. Hence $F_T(t)$ is a piecewise polynomial of degree at most d. The continuity of $F_T(t)$ is clear.

References

[1] Y. Avishai, D. Berend and R. Berkovits, *Statistics of addition spectra of independent quantum systems*, J. Phys. A (Math. Gen.) 31 (1998), 8063–8072.

[2] A. Barvinok, *Computing the volume, counting integral points, and exponential sums*, Discrete Comput. Geom. 10 (1993), 123–141.

[3] B. Büeler, A. Enge and K. Fukuda, *Exact volume computation for polytopes: A practical study*, in: Polytopes—Combinatorics and Computation, G. Kalai and G. M. Ziegler (eds.), Birkhäuser, Basel, 2000, 131–154.

[4] M. E. Dyer and A. M. Frieze, *On the complexity of computing the volume of a polyhedron*, SIAM J. Comput. 17 (1988), 967–974.

[5] W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd ed., Wiley, New York, 1971.

[6] V. Kaibel and M. E. Pfetsch, *Some algorithmic problems in polytope theory*, in: Algebra, Geometry, and Software Systems, M. Joswig and N. Takayama (eds.), Springer, 2003, 23–47. (Online updated version at http://www.zib.de/pfetsch/apropo.)

[7] J. B. Lasserre, *Integration on a convex polytope*, Proc. Amer. Math. Soc. 136 (1998), 2433–2441.

[8] J. B. Lasserre and E. S. Zeron, *Solving a class of multivariate integration problems via Laplace techniques*, Appl. Math. (Warsaw) 28 (2001), 391–405.

[9] M. Schechter, *Integration over a polyhedron: An application of the Fourier–Motzkin elimination method*, Amer. Math. Monthly 105 (1998), 246–251.

[10] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, New York, 1986.

Departments of Mathematics and Computer Science
Ben-Gurion University of the Negev
Beer-Sheva 84105, Israel
E-mail: berend@math.bgu.ac.il

Received on 23.9.2004 (1761)